

MODIFIED VIM FOR WAVE EQUATIONS

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ABSTRACT. During the past few years, there has been a significant amount of interests in the Variational Iteration Method (VIM), which is a method that is utilized for the purpose of solving nonlinear differential equations. Minor adjustments have been made to ‘VIM as part of the Modified Variational Iterative Method (MVIM)’. For a specific type of nonlinear differential equation, the implementation of the MVIM method results in unnecessary calculations and additional time spent on repeated calculations for series solutions. The drawbacks of VIM are addressed by introducing a modified version, and its usefulness is demonstrated through a few examples.

1. INTRODUCTION

The variational iteration approach (VIM), which He constructed and published in 1999 [5], will be utilized to investigate the linear wave equation, the nonlinear wave equation, and the wave-like equation in both limited and unbounded domains. Numerous researchers [11, 12] have proved the dependability and usefulness of the approach for a wide variety of scientific applications, including linear and nonlinear applications within the scientific community. Numerous writers have demonstrated the superiority of this approach over other current methods, like the perturbation method, the Adomian method etc. If there is an exact solution, the approach provides fast convergent successive approximations of it; if not, for numerical purposes, a couple of estimates can be employed. The method is effectively implemented in the references [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] and their corresponding works. The computing burden of the perturbation method is detrimental, particularly as the degree of nonlinearity rises. In addition, the Adomian approach is hampered by the intricate methods required to compute the Adomian polynomials, which are required for nonlinear situations.

For nonlinear operators, the MVIM does not have any particular constraints, such as linearization or minimal parameters. Extensive research has been devoted to the study of linear and nonlinear wave equations, wave-like equations [1] in confined domains, and the wave equation given in an unbounded domain.

There are three types of functions: source, nonlinear, and linear, respectively: $g(x, t)$, $f(u)$, and $F(u)$. In many different physical issues, the wave equation is

Date: Received: Feb 3, 2024; Accepted: Apr 4, 2024.

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2010 *Mathematics Subject Classification.* Primary 14N05; Secondary 14E99; 14F99.

Key words and phrases. modified variational iteration method (MVIM), wave equation, analytical solution.

crucial. Numerous branches of science and engineering require an understanding of the wave equation.

Wave equations have been the subject of extensive research. The D'Alembert approach, the separation of variables method, and many more are examples of conventional techniques. The purpose of this study is to find precise solutions by making good use of the Modified Variational Iteration Technique (MVIM). As previously mentioned, the approach effortlessly manages linear and nonlinear problems, whether they are homogeneous or heterogeneous, in a bounded or unbounded domain. It is dependable and effective in this regard. In contrast to the Adomian decomposition method, which typically employs computational algorithms to handle the nonlinear terms, the MVIM employs the nonlinear terms directly without any stipulations or restrictive assumptions.

Many numerical methods for solving PDEs have been developed within two decades. These methods include the Adomian decomposition method (ADM) [2] for obtaining a more meaningful solution. In the references [5, 8, 15], the authors applied the homotopy method for PDE. Some authors employed fractional wave equation, burgers equation, and wave equation. They proved and compared various numerical methods such as ADM, HAM, VIM, HPM as effective and promising methods.

We have introduced a modified variational iteration method (MVIM). The lack of rounding mistakes and the modest resource requirements of this method make it the preferable numerical method. He [6, 7] has used the variation iteration method to derive analytical solutions to nonlinear partial differential equations with variable coefficients and autonomous ordinary differential equations.

It is noteworthy to mention that none of the open literature presents the solution of Wave equation employing MVIM. Also, the examples considered in this paper are missing from the literature.

There are two objectives to this work. Our initial objective is to examine the physical wave models in order to derive precise solutions, while avoiding any constraining that could potentially alter the solution's physical characteristics. The validation of the efficacy of modified variational iteration in resolving scientific and engineering challenges constitutes our second objective.

2. MODIFIED VARIATIONAL ITERATION METHOD

Modified Variational Iteration technique represents a marginal modification to the initial Variational Iteration methodology. Understanding the subsequent partial differential equation will assist in comprehending the fundamental principle underlying the modified variational iteration method (MVIM).

$$Lu(x, t) + Ru(x, t) + Nu(x, t) = g(x, t) \quad (2.1)$$

$$u(x, 0) = f(x), \quad (2.2)$$

Where

$$L = \partial/\partial t, \quad (2.3)$$

R is a linear operator which has partial derivatives with respect to x , $Nu(x, t)$ is a nonlinear term and $g(x, t)$ is an inhomogeneous term. Partial differential equation covers a large branch of applications such as soliton equations like Burger's equations, coupled Burger's equations, Schrödinger equations, $K - dV$ equations, modified $K - dV$ equations and many others important equations. The subsequent iteration formula was derived using the variational iteration method.

$$U_{n+1} = U_n - \int_0^t \{L(U_n) + R(U_n) + NU_n - g\} d\tau \quad (2.4)$$

We can rewrite the equation as

$$U_{n+1} = U_n - \int_0^t \{R(U_n - U_{n-1}) + (G_n - G_{n-1})\} d\tau \quad (2.5)$$

where,

$$U_{-1} = 0, U_0 = f(x), U_1 = U_0 - \int_0^t \{R(U_0 - U_{-1}) + (G_0 - G_{-1}) - g\} d\tau \quad \text{and} \quad G_0(x, t) \quad (2.6)$$

is obtained from

$$NU_n(x, t) = G_n(x, t) + O(t^{n+1}) \quad (2.7)$$

By solving the equation iteratively, one can acquire an approximation of the solution in the form

$$u(x, t) \cong U_n(x, t), \quad (2.8)$$

where n is the final iteration step. This is the required Modified Variational iteration method (MVIM)

3. HOMOGENEOUS WAVE EQUATIONS

We will now consider the below homogeneous wave equation

$$u_{tt} = u_{xx} - 3u, 0 < x < \pi, t > 0 \quad (3.1)$$

Where the initial condition are

$$u(x, 0) = 0, u_t(x, 0) = 2 \cos x \quad (3.2)$$

$$RU_n = R(-U_n)_{xx} + R(3U_n) \quad (3.3)$$

$$U_0 = 2t \cos x \quad (3.4)$$

$$R(U_n) = R(U_n)_{xx} + R(3U_n) \quad (3.5)$$

$$R(U_0) = R(U_0)_{xx} + R(3U_0) \quad (3.6)$$

$$= -4t \cos x \quad (3.7)$$

$$U_1 = U_0 - \int_0^t (t - \tau) R(U_0 - U_{-1}) d\tau \quad (3.8)$$

$$= U_0 + (4t^3 \cos x)/6 \quad (3.9)$$

Similarly,

$$U_2 = U_1 - \int_0^t (t - \tau) R(U_1 - U_0) d\tau \quad (3.10)$$

$$U_2 = U_1 - (t^5 \cos x)/15 \quad (3.11)$$

$$u_n(x, t) = \cos x(2t - (2t)^3/3! + (2t)^5/5! - (2t)^7/7! + \dots) \quad (3.12)$$

Recall that

$$u = \lim_{n \rightarrow \infty} u_n \quad (3.13)$$

This provides the precise solution.

$$u(x, t) = \cos x \sin 2t. \quad (3.14)$$

4. THE WAVE-LIKE EQUATION

Now let's look at the wave-like equation below

$$U_{(tt)} = x^2/2u_{xx}, 0 < x < 1, t > 0 \quad (4.1)$$

Initial condition,

$$u(x, 0) = 0, u_t(x, 0) = x^2 \quad (4.2)$$

Stationary conditions is being obtained.

$$1 - \lambda' = 0 \quad (4.3)$$

$$\lambda(\xi = t) = 0 \quad (4.4)$$

$$\lambda' = 0 \quad (4.5)$$

So that

$$\lambda = \xi - t \quad (4.6)$$

The iteration formula can be obtained by replacing the value of the Lagrangian multiplier in functional (2.4). Taking into account the initial values provided, we can choose

$$u_0(x, t) = x_2 t \quad (4.7)$$

. Applying this choice in (2.7) results in the consecutive approximations that follow:

$$u_0(x, t) = x^2 t \quad (4.8)$$

$$u_1(x, t) = x^2 t + x^2 t^3/3! \quad (4.9)$$

$$u_n(x, t) = x^2(t + t^3/3! + t^5/5! + t^7/7! + \dots) \quad (4.10)$$

and using the fact that

$$u = \lim_{n \rightarrow \infty} u_n \quad (4.11)$$

that leads to the exact solution

$$u(x, t) = x^2 \sinh t \quad (4.12)$$

5. WAVE EQUATION IN UNBOUNDED DOMAIN

In conclusion, an unbounded domain is examined regarding the wave equation.

$$u_t t = u_x x, -\infty < x < \infty, t > 0 \quad (5.1)$$

Initial condition,

$$u(x, 0) = \sin x, u_t(x, 0) = 0 \quad (5.2)$$

It is important to mention that the D'Alembert method is commonly employed to solve this equation. Due to the unbounded nature of the domain, boundary conditions are not specified. The equation's correction function is

$$u_{(n+1)}(x, t) = u_n(x, t) + \int_0^1 \lambda(\xi) ((\partial^2 u_n(x, \xi))/\partial \xi^2 - (\partial^2(\tilde{u}_n)(x, \xi))/\partial x^2) d\xi \quad (5.3)$$

The conditions generated is stable.

$$1 - \lambda' = 0 \quad (5.4)$$

$$\lambda(\xi = t) = 0 \quad (5.5)$$

$$\lambda' = 0 \quad (5.6)$$

So that

$$\lambda = \xi - t \quad (5.7)$$

By plugging this Lagrangian multiplier value into the functional (3.2) we get the iteration formula.

$$u_{(n+1)}(x, t) = u_n(x, t) + \int_0^1 (\xi - t) ((\partial^2 u_n(x, \xi))/\partial \xi^2 - (\partial^2(\tilde{u}_n)(x, \xi))/\partial x^2) d\xi, n \geq 0. \quad (5.8)$$

Given the starting values, we can choose $u_0(x, t) = \sin x$. By utilizing this particular choice in (3.5), we derive the subsequent iterative approximations:

$$u_0(x, t) = \sin x \quad (5.9)$$

$$(u_1(x, t) = \sin x - t^2/2! \sin x \quad (5.10)$$

$$u_2(x, t) = \sin x - t^2/2! \sin x + (t^4/4!) \sin x \quad (5.11)$$

$$u_2(x, t) = \sin x - t^2/2! \sin x + t^4/4! \sin x - t^6/6! \sin x \quad (5.12)$$

$$u_n(x, t) = \sin x (1 - t^2/2! + t^4/4! - t^6/6! + \dots) \quad (5.13)$$

and using the fact that

$$u = \lim_{n \rightarrow \infty} u_n \quad (5.14)$$

that leads to the exact solution

$$u(x, t) = \sin x \cos t. \quad (5.15)$$

6. CONCLUSIONS

In this work, we have examined a few of MVIM-related issues in depth, and we've drawn the conclusion that the method can be used to mitigate some of VIM's drawbacks. By removing unnecessary terminology and redundant calculations, MVIM streamlines the VIM language. When it comes to reducing workload and speeding up computations, MVIM is clearly the more appropriate. Engineers and non-experts alike can benefit from MVIM since it provides a closed-form approximation of a solution to a nonlinear condition. Utilizing variational iteration theory, the technique for finding a solution is straightforward, and only a small number of iterations are required to obtain precise results. Specifically, Variational Iteration Method (VIM) includes repetitive calculations and the computation of unnecessary terms. Therefore, by utilising the Modified Variational Iteration Method (MVIM), we are able to overcome the shortcomings of the Variational Iteration Method (VIM), which in turn prevents the repetition of old computations and thus minimising extra effort and calculations.

Acknowledgement. The author Dr. Nahid Fatima would like to thank Prince Sultan University for the help and this work was supported by the research grants SEED project Prince Sultan University, Saudi Arabia [Grant Number SEED-CHS-2023-136].

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