

MAJORIZATION PROPERTY FOR SPECIFIC CLASSES OF ANALYTIC FUNCTIONS

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ABSTRACT. The majorization property plays a significant role in understanding the behavior of certain classes of analytic functions and their mappings. In this research, we introduce two novel classes of analytic functions, denoted as $S_\zeta[E, F; \mu; \gamma]$ and $T_\zeta(\theta, \mu, \gamma)$ using the usual operator H_s . Our primary focus is to investigate the majorization properties of these newly defined classes of analytic function. Some applications of the results are discussed in the corollaries with different operators.

1. INTRODUCTION AND DEFINITIONS

The majorization property is a concept used in geometric function theory, a branch of complex analysis that deals with the study of holomorphic functions, their properties, and their geometric behavior. In the context of geometric function theory, majorization refers to a specific kind of order relation between two complex-valued functions defined on some domain in the complex plane. We consider two analytic functions, u and v , on a domain D . We say that $u(\xi)$ majorizes $v(\xi)$ on D , denoted as (see [10])

$$u(\xi) \prec\prec v(\xi); \quad (\xi \in D),$$

if there is an analytic function $\psi(\xi)$, such that

$$|\psi(\xi)| \leq 1 \text{ and } u(\xi) = \psi(\xi) v(\xi); \quad (\xi \in D), \quad (1.1)$$

where $D = \{\xi \in \mathbb{C} : |\xi| < 1\}$ is an open unit disk.

In complex analysis, a function is called univalent (or one-to-one) in a region, if it never takes the same value twice within that region. Subordination is a fundamental concept used to compare the behavior of two univalent functions.

We have two univalent functions u and v , defined in D , then the function u is said to be subordinate to the function v (denoted as $u(\xi) \prec v(\xi)$), if there exists a *Schwarz function* w , which is analytic in D and satisfies $|w(\xi)| < 1$, $w(0) = 0$, $\xi \in D$ such that $u(\xi) = v(w(\xi))$, $\forall \xi \in D$.

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As a result, we can define quasi-subordination as follows by combining subordination and majorization:

We define the quasi-subordination relationship between the functions u and v as follows (see [15]): We say that the function $u(\xi)$ is quasi-subordinate to the function $v(\xi)$, denoted as $u(\xi) \prec_q v(\xi)$, if there are two analytic functions $\phi(\xi)$ and $w(\xi)$ in D such that $\frac{u(\xi)}{\phi(\xi)}$ is analytic in D and

$$|\phi(\xi)| \leq 1 \text{ and } w(0) = 0, \quad |w(\xi)| \leq |\xi| \leq 1; \quad (\xi \in D).$$

Additionally, the quasi-subordination relationship is established by the equation

$$u(\xi) = \phi(\xi) v(w(\xi)); \quad (\forall \xi \in D). \quad (1.2)$$

Remark 1.1. (i) If we set $\phi(\xi) = 1$ in (1.2), the definition of quasi-subordination simplifies to the conventional definition of subordination.

(ii) If we put $w(\xi) = \xi$ in (1.2), the definition of quasi-subordination simplifies to the conventional definition of majorization.

Let $\mathcal{A}$ be the class of all functions of the form

$$f(\xi) = \xi + \sum_{\mathfrak{K}=2}^{\infty} a_{\mathfrak{K}} \xi^{\mathfrak{K}}; \quad (\xi \in D), \quad (1.3)$$

which are analytic in open unit disk D , and consider $H_s : \mathcal{A} \rightarrow \mathcal{A}$ be an operator such that $\frac{\xi H'_s(f)(\xi)}{H_s(f)(\xi)}$ is analytic in D with

$$\left. \frac{\xi H'_s(f)(\xi)}{H_s(f)(\xi)} \right|_{\xi=0} = \mu + \sigma + \gamma,$$

and satisfies

$$\xi H'_s(f)(\xi) = \sigma H_{s+1}(f)(\xi) + \zeta H_s(f)(\xi), \quad \forall f \in \mathcal{A}, \quad (1.4)$$

for some $\gamma, \zeta, \sigma \in \mathbb{C}$, and μ is a real number with $\mu > 0$ (see [2]).

Remark 1.2. (i) If we take $\sigma = \lambda + 1$, $\zeta = -\lambda$, $\mu = 1 - \eta$, and $\gamma = \eta + \lambda$ for some integers $\lambda > -1$ and $0 \leq \eta < 1$, then the operator H_s is reduced to the integral operator $\mathcal{I}^\lambda(\beta, \rho)$ with $\beta, \rho \in \mathbb{R} \setminus \mathbb{Z}_0^-$ introduced by Cho et. al. in [4].

(ii) If we take $\sigma = 1 - \lambda$, $\mu = 1 - \alpha$, $\gamma = \alpha - \lambda$ and $\zeta = \lambda$, for $-\infty < \lambda < 2$, $0 \leq \alpha < 1$, then the operator H_s is reduced to the fractional differintegral operator Ω_ξ^λ introduced by Owa and Srivastava in [12].

Now, using the operator H_s , we express the analytic function classes listed below.

Definition 1.3. The function $f \in \mathcal{A}$ is in the class $S_\zeta[E, F; \mu; \gamma]$ iff

$$1 + \frac{1}{\mu} \left(\frac{\xi (H_s f(\xi))'}{H_s f(\xi)} - \zeta - \gamma \right) \prec \frac{1 + E \xi}{1 + F \xi}, \quad (1.5)$$

with $\zeta, \gamma \in \mathbb{C}$ and $-1 \leq F < E \leq 1$.

If we take the value of ζ , μ and γ as defined in Remark (1.2)(i), then this class becomes $S_{\beta,\rho}^\lambda[\eta, E, F]$ which is defined by Patel et. al. in [13].

Again if we take the value of ζ , μ and γ as defined in Remark (1.2)(ii), then this class becomes $S^\lambda(\alpha, E, F)$ which is defined by Patel and Mishra in [14].

Definition 1.4. The function $f \in \mathcal{A}$ is in the class $T_\zeta(\theta, \mu, \gamma)$ iff

$$\frac{e^{i\theta}}{\mu + \zeta + \gamma} \left(\frac{\xi(H_s f(\xi))'}{H_s f(\xi)} \right) \prec e^{\xi \cos \theta} + i \sin \theta; \quad (\xi \in D), \quad (1.6)$$

where $\zeta, \gamma \in \mathbb{C}$, and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

If we take the value of σ , ζ , μ and γ as defined in Remark (1.2)(i), then this class become $T_\lambda[\theta; \eta]$.

If we take the value of σ , ζ , μ and γ as defined in Remark (1.2)(ii), then this class become $T_{\lambda, \alpha}$.

Several mathematicians have lately addressed various majorization issues for univalent and multivalent functions, as well as meromorphic and multivalent functions including separate operators and groups, (see [1], [5], [6], [7], [8], [9], [17], [18]).

The majorization problems of the classes $S_\zeta[E, F; \mu; \gamma]$ and $T_\zeta(\theta, \mu, \gamma)$ are explored in this study in the following sections:

2. MAIN RESULTS

Theorem 2.1. We consider a function $f \in \mathcal{A}$ and an another function $g \in S_\zeta[E, F; \mu; \gamma]$. If $H_s f(\xi)$ is majorized by $H_s g(\xi)$ in the domain D , then

$$|H_{s+1} f(\xi)| \leq |H_{s+1} g(\xi)|, \quad \text{for } |\xi| \leq \epsilon_0, \quad (2.1)$$

where the least positive root of following equation is ϵ_0 .

$$\begin{aligned} & |\mu(E - F) + \gamma F| \epsilon^3 - (2|F| + |\gamma|) \epsilon^2 - [2 + |\mu(E - F) + \gamma F|] \epsilon \\ & + |\gamma| = 0, \end{aligned} \quad (2.2)$$

and $-1 \leq F < E \leq 1$, $\sigma, \gamma, \zeta \in \mathbb{C}$, μ is a real number with $\mu > 0$.

Proof. Since $g \in S_\zeta[E, F; \mu; \gamma]$ then, from (1.5) and definition of majorization

$$1 + \frac{1}{\mu} \left(\frac{\xi(H'_s g(\xi))}{H_s g(\xi)} - \zeta - \gamma \right) = \frac{1 + E u(\xi)}{1 + F u(\xi)},$$

with $u(0) = 0$ and $|u(\xi)| \leq |\xi| < 1$, $\forall \xi \in D$.

Now, from the above equality

$$\frac{\xi(H'_s g(\xi))}{H_s g(\xi)} = \frac{(\zeta + \gamma) + (\mu(E - F) + (\zeta + \gamma)F) u(\xi)}{1 + F u(\xi)}. \quad (2.3)$$

Using the relation (1.4), that is,

$$\xi(H'_s g(\xi)) = \sigma H_{s+1} g(\xi) + \zeta H_s g(\xi),$$

for $\sigma, \zeta \in \mathbb{C}$, we have from (2.3) as

$$\frac{H_{S+1} g(\xi)}{H_s g(\xi)} = \frac{\gamma + (\mu(E - F) + \gamma F) u(\xi)}{\sigma (1 + F u(\xi))},$$

which implies that

$$|H_s g(\xi)| \leq \frac{|\sigma| (1 + |F| |\xi|) |H_{S+1} g(\xi)|}{|\gamma| - |\mu(E - F) + \gamma F| |\xi|}. \quad (2.4)$$

As $H_s f(\xi)$ is majorized by $H_s g(\xi)$ in open unit disk D , then

$$H_s f(\xi) = \phi(\xi) H_s g(\xi). \quad (2.5)$$

Multiplying (2.5) by ξ after differentiating with respect to ξ , we get

$$\xi (H'_s f(\xi)) = \xi \phi(\xi) (H'_s g(\xi)) + \xi \phi'(\xi) H_s g(\xi),$$

on using relation (1.4), we have

$$\sigma H_{s+1} f(\xi) = \xi \phi'(\xi) H_s g(\xi) + \sigma \phi(\xi) H_{s+1} g(\xi),$$

that implies

$$|\sigma| |H_{s+1} f(\xi)| \leq |\xi| |\phi'(\xi)| |H_s g(\xi)| + |\sigma| |\phi(\xi)| |H_{s+1} g(\xi)|. \quad (2.6)$$

As a consequence, considering that the ϕ (*Schwarz function*) meets the inequality, (see [11])

$$|\phi'(\xi)| \leq \frac{1 - |\phi(\xi)|^2}{1 - |\xi|^2}; \quad (\xi \in D), \quad (2.7)$$

on using (2.4) and (2.7) in (2.6), we have

$$|H_{s+1} f(\xi)| \leq \left[\frac{|\xi| (1 - |\phi(\xi)|^2) (1 + |F| |\xi|)}{(1 - |\xi|^2) (|\gamma| - |\mu(E - F) + \gamma F| |\xi|)} + |\phi(\xi)| \right] |H_{s+1} g(\xi)|. \quad (2.8)$$

Setting $|\xi| = \epsilon$, $|\phi(\xi)| = \nu$, then inequality (2.8) leads to

$$|H_{s+1} f(\xi)| \leq \frac{\zeta_2(\epsilon, \kappa) |H_{s+1} g(\xi)|}{(1 - \epsilon^2) (|\gamma| - |\mu(E - F) + \gamma F| \epsilon)}, \quad (2.9)$$

where

$$\zeta_2(\epsilon, \kappa) = \epsilon(1 - \kappa^2)(1 + |F|\epsilon) + \kappa(1 - \epsilon^2)[|\gamma| - |\mu(E - F) + \gamma F|\epsilon].$$

Then, from (2.9)

$$|H_{s+1} f(\xi)| \leq \mathfrak{T}(\epsilon, \kappa) |H_{s+1} g(\xi)|, \quad (2.10)$$

where

$$\mathfrak{T}(\epsilon, \kappa) = \frac{\zeta_2(\epsilon, \kappa)}{(1 - \epsilon^2) (|\gamma| - |\mu(E - F) + \gamma F| \epsilon)}, \quad (2.11)$$

from relation (2.10), in an attempt to prove our result, we have to specify

$$\begin{aligned} \epsilon_0 &= \max\{\epsilon \in [0, 1); \mathfrak{T}(\epsilon, \kappa) \leq 1; \quad \forall \kappa \in [0, 1]\} \\ &= \max\{\epsilon \in [0, 1); G(\epsilon, \kappa) \geq 0; \quad \forall \kappa \in [0, 1]\}, \end{aligned}$$

where

$$G(\epsilon, \kappa) = (1 - \epsilon^2)(1 - \kappa) \left[|\gamma| - |\mu(E - F) + \gamma F| \epsilon \right] \\ - \epsilon(1 - \kappa^2)(1 + |F| \epsilon).$$

A simply calculation shows that the inequality $G(\epsilon, \kappa) \geq 0$ is equivalent to

$$u(\epsilon, \kappa) = \left[|\gamma| - |\mu(E - F) + \gamma F| \epsilon \right] (1 - \epsilon^2) \\ - \epsilon(1 + \kappa)(1 + |F| \epsilon) \geq 0,$$

while the function $u(\epsilon, \kappa)$ has the least value at $\kappa = 1$, i.e.

$$\min\{u(\epsilon, \kappa) : \kappa \in [0, 1]\} = u(\epsilon, 1) = v(\epsilon),$$

where

$$v(\epsilon) = |\mu(E - F) + \gamma F| \epsilon^3 - (2|F| + |\gamma|) \epsilon^2 \\ - [2 + |\mu(E - F) + \gamma F|] \epsilon + |\gamma| = 0,$$

it follows that $v(\epsilon) \geq 0$; $\forall \epsilon \in [0, \epsilon_0]$, where $\epsilon_0 = \epsilon_0(\mu, \gamma, E, F)$ is the least positive root of equation (2.2), which proves the conclusion of (2.1). $\square$

Theorem 2.2. *We consider a function $f \in \mathcal{A}$ and an another function $g \in T_\zeta(\theta, \mu, \gamma)$. If $H_s f(\xi)$ is majorized by $H_s g(\xi)$ in domain D , then*

$$|H_{s+1} f(\xi)| \leq |H_{s+1} g(\xi)| \quad \text{for } |\xi| \leq \epsilon_1, \quad (2.12)$$

where the least positive root of following equation is ϵ_1 .

$$\epsilon^2(|\mu + \zeta + \gamma| e^\epsilon - |\zeta| - |\mu + \gamma| |\tan \theta|) + 2\epsilon |\sec \theta| - (|\mu + \zeta + \gamma| e^\epsilon - |\zeta| - |\mu + \gamma| |\tan \theta|) = 0, \quad (2.13)$$

and $\gamma, \zeta \in \mathbb{C}$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, μ is a real number with $\mu > 0$.

Proof. Since, $g \in T_\zeta(\theta, \mu, \gamma)$ then, from (1.1) and the subordination relation

$$\frac{e^{i\theta}}{\mu + \zeta + \gamma} \left(\frac{\xi(H'_s g(\xi))}{H_s g(\xi)} \right) = e^{u(\xi)} \cos \theta + i \sin \theta, \quad (2.14)$$

with $u(0) = 0$ and $|u(\xi)| \leq |\xi| \leq 1 \quad \forall \xi \in D$.

From (2.14), we have

$$\frac{\xi H'_s g(\xi)}{H_s g(\xi)} = (\mu + \zeta + \gamma) \left(\frac{e^{u(\xi)} + i \tan \theta}{1 + i \tan \theta} \right). \quad (2.15)$$

Now, using (1.4) in (2.15), for $\gamma, \zeta, \sigma \in \mathbb{C}$ and $\mu > 0$, we have the following:

$$\frac{H_{s+1} g(\xi)}{H_s g(\xi)} = \frac{(\mu + \zeta + \gamma) e^{u(\xi)} - \zeta + (\gamma + \mu) i \tan \theta}{\sigma(1 + i \tan \theta)},$$

which implies that

$$|H_s g(\xi)| \leq \frac{|\sigma| |\sec \theta|}{(|\mu + \zeta + \gamma| e^{|\xi|} - |\zeta| - |\mu + \gamma| |\tan \theta|)} |H_{s+1} g(\xi)|. \quad (2.16)$$

Now, since $H_s f(\xi)$ is majorized by $H_s g(\xi)$ in D , we have

$$H_s f(\xi) = \phi(\xi) H_s g(\xi). \quad (2.17)$$

Multiplying (2.17) by ξ after differentiating with respect to ξ , we get

$$\xi(H'_s f(\xi)) = \xi \phi(\xi) (H'_s g(\xi)) + \xi \phi'(\xi) H_s g(\xi),$$

on using relation (1.4), we have

$$\sigma H_{s+1} f(\xi) = \xi \phi'(\xi) H_s g(\xi) + \sigma \phi(\xi) H_{s+1} g(\xi),$$

that implies

$$|\sigma| |H_{s+1} f(\xi)| \leq |\xi| |\phi'(\xi)| |H_s g(\xi)| + |\sigma| |\phi(\xi)| |H_{s+1} g(\xi)|. \quad (2.18)$$

As a consequence, considering that the ϕ (*Schwarz function*) meets the inequality, (see [11])

$$|\phi'(\xi)| \leq \frac{1 - |\phi(\xi)|^2}{1 - |\xi|^2}; \quad (\xi \in D), \quad (2.19)$$

using (2.16) and (2.19) in (2.18), we have

$$|H_{s+1} f(\xi)| \leq \left(\frac{|\xi| (1 - |\phi(\xi)|^2) |\sec\theta|}{(1 - |\xi|^2) (|\mu + \zeta + \gamma| e^{|\xi|} - |\zeta| - |\mu + \gamma| |\tan\theta|)} + |\phi(\xi)| \right) |H_{s+1} g(\xi)|. \quad (2.20)$$

Setting $|\xi| = \epsilon$, $|\phi(\xi)| = \kappa$ ($0 \leq \kappa \leq 1$), then inequality (2.20) leads to

$$|H_{s+1} f(\xi)| \leq \frac{\zeta_1(\epsilon, \kappa)}{(1 - \epsilon^2) (|\mu + \zeta + \gamma| e^\epsilon - |\zeta| - |\mu + \gamma| |\tan\theta|)} |H_{s+1} g(\xi)|, \quad (2.21)$$

where

$$\zeta_1(\epsilon, \kappa) = \epsilon(1 - \kappa^2) |\sec\theta| + \kappa(1 - \epsilon^2) (|\mu + \zeta + \gamma| e^\epsilon - |\zeta| - |\mu + \gamma| |\tan\theta|).$$

Then, from (2.21)

$$|H_{s+1} f(\xi)| \leq \mathfrak{T}_1(\epsilon, \kappa) |H_{s+1} g(\xi)|, \quad (2.22)$$

where

$$\mathfrak{T}_1(\epsilon, \kappa) = \frac{\zeta_1(\epsilon, \kappa)}{(1 - \epsilon^2) (|\mu + \zeta + \gamma| e^\epsilon - |\zeta| - |\mu + \gamma| |\tan\theta|)}. \quad (2.23)$$

From relation (2.22), in order to prove our result, we have to specify

$$\begin{aligned} \epsilon_1 &= \max\{\epsilon \in [0, 1]; \mathfrak{T}_1(\epsilon, \kappa) \leq 1 \quad \forall \kappa \in [0, 1]\} \\ &= \max\{\epsilon \in [0, 1]; G_1(\epsilon, \kappa) \geq 0 \quad \forall \kappa \in [0, 1]\}, \end{aligned}$$

where

$$G_1(\epsilon, \kappa) = (1 - \epsilon^2)(1 - \kappa) (|\mu + \zeta + \gamma| e^\epsilon - |\zeta| - |\mu + \gamma| |\tan\theta|) - \epsilon(1 - \kappa^2) |\sec\theta|.$$

A quick calculation illustrates that the inequality $G_1(\epsilon, \kappa) \geq 0$ is equivalent to

$$u_1(\epsilon, \kappa) = (1 - \epsilon^2) (|\mu + \zeta + \gamma| e^\epsilon - |\zeta| - |\mu + \gamma| |\tan\theta|) - \epsilon(1 + \kappa) |\sec\theta| \geq 0,$$

while the function $u_1(\epsilon, \kappa)$ takes its lowest value at $\kappa = 1$, that is,

$$\min\{u_1(\epsilon, \kappa) : \kappa \in [0, 1]\} = u_1(\epsilon, 1) = v_1(\epsilon),$$

where

$$v_1(\epsilon) = (1 - \epsilon^2) (|\mu + \zeta + \gamma| e^\epsilon - |\zeta| - |\mu + \gamma| |\tan\theta|) - 2\epsilon |\sec\theta| = 0.$$

It follows that $v_2(\epsilon) \geq 0 \quad \forall \epsilon \in [0, \epsilon_1]$, where $\epsilon_1 = \epsilon_1(\theta, \gamma, \mu, \zeta)$ is the least positive root of equation (2.13), which proves the conclusion of (2.12). $\square$

3. COROLLARIES AND CONSEQUENCES

Corollary 3.1. *We consider a function $f \in \mathcal{A}$ and an another function $g \in S_{\beta, \rho}^\lambda[\eta, E, F]$. If $\mathcal{I}^\lambda(\beta, \rho) f(\xi)$ is majorized by $\mathcal{I}^\lambda(\beta, \rho) g(\xi)$ in domain D , then*

$$|\mathcal{I}^{\lambda+1}(\beta, \rho) f(\xi)| \leq |\mathcal{I}^{\lambda+1}(\beta, \rho) g(\xi)| \quad \text{for } |\xi| \leq \epsilon_2, \quad (3.1)$$

where the least positive root of following equation is ϵ_2 .

$$|(1-\eta)E + (\lambda+2\eta-1)F| \epsilon^3 - (2|F| + |\lambda+\eta|) \epsilon^2 - (2 + |(1-\eta)E + (\lambda+2\eta-1)F|) \epsilon + |\eta+\lambda| = 0, \quad (3.2)$$

and $-1 \leq F < E \leq 1$, $\lambda > -1$, $0 \leq \eta < 1$, $\beta, \rho \in \mathbb{R} \setminus \mathbb{Z}_0^-$.

Corollary 3.2. *We consider a function $f \in \mathcal{A}$ and an another function $g \in T_\lambda[\theta; \eta]$. If $\mathcal{I}^\lambda(\beta, \rho) f(\xi)$ is majorized by $\mathcal{I}^\lambda(\beta, \rho) g(\xi)$ in domain D , then*

$$|\mathcal{I}^{\lambda+1}(\beta, \rho) f(\xi)| \leq |\mathcal{I}^{\lambda+1}(\beta, \rho) g(\xi)| \quad \text{for } |\xi| \leq \epsilon_3, \quad (3.3)$$

where the least positive root of following equation is ϵ_3 .

$$(e^\epsilon - |\lambda| - |1 + \lambda||\tan\theta|) \epsilon^2 - 2|\sec\theta| \epsilon - (e^\epsilon - |\lambda| - |1 + \lambda||\tan\theta|) = 0, \quad (3.4)$$

and $\lambda > -1$, $0 \leq \eta < 1$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\beta, \rho \in \mathbb{R} \setminus \mathbb{Z}_0^-$.

Corollary 3.3. *We consider a function $f \in \mathcal{A}$ and an another function $g \in S^\lambda[\alpha, E, F]$. If $\Omega^\lambda f(\xi)$ is majorized by $\Omega^\lambda g(\xi)$ in domain D , then*

$$|\Omega^{\lambda+1} f(\xi)| \leq |\Omega^{\lambda+1} g(\xi)| \quad \text{for } |\xi| \leq \epsilon_4, \quad (3.5)$$

where the least positive root of following equation is ϵ_4 .

$$|(1-\alpha)E + (2\alpha - \lambda - 1)F| \epsilon^3 - (2|F| + |\alpha - \lambda|) \epsilon^2 - (2 + |((1-\alpha)E + (2\alpha - \lambda - 1)F)|) \epsilon + |\alpha - \lambda| = 0, \quad (3.6)$$

and $-1 \leq F < E \leq 1$, $-\infty < \lambda < 2$, $0 \leq \alpha < 1$.

Corollary 3.4. *We consider a function $f \in \mathcal{A}$ and an another function $g \in T_{\lambda, \alpha}$. If $\Omega^\lambda f(\xi)$ is majorized by $\Omega^\lambda g(\xi)$ in domain D , then*

$$|\Omega^{\lambda+1} f(\xi)| \leq |\Omega^{\lambda+1} g(\xi)| \quad \text{for } |\xi| \leq \epsilon_5, \quad (3.7)$$

where the least positive root of following equation is ϵ_5 .

$$(e^\epsilon - |1 - \lambda||\tan\theta| - |\lambda|) \epsilon^2 + 2|\sec\theta| \epsilon - (e^\epsilon - |\lambda| - |1 - \lambda||\tan\theta|) = 0, \quad (3.8)$$

and $-\infty < \lambda < 2$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

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