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A WELL-DEFINED GODUNOV-TYPE SCHEME IN A NAVIER-STOKES MODEL WITH A FRICTION TERM

JULES OUYA[1*] AND AROUNA OUEDRAOGO[2]

Abstract. In this paper, particular attention is paid to Godunov-type numerical scheme for solving partial differential equations. We numerically approximate the weak solutions of the Navier-Stokes problem in the compressible case in one dimension of space with a friction term. The solutions of this model exhibit various properties that must be maintained accurately through numerical methods. Indeed, the solutions may satisfy stable regimes, the scheme mush maintain positive density throughout the flow. By developing a suitable approximate Riemann solver, a finite volume method is formulated to preserve as well as possible (or even exactly) those steady states of particular physical interest. Numerical simulations illustrate the effectiveness of the suggested computational method.

1. Introduction

Our focus lies in simulating the 1D unsteady compressible Navier-Stokes system. Assuming the fluid is subject only to the force of friction, the problem is written as follows

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho v) + \partial_x(\rho v^2) + \partial_x p(\rho) = \partial_x\left(\mu \partial_x v\right) - r\rho v^2, \end{cases} \tag{1.1}$$

where, $\rho(t,x)$, $v(t,x)$ and $\mu > 0$ represent, respectively, the density, velocity and viscosity of the fluid; with $(t,x) \in [0,+\infty[\times\mathbb{R}$. We restrict ourselves to the isentropic case, where the fluid pressure p is written as $p(\rho) = k\rho^\alpha$ with $k > 0$ and $\alpha > 1$. Finally, the quantity of motion is given by $q = \rho v$. In (1.1), the first equation expresses the conservation of mass and the second, the evolution of the quantity of motion.

The main purpose of this paper concerns numerical implementation of finite volume schemes associated with Godunov-type solvers (see [18]). The Godunov scheme, proposed by Serguey Godunov in 1959 [16, 26], is a conservative numerical scheme used for solving partial differential equations. This scheme employs a finite volume method to solve exact or approximate Riemann problems between

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* Corresponding author.

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each cell. The simplest form of the Godunov scheme is first-order accuracy in both space and time, but it can be extended to higher-order methods. The Godunov scheme and its linearized extensions, known as Godunov-type schemes, are commonly used to illustrate finite volume schemes.
Motivated by the articles [11], [1, 2, 4, 6] and D. Hoff [19], we use the Godunov-type scheme to approximate the weak solutions of our problem (1.1). In [19], the existence of weak solutions has been established. For additional results and references on the existence theory for the compressible Navier-Stokes system, we recommend consulting [10, 22] and the reference texts [12, 23]. It is worth noting that recent advancements [8] address more intricate pressure laws and introduce novel compactness arguments. For outcomes related to density-dependent viscosities, we suggest referring to [7, 24, 28].
The paper is structured as follows: Section 2 is devoted to the search for stable solutions, in particular those at rest and the establishment of the entropy inequality. In Section 3, we construct the finite volumes scheme, the Godunov scheme and the Godunov-type scheme, focusing on the discretization of the diffusion and friction terms. Finally, in Section 4, we illustrate numerical results with particular emphasis on the value of viscosity.

2. Preliminaries

We introduce the new variables

$$W = \begin{pmatrix} \rho \\ \rho v \end{pmatrix};\ F(W) = \begin{pmatrix} \rho v \\ \rho v^2 + p \end{pmatrix};\ D(W)\partial_x W = \begin{pmatrix} 0 \\ \mu \partial_x v \end{pmatrix};\ S(W) = \begin{pmatrix} 0 \\ -r\rho v^2 \end{pmatrix} \quad (2.1)$$

where the vector W accounts for the conservative variables, $F(W)$ for the component of the convective flux, $D(W)\partial_x W$ designed the viscosity term respectively, and $S(W)$ for the friction term . Thus, (1.1) is written as

$$\partial_t W + \partial_x F(W) = \partial_x \left(D(W)\partial_x W \right) + S(W), \quad (2.2)$$

where W belongs to the set of admissible states Ω defined by

$$\Omega = \{W = (\rho, q)^t \in \mathbb{R}^2;\ \rho \geq 0,\ q \in \mathbb{R}\} \subset \mathbb{R}^2.$$

Remark 2.1. The admissible space Ω contains zones defined by $\rho = 0$ in which, a priori, the system no longer makes sense. So, from now on, let us emphasize we do not consider vacuums regions where $\rho = 0$.

The primary goal of this study is to develop a finite volume scheme that can effectively approximate the weak solutions of equation (1.1), with a specific focus on steady states. Steady states are particularly significant in numerical simulations, as many real-world experiments are expected to eventually converge to these stable states over long periods of time. Therefore, it is crucial for the scheme to accurately approximate these specific solutions. In the following, we approach the weak solutions of (1.1) with a Godunov-type scheme (see [16, 18, 26]).
Indeed, this paper focuses on steady states described by

$$\begin{cases} \partial_x(\rho v) = 0, \\ \partial_x \left(\rho v^2 + p(\rho) - \mu \partial_x v \right) = -r\rho v^2, \end{cases} . \quad (2.3)$$

Our focus is on steady states at rest, where the velocity v is zero. Consequently, the steady states we are interested in are solutions of equation

$$\begin{cases} v = 0, \\ \partial_x \rho = 0, \end{cases} . \tag{2.4}$$

Thus, we obtain the following definition of the steady states: $v = 0$ and $\rho = cst$.

In order to exclude non-physical solutions, the in-homogeneous system is given an entropy inequality.

Lemma 2.2. *A weak solution of the Navier-Stokes system satisfies the following entropy inequality :*

$$\partial_t E(W) + \partial_x G(W) + r\rho v^3 \leq \partial_x \left(\mu \partial_x \frac{v^2}{2} \right), \tag{2.5}$$

where E is the entropy or energy function and G is the entropy flux defined by $E(W) = \rho \frac{v^2}{2} + e(\rho)$ and $G(W) = \rho v \left(\frac{v^2}{2} + e'(\rho) \right)$, with $e''(\rho) = \frac{p'(\rho)}{\rho}$.

3. Finite volumes scheme

The aim of finite volume schemes is to try to approximate weak solutions for the problem

$$\begin{cases} \partial_t W + \partial_x F(W) = 0 \\ W(x, t = 0) = W_0(x), \end{cases} \tag{3.1}$$

with $W_0(x) \in \Omega$.

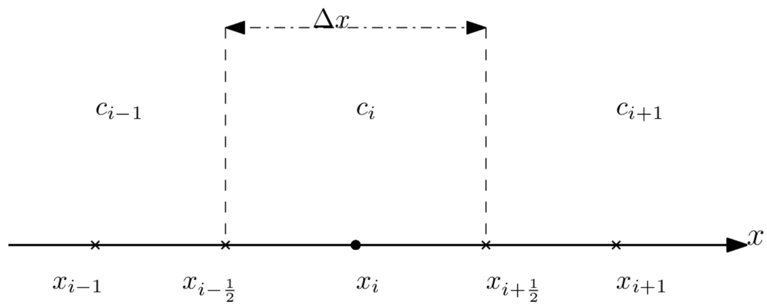


Figure 1. One-dimensional discretization of space.

In this paper, we adopt this method which is more adapted to the hyperbolic character of the Navier-Stokes system. We will briefly present finite volume schemes in a general framework and then recall the basics of the Godunov [27, 18] scheme and Godunov type schemes [27, 26]. Here, we consider the homogeneous model in its (3.1) form. The case with a viscosity term will be dealt with in the next subpoint. The first step in the finite volume method is to discretize the space. This involves dividing the space into cells as shown in Figure 1. Consider a uniform mesh defined by the sequence of points $(x_{i+1/2})_{i\in\mathbb{Z}}$ where

$$x_{i+1/2} = x_{i-1/2} + \Delta x,$$

with Δx the step in space, assumed to be constant. Thus we have the cells $c_i = (x_{i-1/2}, x_{i+1/2})$ centered at x_i. In the same way, we define the sequence $(t^n)_{n\in\mathbb{N}}$ by $t^0 = 0$ and $t^{n+1} = t^n + \Delta t$ where Δt is the time step that will be restricted by a CFL (*Courant-Friedrichs-Lewy*) condition.
We note W_i^n the approximation of the solution $W(x,t^n)$ at date t^n . In fact, this value corresponds to the approximation of the average of the exact solution $W(x,t^n)$ over the cell c_i , as follows:

$$W_i^n \simeq \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} W(x,t^n)dx. \tag{3.2}$$

This approximation is initialized by averaging the initial condition over each cell, given for all $i \in \mathbb{Z}$

$$W_i^0 = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} W_0(x)dx. \tag{3.3}$$

A finite volume scheme is obtained by integrating the system (3.1) over $c_i \times [t^n, t^{n+1}[$,

$$\int_{x_{i-1/2}}^{x_{i+1/2}}\int_{t^n}^{t^{n+1}} \left(\partial_t W(x,t) + \partial_x F(W(x,t))\right) dxdt = 0.$$

This implies that

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \big(W(x,t^{n+1}) - W(x,t^n)\big)\, dx + \big(F(W(x_{i+1/2},t)) - F(W(x_{i-1/2},t))\big)\, dt = 0.$$

Using (3.2) and by choosing the flow approximation, called the numerical flow:

$$F_\Delta\left(W_i^n, W_{i+1}^n\right) \simeq \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} F\left(W(x_{i+1/2},t)\right) dt, \tag{3.4}$$

we obtain the first-order finite volume scheme, given by :

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x}\left[F_\Delta\left(W_i^n, W_{i+1}^n\right) - F_\Delta\left(W_{i-1}^n, W_i^n\right)\right], \tag{3.5}$$

where the numerical flow F_Δ defines an approximation of the physical flows at the interfaces.

3.1. **Godunov scheme.**

In this subsection we present the Godunov scheme [26, 16]. At date t^n, we assume that an approximation to the piece-wise constant solution $W_\Delta(x,t^n) = W_i^n$ is known if $x \in (x_{i-1/2}, x_{i+1/2})$. This approximation is now extended to the date $t^n + \Delta t$ by solving the following problem

$$\begin{cases} \partial_t W(x,t) + \partial_x F(W(x,t)) = 0 \\ W(x,t^n) = W_\Delta(x,t^n). \end{cases} \tag{3.6}$$

Locally, on each interface $x_{i+1/2}$, we have to consider a Riemann problem written in the form

$$\begin{cases} \partial_t W(x,t) + \partial_x F(W(x,t)) = 0 \\ W(x,t^n) = \begin{cases} W_i^n & \text{if} \quad x < 0 \\ W_{i+1}^n & \text{if} \quad x > 0. \end{cases} \end{cases} \tag{3.7}$$

We note $W_{\mathcal{R}}\left(\dfrac{x - x_{i+1/2}}{t}, W_i^n, W_{i+1}^n\right)$ the exact solution of the Riemann problem (3.7). Thus, for Δt sufficiently small, the solution to the (3.6) problem is given by

$$W_\Delta(x, t^n + t) = W_{\mathcal{R}}\left(\frac{x - x_{i+1/2}}{t}, W_i^n, W_{i+1}^n\right), \tag{3.8}$$

where $t \in (0, \Delta t)$ and $x \in (x_i, x_{i+1})$.

Let $\left(\lambda_{i+1/2}^{\pm}\right)_{i\in\mathbb{Z}}$ be the fastest and slowest wave speeds resulting from the exact solution of the Riemann problem (3.7) at the interface $x_{i+1/2}$. By construction of the solutions of the Riemann problem all these wave speeds are bounded whatever $i \in \mathbb{Z}$. We impose the condition CFL (Courant-Friedrichs-Lewy) defined by

$$\frac{\Delta t}{\Delta x}\max_{i\in\mathbb{Z}}\left(|\lambda_{i+1/2}^-|, |\lambda_{i+1/2}^+|\right) \le \frac{1}{2} \Rightarrow \frac{\Delta t}{\Delta x}\max_{i\in\mathbb{Z}}\left(|u_i^n| + \sqrt{p_i'^n}\right) \le \frac{1}{2}, \tag{3.9}$$

it follows that $W_\Delta(x, t^n + \Delta t)$ is a non-interacting juxtaposition of Riemann problems positioned at each interface. We see that on each cell $(x_{i-1/2}, x_{i+1/2})$ the solution $W_\Delta(x, t^n + \Delta t)$, given by (3.8), is not constant over (x_i, x_{i+1}). It is made up of constant states separated by shock waves and rarefaction waves. We project this solution onto the piecewise constant functions in the sense of L^2. To do this, we define the approximation at date $t^n + \Delta t$, as follows:

$$W_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} W_\Delta(x, t^n + \Delta t)dx. \tag{3.10}$$

It is also possible to reformulate the diagram in the conservative three-point form (3.5).

Lemma 3.1. *At $\Delta t > 0$, consider $W_{\mathcal{R}}$ the solution of the Riemann problem (3.7). Then, under the condition CFL (3.9), we have*

$$\frac{2}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_i} W_{\mathcal{R}}\left(\frac{x - x_{i-\frac{1}{2}}}{t}, W_{i-1}, W_i\right)dx = W_i - \frac{2\Delta t}{\Delta x}\Big(F(W_i) - F(W_R(0; W_{i-1}, W_i))\Big) \tag{3.11}$$

and

$$\frac{2}{\Delta x}\int_{x_i}^{x_{i+\frac{1}{2}}} W_{\mathcal{R}}\left(\frac{x - x_{i+\frac{1}{2}}}{t}, W_i, W_{i+1}\right)dx = W_i - \frac{2\Delta t}{-\Delta x}\Big(F(W_i) - F(W_R(0; W_i, W_{i+1}))\Big) \tag{3.12}$$

Proof. Since $W_{\mathcal{R}}$ is the solution to the Riemann problem, then this function verifies

$$\partial_t W_{\mathcal{R}} + \partial_x F(W_{\mathcal{R}}) = 0. \tag{3.13}$$

If we integrate this equation over the block $[x_{i-1/2}, x_i] \times [0, \Delta t]$, we obtain

$$\int_0^{\frac{\Delta x}{2}} \int_0^{\Delta t} \partial_t W_{\mathcal{R}}\left(\frac{x}{t}, W_{i-1}, W_i\right) dxdt + \partial_x F\left(W_{\mathcal{R}}\left(\frac{x}{t}, W_{i-1}, W_i\right)\right) dxdt = 0.$$

This implies

$$\int_0^{\frac{\Delta x}{2}} W_{\mathcal{R}}\left(\frac{x}{\Delta t}, W_{i-1}, W_i\right) dx - \int_0^{\frac{\Delta x}{2}} W_0(x)dx + \int_0^{\Delta t} F\left(W_{\mathcal{R}}\left(\frac{\Delta x/2}{t}, W_{i-1}, W_i\right)\right) dt$$
$$- \int_0^{\Delta t} F\left(W_{\mathcal{R}}\left(0; W_{i-1}, W_i\right)\right) = 0.$$

Since $x > 0$, then we have $W(x, t = 0) = W_i$. Under CFL condition (3.9), we have $F\left(W_{\mathcal{R}}\left(\frac{\Delta x/2}{t}, W_{i-1}, W_i\right)\right) = F(W_i)$. Furthermore, $F\left(W_{\mathcal{R}}\left(0; W_{i-1}, W_i\right)\right)$ is independent of t, so

$$\frac{1}{\Delta x/2} \int_0^{\frac{\Delta x}{2}} W_{\mathcal{R}}\left(\frac{x}{\Delta t}, W_{i-1}, W_i\right) dx = W_i - \frac{\Delta t}{\Delta x/2}\left(F(W_i) - F(W_{\mathcal{R}}(0; W_{i-1}, W_i))\right).$$

Hence (3.11).
Similarly, by integrating the equation (3.13) on the block $[x_i, x_{i+1/2}] \times [0, \Delta t]$, we obtain

$$\int_{-\frac{\Delta x}{2}}^0 W_{\mathcal{R}}\left(\frac{x}{\Delta t}, W_i, W_{i+1}\right) dx - \int_0^{\frac{\Delta x}{2}} W_0(x)dx - \int_0^{\Delta t} F\left(W_{\mathcal{R}}\left(\frac{-\Delta x/2}{t}, W_i, W_{i+1}\right)\right) dt$$
$$+ \int_0^{\Delta t} F\left(W_{\mathcal{R}}\left(0; W_i, W_{i+1}\right)\right) dt = 0.$$

Since $x < 0$, then we have $W(x, t = 0) = W_i$. Under CFL condition (3.9), we have
$F\left(W_{\mathcal{R}}\left(\frac{-\Delta x/2}{t}, W_i, W_{i+1}\right)\right) = F(W_i)$. In addition, $F\left(W_{\mathcal{R}}\left(0; W_{i-1}, W_i\right)\right)$ is independent of t, so

$$\frac{1}{\Delta x/2} \int_{-\frac{\Delta x}{2}}^0 W_{\mathcal{R}}\left(\frac{x}{\Delta t}, W_i, W_{i+1}\right) dx = W_i - \frac{\Delta t}{-\Delta x/2}\left(F(W_i) - F(W_{\mathcal{R}}(0; W_i, W_{i+1}))\right).$$

□

The Godunov scheme (3.10) can be written as :

$$W_i^{n+1} = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_i} W_{\mathcal{R}}\left(\frac{x - x_{i-1/2}}{\Delta t}, W_{i-1}, W_i\right) dx + \frac{1}{\Delta x} \int_{x_i}^{x_{i+1/2}} W_{\mathcal{R}}\left(\frac{x - x_{i+1/2}}{\Delta t}, W_i, W_{i+1}\right) dx.$$

By changing the variable $x - x_{i+1/2} \longleftarrow x$, it follows that

$$W_i^{n+1} = \frac{1}{\Delta x} \int_0^{\frac{\Delta x}{2}} W_{\mathcal{R}}\left(\frac{x}{\Delta t}, W_{i-1}, W_i\right) dx + \frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^0 W_{\mathcal{R}}\left(\frac{x}{\Delta t}, W_i, W_{i+1}\right) dx.$$

Consequently, the relations (3.11) and (3.12) give us

$$\begin{aligned} W_i^{n+1} &= \frac{1}{2}\left(W_i^n - \frac{\Delta t}{\Delta x/2}\left(F(W_i^n)\right) - F(W_{\mathcal{R}}(0;W_{i-1},W_i))\right) \\ &+\frac{1}{2}\left(W_i^n - \frac{\Delta t}{-\Delta x/2}\left(F(W_i^n)\right) - F(W_{\mathcal{R}}(0;W_i,W_{i+1}))\right) \\ &= W_i^n - \frac{\Delta t}{\Delta x}\left(F(W_{\mathcal{R}}(0;W_i,W_{i+1})) - F(W_{\mathcal{R}}(0;W_{i-1},W_i))\right), \end{aligned}$$

which can be reformulated in the conservative form (3.5) with a numerical flow given by

$$F_\Delta(W_i^n, W_{i+1}^n) = F\left(W_{\mathcal{R}}(0;W_i^n,W_{i+1}^n)\right). \tag{3.14}$$

Note that the Godunov scheme is consistent since $W_{\mathcal{R}}(\frac{x}{t}, W, W) = W$ so that $F_\Delta(W,W) = F(W)$.

3.2. **Godunov-type scheme.**

The exact solution of the Riemann problem must be computed for the Godunov scheme. This solution is often difficult to show. In 1983, Harten, Lax and van Leer [18], proposed using an approximation of the solution of the Riemann problem in place of the exact solution to construct a new class of numerical methods. Harten, Lax and van Leer noticed that the consistency of the scheme can be obtained by considering the projection of the exact solution of the Riemann problem onto the constants in the L^2 sense. On the basis of this observation, they proposed substituting the exact solution $W_{\mathcal{R}}(\frac{x}{t}; W_i, W_{i+1})$ by an approximation $\widetilde{W}_{\mathcal{R}}\left(\frac{x}{t}; W_i, W_{i+1}\right)$ satisfying the following *integral consistency* relation:

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} W_{\mathcal{R}}\left(\frac{x}{t}; W_i, W_{i+1}\right) dxdt = \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \widetilde{W}_{\mathcal{R}}\left(\frac{x}{t}; W_i, W_{i+1}\right) dxdt. \tag{3.15}$$

Under the previous consistency condition, it is possible to reformulate the mean of the exact solution of the Riemann problem explicitly. To fix ideas, the approximate Riemann solver will be taken in the following form:

$$\widetilde{W}_{\mathcal{R}}\left(\frac{x}{t}; W_i, W_{i+1}\right) = \begin{cases} W_i & \text{if} \quad x < \lambda_{i+1/2}^L \Delta t \\ \overline{W}(\frac{x}{\Delta t}) & \text{if} \quad \lambda_{i+1/2}^L \Delta t < x < \lambda_{i+1/2}^R \Delta t \\ W_{i+1} & \text{if} \quad x > \lambda_{i+1/2}^R \Delta t \end{cases} \tag{3.16}$$

where $\lambda_{i+1/2}^L$ and $\lambda_{i+1/2}^R$ are characteristic velocities. However, it is important to note that the cone of dependence of the approximate solver defined by $\lambda_{i+1/2}^L$ and $\lambda_{i+1/2}^R$ must contain the cone of dependence of the exact solution defined by $\lambda_{i+1/2}^-$ and $\lambda_{i+1/2}^+$.

Lemma 3.2. *Let* $W_{\mathcal{R}}\left(\frac{x}{\Delta t}; W_i, W_{i+1}\right)$ *the exact solution of the Riemann problem* (3.7)*, then*

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} W_{\mathcal{R}}\left(\frac{x}{\Delta t}; W_i, W_{i+1}\right) dx = \frac{1}{2}\left(W_i + W_{i+1}\right) - \frac{\Delta t}{\Delta x}\left(F(W_{i+1}) - F(W_i)\right). \tag{3.17}$$

under CFL condition defined by

$$\frac{\Delta t}{\Delta x}\max\left(|\lambda^L_{i+1/2}|, |\lambda^R_{i+1/2}|\right) \leq \frac{1}{2}. \tag{3.18}$$

As a consequence of this result, let us now note that, according to (3.17), the condition of *integral consistency* (3.15) becomes

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \widetilde{W}_{\mathcal{R}}\left(\frac{x}{\Delta t}; W_i, W_{i+1}\right) dx = \frac{1}{2}(W_i + W_{i+1}) - \frac{\Delta t}{\Delta x}\left(F(W_{i+1}) - F(W_i)\right) \tag{3.19}$$

Proof. By integrating the Riemann problem (3.7) on the interval $[\frac{-\Delta x}{2}, \frac{-\Delta x}{2}]times[0, \Delta t]$, we obtain the following equality:

$$\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} W_{\mathcal{R}}\left(\frac{x}{\Delta t}; W_i, W_{i+1}\right) dx = \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} W_{\mathcal{R}}(x, 0)dx - \int_0^{\Delta t} F\left(W_{\mathcal{R}}\left(\frac{\Delta x/2}{t}; W_i, W_{i+1}\right)\right) dt + \int_0^{\Delta t} F\left(W_{\mathcal{R}}\left(\frac{-\Delta x/2}{t}; W_i, W_{i+1}\right)\right) dt. \tag{3.20}$$

Using the definition of the initial condition of the Riemann problem (3.7), we have

$$\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} W_{\mathcal{R}}(x, 0)dx = \frac{\Delta x}{2}\left(W_i + W_{i+1}\right). \tag{3.21}$$

The CFL condition ensures that for $t \in]0, \Delta t]$, we have

$$W_{\mathcal{R}}\left(\frac{\Delta x/2}{t}; W_i, W_{i+1}\right) = W_{i+1} \quad \text{and} \quad W_{\mathcal{R}}\left(\frac{-\Delta x/2}{t}; W_i, W_{i+1}\right) = W_i. \tag{3.22}$$

Therefore, (3.20) becomes

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} W_{\mathcal{R}}\left(\frac{x}{\Delta t}; W_i, W_{i+1}\right) dx = \frac{1}{2}\left(W_i + W_{i+1}\right) - \frac{\Delta t}{\Delta x}\left(F(W_{i+1}) - F(W_i)\right).$$

□

Knowing an approximate Riemann solver satisfying the integral consistency condition (3.15), we can construct a conservative scheme consisting of three points. At date t^n we assume that an approximation to the piecewise constant solution is known $\widetilde{W}_\Delta(x, t^n) = W_i^n$ if $x \in (x_{i-1/2}, x_{i+1/2})$. Then, we introduce an approximation to the solution at date $t^n + \Delta t$, given by

$$\widetilde{W}_\Delta(x, t^n + \Delta t) = \widetilde{W}_{\mathcal{R}}\left(\frac{x - x_{i+1/2}}{\Delta t}; W_i^n, W_{i+1}^n\right) \quad \text{if} \quad x \in (x_{i-1/2}, x_{i+1/2}), \tag{3.23}$$

where $\widetilde{W}_{\mathcal{R}}$ satisfies the consistency condition (3.19). Note that under the CFL condition (3.18), the approximate solution $\widetilde{W}_\Delta$ is in fact a non-interacting juxtaposition of the approximate Riemann solvers posed on each interface. At date

$t^n + \Delta t$ we consider the projection in the sense L^2 of $\widetilde{W}_\Delta$ on the piecewise constant functions to define the update as follows:

$$W_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \widetilde{W}_\Delta(x, t^n + \Delta t)dx. \tag{3.24}$$

We will now show that this Godunov-type scheme can be rewritten in the conservative and consistent form (3.5). Indeed

$$W_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_i} \widetilde{W}_\Delta(x, t^n + \Delta t)dx + \frac{1}{\Delta x}\int_{x_i}^{x_{i+1/2}} \widetilde{W}_\Delta(x, t^n + \Delta t)dx, \tag{3.25}$$

then, using the definition (3.23), we obtain

$$\begin{aligned} W_i^{n+1} = &\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_i} \widetilde{W}_\mathcal{R}\left(\frac{x - x_{i-1/2}}{\Delta t}; W_{i-1}^n, W_i^n\right)dx \\ &+ \frac{1}{\Delta x}\int_{x_i}^{x_{i+1}} \widetilde{W}_\mathcal{R}\left(\frac{x - x_{i+1/2}}{\Delta t}; W_i^n, W_{i+1}^n\right)dx \\ &+ \frac{1}{\Delta x}\int_{x_{i+1}}^{x_{i+1/2}} \widetilde{W}_\mathcal{R}\left(\frac{x - x_{i+1/2}}{\Delta t}; W_i^n, W_{i+1}^n\right)dx. \end{aligned} \tag{3.26}$$

The integral consistency condition (3.19) allows us to rewrite the previous approximation as follows:

$$\begin{aligned} W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x}\Big(&\frac{\Delta x}{2\Delta t}(W_i^n - W_{i+1}^n) + \big(F(W_{i+1}^n) - F(W_i^n)\big) \\ &- \frac{1}{\Delta t}\int_{x_{i-1/2}}^{x_i} \widetilde{W}_\mathcal{R}\left(\frac{x - x_{i-1/2}}{\Delta t}; W_{i-1}^n, W_i^n\right)dx \\ &+ \frac{1}{\Delta t}\int_{x_{i+1}}^{x_{i+1/2}} \widetilde{W}_\mathcal{R}\left(\frac{x - x_{i+1/2}}{\Delta t}; W_i^n, W_{i+1}^n\right)dx\Big). \end{aligned} \tag{3.27}$$

This gives the following conservative form:

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x}\left(F_\Delta(W_i, W_{i+1}) - F_\Delta(W_{i-1}, W_i)\right),$$

where the numerical flow function finds the following expression :

$$F_\Delta(W_i, W_{i+1}) = F(W_{i+1}^n) - \frac{\Delta x}{2\Delta t}W_{i+1}^n + \frac{1}{\Delta t}\int_{x_{i+1/2}}^{x_{i+1}} \widetilde{W}_\mathcal{R}\left(\frac{x - x_{i+1/2}}{\Delta t}; W_i^n, W_{i+1}^n\right)dx$$

Or else

$$F_\Delta(W_L, W_R) = F(W_L) + \frac{\Delta x}{2\Delta t}W_L - \frac{1}{\Delta t}\int_{-\frac{\Delta x}{2}}^{0} \widetilde{W}_\mathcal{R}\left(\frac{x}{\Delta t}; W_L, W_R\right)dx$$

or again

$$\begin{aligned} F_\Delta(W_L, W_R) = &\frac{1}{2}\left[F(W_L) + F(W_R)\right] - \frac{\Delta x}{4\Delta t}\left(W_R - W_L\right) \\ &+ \frac{1}{2\Delta t}\int_0^{\frac{\Delta x}{2}} \widetilde{W}_\mathcal{R}\left(\frac{x}{\Delta t}; W_L, W_R\right)dx - \frac{1}{2\Delta t}\int_{-\frac{\Delta x}{2}}^{0} \widetilde{W}_\mathcal{R}\left(\frac{x}{\Delta t}; W_L, W_R\right)dx. \end{aligned}$$

Note that we have a scheme in conservative and consistent form if $\widetilde{W}_{\mathcal{R}}\left(\frac{x}{\Delta t}; W, W\right) = W$ since we then obtain $F_\Delta(W, W) = F(W)$.

3.3. Godunov-type scheme for the Navier-Stokes model with viscosity and friction terms.

The special feature of this scheme, as mentioned earlier, is the use of the approximate solution of the Riemann problem. In the previous subsection, we presented the Godunov-type scheme for the homogeneous system. In this Section, we extend the Godunov-type schemes to take into account the viscosity term and the friction term. We adopt the space and time discretization introduced at the beginning of this Section 3. It is assumed that at time t^n, a piecewise constant approximation to the solution of (1.1) is known and defined as follows:

$$W_\Delta(x, t^n) = W_i^n \quad \text{if} \quad x \in (x_{i-1/2}, x_{i+1/2}).$$

We are looking for an approximation of the solution at time $t^{n+1} = t^n + \Delta t$. To approximate the solution of the system (1.1), according to [16, 9, 17], the update W^{n+1} is given by

$$W_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} W_\Delta\left(x, t^n + \Delta t\right) dx, \tag{3.28}$$

where $W_\Delta\left(x, t^n + \Delta t\right)$ represents the non-interacting juxtaposition of the approximate Riemann solvers defined on each cell (x_i, x_{i+1}) by

$$W_\Delta\left(x, t^n + \Delta t\right) = \widetilde{W}_{\mathcal{R}}\left(\frac{x - x_{i+1/2}}{t}; W_i^n, W_{i+1}^n\right), \quad \text{if} \quad (x,t) \in (x_i, x_{i+1}) \times (0, \Delta t).$$

Note that $\widetilde{W}_{\mathcal{R}}\Big(\frac{x}{t}; W_L, W_R\Big)$ denotes the approximation of the solution of the following Riemann problem:

$$\begin{cases} \partial_t W + \partial_x F(W) = \partial_x\left(D(W)\partial_x W\right) + S(W) \\ W(x, t=0) = \begin{cases} W_L & \text{if} \quad x < 0 \\ W_R & \text{if} \quad x > 0. \end{cases} \end{cases} \tag{3.29}$$

According to [18](see also equality (3.15)), the approximate Riemann solver must satisfy the following *integral consistency* condition:

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \widetilde{W}_{\mathcal{R}}\Big(\frac{x}{\Delta t}; W_L, W_R,\Big) dx = \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} W_{\mathcal{R}}\big(x, \Delta t; W_L, W_R\big) dx, \tag{3.30}$$

where $W_{\mathcal{R}}\big(x, \Delta t; W_L, W_R\big)$ represents the exact solution of the Riemann problem (3.29). However, because of the diffusion term, the Lemma 3.2 can no longer be applied here and the mean of $W_{\mathcal{R}}$ is, a priori, no longer accessible. Indeed, as long as Δt is sufficiently small that

$$W_{\mathcal{R}}\Big(-\frac{\Delta x}{2}, \Delta t; W_L, W_R\Big) = W_L \quad \text{and} \quad W_{\mathcal{R}}\Big(\frac{\Delta x}{2}, \Delta t; W_L, W_R\Big) = W_R,$$

integrating (3.29) over the domain $(-\Delta x/2, \Delta x/2) \times (0, \Delta t)$ gives us

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} W_{\mathcal{R}}\big(x,\Delta t;W_L,W_R\big)dx = \frac{1}{2}\left(W_R+W_L\right) - \frac{1}{\Delta x}\int_0^{\Delta t} F\left(W_{\mathcal{R}}\big(\Delta x/2,t;W_L,W_R\big)\right)dt$$
$$+\frac{1}{\Delta x}\int_0^{\Delta t} F\left(W_{\mathcal{R}}\big(-\Delta x/2,t;W_L,W_R\big)\right)dt + \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}}\int_0^{\Delta t} S\left(W_{\mathcal{R}}\big(x,t;W_L,W_R\big)\right)dxdt$$
$$+\frac{1}{\Delta x}\int_0^{\Delta t} D\left(W_{\mathcal{R}}\big(\Delta x/2,t;W_L,W_R\big)\right)dt - \frac{1}{\Delta x}\int_0^{\Delta t} D\left(W_{\mathcal{R}}\big(-\Delta x/2,t;W_L,W_R\big)\right)dt, \tag{3.31}$$

where we use the notation $D(W_{\mathcal{R}})$ instead of $D(W_{\mathcal{R}})\partial_x W_{\mathcal{R}}$ for simplicity.

The consistency condition (3.31) (due to (3.30)), satisfied by the approximate Riemann solver $\widetilde{W}_{\mathcal{R}}\Big(\frac{x}{\Delta t};W_L,W_R\Big)$ allows us to reformulate the update W_i^{n+1} as a usual finite volumes scheme. Indeed, from (3.28), we obtain

$$W_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_i} \widetilde{W}_{\mathcal{R}}\Big(\frac{x-x_{i-1/2}}{\Delta t};W_{i-1}^n,W_i^n\Big)dx$$
$$+\frac{1}{\Delta x}\int_{x_i}^{x_{i+1/2}} \widetilde{W}_{\mathcal{R}}\Big(\frac{x-x_{i+1/2}}{\Delta t};W_i^n,W_{i+1}^n\Big)dx. \tag{3.32}$$

At the interface $x_{i+1/2}$, we perform the following change of variable
$x \longrightarrow x - x_{i+1/2}$, to obtain

$$\frac{1}{\Delta x}\int_{x_i}^{x_{i+1/2}} \widetilde{W}_{\mathcal{R}}\Big(\frac{x-x_{i+1/2}}{\Delta t};W_i^n,W_{i+1}^n\Big)dx = \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{0} \widetilde{W}_{\mathcal{R}}\Big(\frac{x}{\Delta t};W_i^n,W_{i+1}^n\Big)dx,$$

and

$$\frac{1}{\Delta x}\int_{x_{i+1/2}}^{x_{i+1}} \widetilde{W}_{\mathcal{R}}\Big(\frac{x-x_{i+1/2}}{\Delta t};W_i^n,W_{i+1}^n\Big)dx = \frac{1}{\Delta x}\int_0^{\frac{\Delta x}{2}} \widetilde{W}_{\mathcal{R}}\Big(\frac{x}{\Delta t};W_i^n,W_{i+1}^n\Big)dx.$$

To simplify notation, we define,

$$I_{i+1/2}^- = \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{0} \widetilde{W}_{\mathcal{R}}\Big(\frac{x}{\Delta t};W_i^n,W_{i+1}^n\Big)dx, \quad I_{i+1/2}^+ = \frac{1}{\Delta x}\int_0^{\frac{\Delta x}{2}} \widetilde{W}_{\mathcal{R}}\Big(\frac{x}{\Delta t};W_i^n,W_{i+1}^n\Big)dx.$$

Then (3.32) is rewritten as follows:

$$\begin{aligned} W_i^{n+1} &= I_{i-1/2}^+ + I_{i+1/2}^- \\ &= \frac{1}{2}\left(I_{i-1/2}^+ + I_{i-1/2}^-\right) + \frac{1}{2}\left(I_{i+1/2}^- + I_{i+1/2}^+\right) \\ &+ \frac{1}{2}\left(I_{i-1/2}^+ - I_{i-1/2}^-\right) + \frac{1}{2}\left(I_{i+1/2}^- - I_{i+1/2}^+\right) \end{aligned}$$

We have, for all $i \in \mathbb{Z}$

$$\frac{1}{2}\left(I_{i+1/2}^- + I_{i+1/2}^+\right) = \frac{1}{2\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \widetilde{W}_{\mathcal{R}}\Big(\frac{x}{\Delta t};W_i^n,W_{i+1}^n\Big)dx, \tag{3.33}$$

and

$$\frac{1}{2}\left(I_{i-1/2}^{-}+I_{i-1/2}^{+}\right)=\frac{1}{2\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}}\widetilde{W}_{\mathcal{R}}\left(\frac{x}{\Delta t};W_{i-1}^{n},W_{i}^{n}\right)dx. \tag{3.34}$$

Using (3.31) and the equalities (3.33)-(3.34), we obtain

$$\begin{aligned}W_i^{n+1}=W_i^n-&\frac{\Delta t}{\Delta x}\Big(F_\Delta\left(W_i^n,W_{i+1}^n\right)-F_\Delta\left(W_{i-1}^n,W_i^n\right)\Big)\\&+\frac{\Delta t}{\Delta x}\left(\overline{D}_{i+1/2}-\overline{D}_{i-1/2}\right)+\frac{\Delta t}{2}\left(\overline{S}_{i-1/2}+\overline{S}_{i+1/2}\right),\end{aligned} \tag{3.35}$$

where we approached the integral on $(0,\Delta t)$ of $D\left(W_{\mathcal{R}}\left(\frac{-\Delta x/2}{\Delta t};W_L,W_R\right)\right)$ and $D\left(W_{\mathcal{R}}\left(\frac{\Delta x/2}{\Delta t};W_L,W_R\right)\right)$ by constants given by $\overline{D}_L$ and $\overline{D}_R$, with the numerical flow function defined by

$$\begin{aligned}F_\Delta\Big(W_L,W_R\Big)=&\frac{1}{2}\left(F(W_L)+F(W_R)\right)\\&-\frac{\Delta x}{4\Delta t}\left(W_R-W_L\right)+\frac{\Delta x}{2\Delta t}\left(I_{LR}^{+}-I_{LR}^{-}\right),\end{aligned} \tag{3.36}$$

where

$$I_{LR}^{-}=\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{0}\widetilde{W}_{\mathcal{R}}\left(\frac{x}{\Delta t};W_L,W_R\right)dx,\quad I_{LR}^{+}=\frac{1}{\Delta x}\int_{0}^{\frac{\Delta x}{2}}\widetilde{W}_{\mathcal{R}}\left(\frac{x}{\Delta t};W_L,W_R\right)dx.$$

Consequently, the scheme (3.35) will be fully determined once the Riemann solver $\widetilde{W}_{\mathcal{R}}$ and the approximation of the viscosity term (diffusion term) and the friction term have been chosen. The characterization of these functions is the subject of the following subsections.

3.3.1. *Discretization of diffusion and friction terms.*
The construction of the local approximation of the terms $\overline{D}_{LR}\left(W_L,W_R\right)$ and $\overline{S}_{LR}\left(W_L,W_R\right)$ must allow the scheme (3.35) to preserve stationary solutions. It is important to note that this function $\overline{D}_{LR}\left(W_L,W_R\right)$ and $\overline{S}_{LR}\left(W_L,W_R\right)$ are a local approximation on each interface of the viscosity term and friction term, respectively. It is therefore necessary to define the stationary solutions interface by interface. In line with the stationary solutions (2.4), we introduce the definition of local stationarity as follows.

Definition 3.3. A solution is locally stationary if for all states W_L and W_R , we have

$$\rho_L v_L=\rho_R v_R \quad \text{and} \quad \mathcal{B}\left(W_L,D_L,S_L\right)=\mathcal{B}\left(W_R,D_R,S_R\right) \tag{3.37}$$

where $\mathcal{B}\left(W,D,S\right)=\rho v^2+p(\rho)$.

Now, if $\Big(W_L,W_R,D_L,D_R,S_L,S_R\Big)$ verifies the local equilibrium (3.37), $\overline{D}_{LR}\left(W_L,W_R\right)$ and $\overline{S}_{LR}\left(W_L,W_R\right)$ must be fixed so that $\widetilde{W}_{\mathcal{R}}\left(\frac{x}{\Delta t};W_L,W_R\right)$ remains stationary;

i.e.

$$\widetilde{W}_{\mathcal{R}}\Big(\frac{x}{\Delta t}; W_L, W_R\Big) = \begin{cases} W_L & \text{if} \quad x < 0 \\ W_R & \text{if} \quad x > 0. \end{cases} \tag{3.38}$$

The stationarity relation (3.38) allows us to propose a discretization of the viscosity term. Note that the integral consistency condition (3.31), for a stationary solution in the form (3.38), is rewritten as follows:

$$2\Delta x \left(\overline{D}_R - \overline{D}_L\right) + \left(\overline{S}_L + \overline{S}_R\right) = F(W_R) - F(W_L). \tag{3.39}$$

(We will establish the proof of (3.39) after defining the Riemann solver).

Consequently, as soon as $(W_L, W_R, D_L, D_R, S_L, S_R)$ satisfies (3.37), the approximate diffusion term $\overline{D}_{LR} = \left(\overline{D}^{\rho}_{LR}, \overline{D}^{q}_{LR}\right)^t$ and the approximate friction term $\overline{S}_{LR} = \left(\overline{S}^{\rho}_{LR}, \overline{S}^{q}_{LR}\right)^t$ must satisfy

$$\Delta x\overline{D}^{\rho}_{LR} = \Delta x\overline{S}^{\rho}_{LR} = 0, \;\; 2\Delta x\overline{D}^{q}_{LR} + \overline{S}^{q}_{LR} = \left(\rho_R v_R^2 + p_R\right) - \left(\rho_L v_L^2 + p_L\right). \tag{3.40}$$

Since the equation of continuity (evolution of density) does not contain neither a viscosity term nor a friction term, we impose $\overline{S}^{\rho}_{LR} = \overline{D}^{\rho}_{LR} = 0$, for all $(W_L, W_R, D_L, D_R, S_L, S_R)$. We will now consider $\overline{D}^{q}_{LR}$ and $\overline{S}^{q}_{LR}$ to propose a definition consistent with the second member $\partial_x\left(\mu\partial_x v\right) - r\rho v^2$ and such that (3.40) is verified as soon as $(W_L, W_R, D_L, D_R, S_L, S_R)$ defines a local stationary solution (3.37). To solve such a problem, we now state a condition that must be satisfied by $\overline{D}^{q}_{LR}$ and $\overline{S}^{q}_{LR}$.

3.3.2. *Approximate Riemann solver.*

To complete the characterization of the numerical scheme (3.35), we need to define appropriately the Riemann solver $\widetilde{W}_{\mathcal{R}}\left(x/t, W_L, W_R\right)$ in accordance with the consistency condition (3.31). We utilize an approximate Riemann solver that involves two constant intermediate states, denoted as $W_L^\star$ and $W_R^\star$, in the following manner

$$\widetilde{W}_{\mathcal{R}}\left(x/t, W_L, W_R\right) = \begin{cases} W_L & \text{if} \quad x/t < \lambda_L, \\ W_L^\star & \text{if} \quad \lambda_L < x/t < 0 \\ W_R^\star & \text{if} \quad 0 < x/t < \lambda_R \\ W_R & \text{if} \quad x/t > \lambda_R, \end{cases} \tag{3.41}$$

where $\lambda_L < 0$ and $\lambda_R > 0$ are the wave velocities.

In order to avoid instability problems (see [18, 27]), we impose a sufficiently wide cone of dependence on the approximate Riemann solver. Consequently, the characteristic velocities λ_L and λ_R are fixed as follows:

$$\lambda_L \le \min\Big(-|v_L| - \sqrt{p'_L}, -|v_R| - \sqrt{p'_R}\Big)$$
$$\lambda_R \ge \max\Big(|v_L| + \sqrt{p'_L}, |v_R| + \sqrt{p'_R}\Big),$$

so that the cone defined by $\lambda_L < x/t < \lambda_R$ contains the cone of dependence of the exact solution of the Riemann problem (3.29). Moreover, using the definition (3.41), we give the expressions of I^-_{LR} and I^+_{LR} to deduce the new expressions of the numerical flux given by (3.36). We will consider cases where the waves are symmetrical, i.e. $\lambda_R = \lambda = -\lambda_L$. For two intermediate states $W^\star_L$ and $W^\star_R$, we have

$$I^-_{LR} = \frac{\lambda\Delta t}{\Delta x}W^\star_L + \left(\frac{1}{2} - \frac{\lambda\Delta t}{\Delta x}\right)W_L \text{ and } I^+_{LR} = \frac{\lambda\Delta t}{\Delta x}W^\star_R + \left(\frac{1}{2} - \frac{\lambda\Delta t}{\Delta x}\right)W_R, \quad (3.42)$$

Then we have

$$I^+_{LR} - I^-_{LR} = \frac{\lambda\Delta t}{\Delta x}\left(W^\star_R - W^\star_L\right) + \left(\frac{1}{2} - \frac{\lambda\Delta t}{\Delta x}\right)\left(W_R - W_L\right). \quad (3.43)$$

From this, (3.36) derives

$$F_\Delta\Big(W_L, W_R\Big) = \frac{1}{2}\Big(F(W_L) + F(W_R)\Big) + \frac{\lambda}{2}\left(W_L - W^\star_L\right) + \frac{\lambda}{2}\left(W^\star_R - W_R\right). \quad (3.44)$$

For one intermediate state $W^\star_L = W^\star_R = W^\star = W^{HLL}$, we have

$$I^+_{LR} - I^-_{LR} = \left(W_R - W_L\right)\left(\frac{1}{2} - \frac{\lambda\Delta t}{\Delta x}\right). \quad (3.45)$$

Consequently, the numerical flow given by (3.36) is written as

$$F_\Delta\Big(W_L, W_R\Big) = \frac{1}{2}\Big(F(W_L) + F(W_R)\Big) - \frac{\lambda}{2}\left(W_R - W_L\right). \quad (3.46)$$

In this case of one intermediate state, the integral consistency

$$\begin{aligned}\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \widetilde{W}_{\mathcal{R}}\Big(\frac{x}{\Delta t}; W_L, W_R\Big)dx &= \frac{1}{2}\left(W_L + W_R\right) \\ - \frac{\Delta t}{\Delta x}\left(F(W_R) - F(W_L)\right) + \frac{\Delta t}{\Delta x}\left(\overline{D}_R - \overline{D}_L\right) &+ \frac{\Delta t}{2}\left(\overline{S}_L + \overline{S}_R\right)\end{aligned} \quad (3.47)$$

is rewritten as follows (using (3.42))

$$\begin{aligned}\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \widetilde{W}_{\mathcal{R}}\Big(\frac{x}{\Delta t}; W_L, W_R\Big)dx &= I^+_{LR} + I^-_{LR} \\ &= 2\lambda\frac{\Delta t}{\Delta x}W^\star + \frac{1}{2}\left(W_R + W_L\right) - \lambda\frac{\Delta t}{\Delta x}\left(W_R + W_L\right).\end{aligned} \quad (3.48)$$

By identifying (3.47) and (3.48) we obtain

$$\begin{aligned}W^\star =& \frac{1}{2}\left(W_R + W_L\right) - \frac{1}{2\lambda}\Big(F(W_R) - F(W_L)\Big) \\ &+ \frac{1}{2\lambda}\Big(\overline{D}_R - \overline{D}_L\Big) + \frac{1}{4\lambda\Delta x}\left(\overline{S}_L + \overline{S}_R\right).\end{aligned} \quad (3.49)$$

With two intermediate states, we have

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \widetilde{W}_R\Big(\frac{x}{\Delta t}; W_L, W_R\Big)dx = I^+_{LR} + I^-_{LR}$$

$$= \left(\frac{1}{2} - \lambda\frac{\Delta t}{\Delta x}\right) W_R + \lambda W_R^\star \frac{\Delta t}{\Delta x} + \left(\frac{1}{2} - \lambda\frac{\Delta t}{\Delta x}\right) W_L + \lambda W_L^\star \frac{\Delta t}{\Delta x}. \tag{3.50}$$

By identifying (3.48) and (3.50) we obtain

$$W^\star = \frac{1}{2}\left(W_L^\star + W_R^\star\right). \tag{3.51}$$

From expression of $W^\star$ given by (3.49), we have

$$\rho^\star = \frac{\rho_R + \rho_L}{2} - \frac{q_R + q_L}{2\lambda} = \rho^{HLL}$$

and

$$\begin{aligned} q^\star &= \frac{q_R + q_L}{2} - \frac{\rho_R v_R^2 + p_R - \rho_L v_L^2 - p_L}{2\lambda} + \frac{1}{2\lambda}\overline{D}^q_{LR} + \frac{1}{4\lambda\Delta x}\overline{S}_{LR} \\ &= q^{HLL} + \frac{1}{2\lambda}\overline{D}^q_{LR} + \frac{1}{4\lambda\Delta x}\overline{S}_{LR}. \end{aligned}$$

At steady state, we have $q_R = q_L = q$; $W_R^\star = W_R$ and $W_L^\star = W_L$, (3.51) becomes

$$W^\star = \frac{1}{2}\left(W_R + W_L\right). \tag{3.52}$$

Identifying (3.52) with (3.49) we obtain (3.39) given by

$$2\Delta x\left(\overline{D}_R - \overline{D}_L\right) + \left(\overline{S}_L + \overline{S}_R\right) = F(W_R) - F(W_L).$$

We state the main properties satisfied by the scheme (3.35).

Theorem 3.4. *Let the following type CFL restriction :*

$$\frac{\Delta t}{\Delta x}\max_{i\in\mathbb{Z}}\left(|\lambda^L_{i+1/2}|, |\lambda^R_{i+1/2}|\right) \le \frac{1}{2}.$$

The scheme constructed (3.35)-(3.51)-(3.39) is consistent with (1.1). Let W_i^n be in Ω for all $i \in \mathbb{Z}$. The updated state W_i^{n+1}, given by the scheme (3.35)-(3.51)-(3.39), satisfies the following properties:

1. Positiveness preservation:: $\rho_i^n \ge 0$ for all $i \in \mathbb{Z}$,

2. Preservation of all smooth steady states: $W_i^{n+1} = W_i^n$ for all $i \in \mathbb{Z}$ as soon as $(W_i^n)_{i\in\mathbb{Z}}$ verifies

   $$u_i^n = 0 \quad \text{and} \quad \rho_i^n = c \quad \text{for all} \quad i \in \mathbb{Z}.$$

   for given constant c.

3. A discrete entropy inequality in the form:

$$\frac{E_i^{n+1} - E_i^n}{\Delta t} + \frac{G_{i+1/2} - G_{i-1/2}}{\Delta x} \le -\overline{S}_i^E + \overline{D}_i^E, \tag{3.53}$$

where

$$G_{i+1/2} = \frac{G(W_{i+1}^n) + G(W_i^n)}{2} - \frac{\Delta x}{4\Delta t}\left(E(W_{i+1}^n) - E(W_i^n)\right) + \frac{\Delta x}{2\Delta t}\left(I_{i+1/2}^{E+} - I_{i-1/2}^{E-}\right),$$

with

$$I_{i+1/2}^{E+} = \frac{1}{\Delta x}\int_0^{\frac{\Delta x}{2}} E\Big(\widetilde{W}_{\mathcal{R}}\left(\frac{x}{\Delta t}; W_i^n, W_{i+1}^n\right)\Big)\,dx,$$

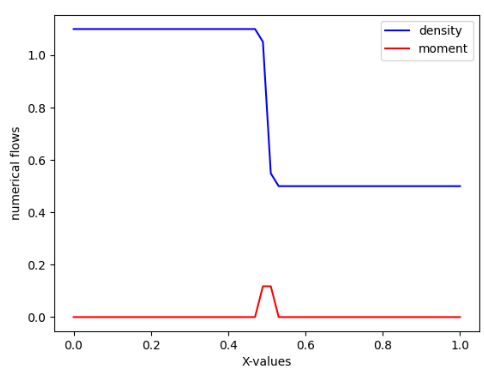
$$I_{i+1/2}^{E-} = \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{0} E\Big(\widetilde{W}_{\mathcal{R}}\left(\frac{x}{\Delta t}; W_i^n, W_{i+1}^n\right)\Big)\,dx,$$

and the diffusion and friction terms satisfy

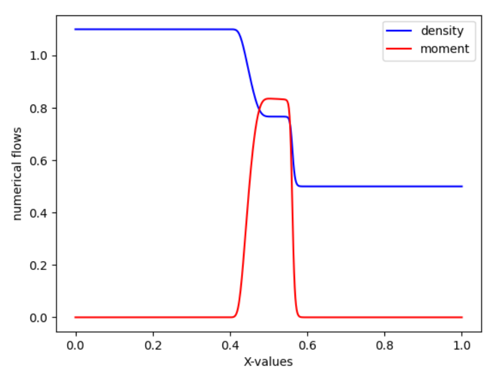
$$2\Delta x\overline{D(W_{\mathcal{R}})}_{i+1/2}^{n} + \overline{S(W_{\mathcal{R}})}_{i+1/2}^{n} = F(W_{i+1}^n) - F(W_i^n).$$

4. NUMERICAL RESULTS

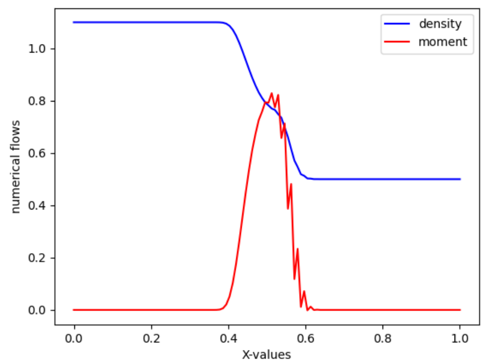
Some numerical results are given here to illustrate how the proposed scheme behaves.
Using one intermediate state (being at equilibrium). For fixed values of $\alpha = 2$, $k = 5$, $r = 1.5$ and a number of control volumes varying between 50 and 600, and by varying the value of the viscosity of the fluid, we obtain, at different times, the graphs below. We repeat the evolution of the density and that of the momentum.



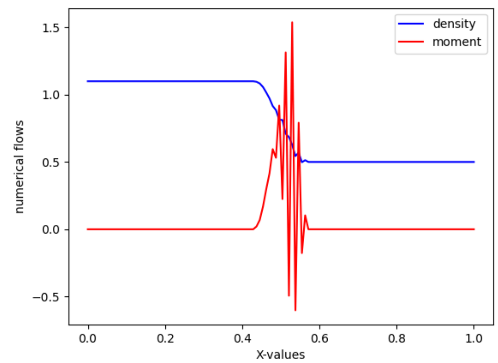
(A) Approximate solution for $N = 50$, $\mu = 0.001$, $t = 0.001$



(B) Approximate solution for $N = 600$, $\mu = 0.0015$, $t = 0.02$



(A) Approximate solution for $N = 120$ $\mu = 0.01$, $t = 0.02$



(B) Approximate solution for $N = 500$, $\mu = 0.02$, $t = 0.02$

Comments: We note that:

- For a fixed number of cells: when the value of the viscosity μ increases progressively, up to a value of about 0.01, instabilities start to appear and become significant as time progresses (the last two figures, for instance). This can be explained by the fact that our constructed scheme is consistent to one order in space, but the viscous term is of second order, so when μ increases at a given moment it is the viscous term that becomes dominant. As a result, instabilities appear especially for a "slightly long" time.
  Another reason is that, naturally, compressible fluids are generally less viscous. The viscosity of the reference fluid, air, is around 10^{-5}.
- As for the simulation time, if it is long, close to 0.1, we are forced to take a number of cells less than 100, otherwise the values will explode, unless the viscosity is very small, less than 10^{-3}. The gas diffuses fairly quickly.
- The density remains positive throughout the flow.

5. Conclusion and perspectives

In this paper, a numerical study has been carried out on the compressible 1D Navier-stokes equations. Taking into account the effects of friction, the quadratic friction term, a Godunov-type scheme has been constructed to approximate the weak solutions of the problem under the CFL condition (meaning that a wave can only travel through half a mesh). The results show that the scheme remains stable and captures the quiescent states for very small values of viscosity.
We plan to make significant corrections to this first-order scheme (e.g. extending it to second order, a more appropriate discretization of friction and diffusion terms) in order to deal with cases of "slightly" higher viscosity. Considering the problem in a higher dimension would also be useful, since real problems are generally of this kind.

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[1] Laboratoire de Mathématiques, Informatique et Applications (LAMIA), Université Norbert ZONGO, BP 376 Koudougou, Burkina Faso.
Email address: ouyajules6@gmail.com

[2] Laboratoire de Mathématiques, Informatique et Applications (LAMIA), Université Norbert ZONGO, BP 376 Koudougou, Burkina Faso
Email address: arounaoued2002@yahoo.fr