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# BEREZIN RADIUS INEQUALITIES FOR FINITE SUMS OF FUNCTIONAL HILBERT SPACE OPERATORS

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Abstract. This work presents applications of certain functions to Berezin radius inequalities, as well as further proofs of inequalities linking them. We also establish new inequalities using the operator on a functional Hilbert space concerning the Berezin radius and Berezin norm of specific finite sums. There are also several Berezin radius inequalities provided.

## 1. Introduction

A Hilbert space $\mathcal{H} = \mathcal{H}(\digamma)$ containing complex-valued functions on a set $\digamma$ such that evaluation at any point of $\digamma$ is a continuous functional on $\mathcal{H}$ is referred to as a functional Hilbert space (abbreviated FHS). A reproducing kernel, or a function $k_\xi(z) \in \mathcal{H}$ such that $\langle f, k_\xi \rangle = f(\xi)$ for any $f \in \mathcal{H}$ and $\xi \in \digamma$ in the Hilbert function space $\mathcal{H}$ thanks to the Riesz representation theorem. This function is called reproducing kernel of the space $\mathcal{H}$. We denote by $\widehat{k}_\xi := \frac{k_\xi}{||k_\xi||_{\mathcal{H}}}$ the normalized reproducing kernel of $\mathcal{H}$. The prototypical FHSs are the Hardy space $H^2(\mathbb{D})$, the Bergman space $L^2_a(\mathbb{D})$, the Dirichlet space $\mathcal{D}^2(\mathbb{D})$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc, and the Fock space $\mathcal{F}(\mathbb{C})$. For example, Aronzajn [2] provides a thorough explanation of the theory of reproducing kernels and FHSs. In several fields of pure and applied mathematics, such as frame theory, wavelets, signals and fractal theory, reproducing kernels are crucial (see, for example, Jorgensen's book [18] and its references).

The Berezin symbol $\widetilde{\mathsf{C}}$ for any bounded linear operator $\mathsf{C}$ on $\mathcal{H}$, meaning $\mathsf{C} \in \mathcal{B}(\mathcal{H})$, is defined as (see, Berezin [6])

$$\widetilde{\mathsf{C}}(\xi) := \langle \mathsf{C}\widehat{k}_\xi, \widehat{k}_\xi \rangle, \ \xi \in \digamma.$$

Put otherwise, the function on $\digamma$ defined by restricting the quadratic form $\langle \mathsf{C}a, a \rangle$ with $a \in \mathcal{H}$ to the subset of all normalized reproducing kernels of the unit sphere in $\mathcal{H}$ is represented by the Berezin symbol $\widetilde{\mathsf{C}}$. The Cauchy-Schwarz inequality makes it evident that $\widetilde{\mathsf{C}}$ is the bounded function on $\digamma$ whose values fall inside the

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operator $\widetilde{\mathsf{C}}$'s numerical range. Accordingly,

$$\operatorname{ber}(\mathsf{C}) := \sup_{\xi \in F} \left| \widetilde{\mathsf{C}}(\xi) \right|,$$

and

$$\operatorname{Ber}(\mathsf{C}) := \operatorname{Range}(\widetilde{\mathsf{C}}) = \{ \widetilde{\mathsf{C}}(\xi) : \xi \in F \}$$

define the Berezin radius $\operatorname{ber}(\mathsf{C})$ and the Berezin set $\operatorname{Ber}(\mathsf{C})$ of operator $\widetilde{\mathsf{C}}$, respectively (see Karaev [4, 19, 20]). Obviously, $\operatorname{ber}(\mathsf{C}) \leq w(\mathsf{C}) \leq \|\mathsf{C}\|$ and $\operatorname{Ber}(\mathsf{C}) \subset W(\mathsf{C})$ hold, where $W(\mathsf{C})$ represents the numerical range of operator $\mathsf{C}$ and $w(\mathsf{C})$ indicates the numerical radius. So, the study of the new Berezin characteristics $\operatorname{Ber}(\mathsf{C})$, $\operatorname{ber}(\mathsf{C})$ and $\|\mathsf{C}\|_{\text{ber}}$ is important firstly for the deep study of the Berezin set and the Berezin radius of operators on the FHSs. There is a huge literature devoted to the investigation of the above mentioned Berezin characteristics of operators and their relationships, see, for instance, [5, 7, 8, 9, 10, 11, 12, 13, 14, 17, 24, 25, 26].

According to Huban et al. [16], if $\mathsf{C} \in \mathcal{B}(\mathcal{H})$, then

$$\left(\operatorname{ber}(\mathsf{C})\right)^2 \leq \frac{1}{2} \left\| \mathsf{C}^*\mathsf{C} + \mathsf{C}\mathsf{C}^* \right\|_{\text{ber}}. \tag{1.1}$$

Motivated by the article [3], we give several generalizing inequalities (1.1) in this study . Among these generalizations are the roles played by certain characters. We provide Berezin radius inequality for sums of two operators, which includes a number of specific inequalities, by employing an alternative approach. There are also some Berezin radius inequalities provided.

## 2. Auxiliary Theorems

In order to achieve our goal, our analysis depends on the following well-known lemmas.

**Lemma 2.1.** *If $u$ and $v$ are two real numbers and $\iota \geq 2$, then we get*

$$|u+v|^\iota + |u-v|^\iota \geq 2\left(|u|^\iota + |v|^\iota\right). \tag{2.1}$$

The formula for Minkowski's inequality is

$$\left( \sum_{i=1}^{n} (u_i + v_i)^\iota \right)^{\frac{1}{\iota}} \leq \left( \sum_{i=1}^{n} u_i^\iota \right)^{\frac{1}{\iota}} + \left( \sum_{i=1}^{n} v_i^\iota \right)^{\frac{1}{\iota}}, \tag{2.2}$$

where $u_i$, $v_i > 0$ for $i = 1, 2, ..., n$ and $\iota \geq 1$.

We require the so-called Hölder-McCarthy inequality for the following. The Jensen inequality and the spectral theorem for positive operators make it simple to determine. Let $\mathsf{C} \in \mathbb{B}(\mathcal{H})$ be positive. Then we have

$$\left\langle \mathsf{C}u, u \right\rangle^\iota \leq (\geq) \left\langle \mathsf{C}^\iota u, u \right\rangle, \quad \iota \geq 1 \;\; (\iota \in [0,1]) \tag{2.3}$$

for any unit vector $u \in \mathcal{H}$ (see [23, p. 20]).

**Lemma 2.2.** *If $\mathsf{C} \in \mathcal{B}(\mathcal{H})$ is self-adjoint operator and $u \in \mathcal{H}$, then we get*

$$\left| \left\langle \mathsf{C}u, u \right\rangle \right| \leq \left\langle |\mathsf{C}| \, u, u \right\rangle. \tag{2.4}$$

The Cauchy-Schwarz inequality was extended to a bounded linear operator $\mathsf{C} \in \mathcal{B}(\mathcal{H})$ by Kato [21]. He demonstrated that for any $u, v \in \mathcal{H}$, and $0 \leq \iota \leq 1$,

$$\left|\left\langle \mathsf{C}u, v\right\rangle\right|^2 \leq \left\langle \left|\mathsf{C}\right|^{2\iota} u, u\right\rangle \left\langle \left|\mathsf{C}^*\right|^{2(1-\iota)} v, v\right\rangle \tag{2.5}$$

As a consequence, we have

$$\left|\left\langle \mathsf{C}u, v\right\rangle\right|^2 \leq \left\langle \left|\mathsf{C}\right| u, u\right\rangle \left\langle \left|\mathsf{C}^*\right| v, v\right\rangle \tag{2.6}$$

for every $u, v \in \mathcal{H}$ (see [15, Theorem 1]).

Kittaneh [22, Theorem 1] provides an expansion of Kato's inequality.

**Proposition 2.3.** *Let $u, v \in \mathcal{H}$ and $\mathsf{C} \in \mathcal{B}(\mathcal{H})$.*

$$\left|\left\langle \mathsf{C}u, v\right\rangle\right|^2 \leq \left\langle f^2\left(\left|\mathsf{C}\right|\right) u, u\right\rangle \left\langle g^2\left(\left|\mathsf{C}^*\right|\right) v, v\right\rangle. \tag{2.7}$$

*is the case where $f$ and $g$ are two nonnegative continuous functions on $[0, \infty)$ that fulfill $f(\varpi)g(\varpi) = \varpi$, $\varpi \geq 0$.*

## 3. Main results

We establish Berezin radius inequalities for finite sums of operators in this section. Berezin radius inequality for finite sums of operators in $\mathcal{B}(\mathcal{H})$ is the first result in this work. Unless otherwise indicated, let us assume that $f$ and $g$ are nonnegative continuous functions on $[0, \infty)$ such that $f(\varpi)g(\varpi) = \varpi$ for any $\varpi \in [0, \infty)$,

**Theorem 3.1.** *If $X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n$ are operators in $\mathcal{B}(\mathcal{H})$ and $\iota_i \geq 0, i \in \{1, 2, ..., n\}, \sum_{i=1}^n \iota_i = 1$, then we have*

$$\operatorname{ber}^{\iota}\left(\sum_{i=1}^{n} \iota_i \left(X_i + Y_i\right)\right) \leq 2^{\iota-2} \left\|\sum_{i=1}^{n} \iota_i \left(f^{2\iota}\left|X_i\right| + f^{2\iota}\left|Y_i\right| + g^{2\iota}\left|X_i^*\right| + g^{2\iota}\left|Y_i^*\right|\right)\right\|_{\text{ber}}. \tag{3.1}$$

*for $\geq 2$.*

*Proof.* Employing the triangle inequality and the Minkowski's inequality (2.2), we get

$$\begin{aligned}
\left|\left\langle \sum_{i=1}^{n} \iota_i \left(X_i + Y_i\right) \widehat{k}_\xi, \widehat{k}_\xi\right\rangle\right|^{\iota} &\leq \sum_{i=1}^{n} \iota_i \left|\left\langle \left(X_i + Y_i\right) \widehat{k}_\xi, \widehat{k}_\xi\right\rangle\right|^{\iota} (Convexity) \qquad (3.2)\\
&= \sum_{i=1}^{n} \iota_i \left|\left\langle X_i \widehat{k}_\xi, \widehat{k}_\xi\right\rangle + \left\langle Y_i \widehat{k}_\xi, \widehat{k}_\xi\right\rangle\right|^{\iota}\\
&\leq \sum_{i=1}^{n} \left(\iota_i^{\frac{1}{\iota}} \left|\left\langle X_i \widehat{k}_\xi, \widehat{k}_\xi\right\rangle\right| + \iota_i^{\frac{1}{\iota}} \left|\left\langle Y_i \widehat{k}_\xi, \widehat{k}_\xi\right\rangle\right|\right)^{\iota}\\
&\leq \left[\left(\sum_{i=1}^{n} \iota_i \left|\left\langle X_i \widehat{k}_\xi, \widehat{k}_\xi\right\rangle\right|^{\iota}\right)^{1/\iota} + \left(\sum_{i=1}^{n} \iota_i \left|\left\langle Y_i \widehat{k}_\xi, \widehat{k}_\xi\right\rangle\right|^{\iota}\right)^{1/\iota}\right]^{\iota}.
\end{aligned}$$

Utilizing Proposition 2.3, AM-GM inequality and the inequality (2.3), we deduce that

$$
\begin{aligned}
&\left[\left(\sum_{i=1}^{n}\iota_i\left|\left\langle X_i\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right|^{\iota}\right)^{1/\iota}+\left(\sum_{i=1}^{n}\iota_i\left|\left\langle Y_i\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right|^{\iota}\right)^{1/\iota}\right]^{\iota}\\
&\le 2^{\iota-1}\left[\sum_{i=1}^{n}\iota_i\left|\left\langle X_i\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right|^{\iota}+\sum_{i=1}^{n}\iota_i\left|\left\langle Y_i\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right|^{\iota}\right]\\
&\le 2^{\iota-1}\left[\sum_{i=1}^{n}\iota_i\left\langle f^2\left|X_i\right|\widehat{k}_\xi,\widehat{k}_\xi\right\rangle^{\iota/2}\left\langle g^2\left|X_i^*\right|\widehat{k}_\xi,\widehat{k}_\xi\right\rangle^{\iota/2}+\sum_{i=1}^{n}\iota_i\left\langle f^2\left|Y_i\right|\widehat{k}_\xi,\widehat{k}_\xi\right\rangle^{\iota/2}\left\langle g^2\left|Y_i^*\right|\widehat{k}_\xi,\widehat{k}_\xi\right\rangle^{\iota/2}\right]\\
&\le 2^{\iota-2}\left[\sum_{i=1}^{n}\iota_i\left(\left\langle f^2\left|X_i\right|\widehat{k}_\xi,\widehat{k}_\xi\right\rangle^{\iota}+\left\langle g^2\left|X_i^*\right|\widehat{k}_\xi,\widehat{k}_\xi\right\rangle^{\iota}\right)+\sum_{i=1}^{n}\iota_i\left(\left\langle f^2\left|Y_i\right|\widehat{k}_\xi,\widehat{k}_\xi\right\rangle^{\iota}+\left\langle g^2\left|Y_i^*\right|\widehat{k}_\xi,\widehat{k}_\xi\right\rangle^{\iota}\right)\right]\\
&\le 2^{\iota-2}\left[\sum_{i=1}^{n}\iota_i\left(\left\langle f^{2\iota}\left|X_i\right|\widehat{k}_\xi,\widehat{k}_\xi\right\rangle+\left\langle g^{2\iota}\left|X_i^*\right|\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right)+\sum_{i=1}^{n}\iota_i\left(\left\langle f^{2\iota}\left|Y_i\right|\widehat{k}_\xi,\widehat{k}_\xi\right\rangle+\left\langle g^{2\iota}\left|Y_i^*\right|\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right)\right]\\
&\le 2^{\iota-2}\left[\sum_{i=1}^{n}\iota_i\left(\left\langle f^{2\iota}\left|X_i\right|\widehat{k}_\xi,\widehat{k}_\xi\right\rangle+\left\langle g^{2\iota}\left|X_i^*\right|\widehat{k}_\xi,\widehat{k}_\xi\right\rangle+\left\langle f^{2\iota}\left|Y_i\right|\widehat{k}_\xi,\widehat{k}_\xi\right\rangle+\left\langle g^{2\iota}\left|Y_i^*\right|\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right)\right]\\
&\le 2^{\iota-2}\left[\sum_{i=1}^{n}\iota_i\left\langle\left(f^{2\iota}\left|X_i\right|+f^{2\iota}\left|Y_i\right|+g^{2\iota}\left|X_i^*\right|+g^{2\iota}\left|Y_i^*\right|\right)\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right]. \qquad (3.3)
\end{aligned}
$$

Utilizing the inequality (3.2) and (3.3), we reach the required inequality

$$
\left|\left\langle\sum_{i=1}^{n}\iota_i\left(X_i+Y_i\right)\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right|^{\iota}\le 2^{\iota-2}\left[\sum_{i=1}^{n}\iota_i\left\langle\left(f^{2\iota}\left|X_i\right|+f^{2\iota}\left|Y_i\right|+g^{2\iota}\left|X_i^*\right|+g^{2\iota}\left|Y_i^*\right|\right)\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right].
$$

Now taking the supremum over all $\xi\in\digamma$ in the above inequality, we have

$$
\sup_{\xi\in\digamma}\left|\left\langle\sum_{i=1}^{n}\iota_i\left(X_i+Y_i\right)\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right|^{\iota}\le 2^{\iota-2}\sup_{\xi\in\digamma}\left[\left\langle\sum_{i=1}^{n}\iota_i\left(f^{2\iota}\left|X_i\right|+f^{2\iota}\left|Y_i\right|+g^{2\iota}\left|X_i^*\right|+g^{2\iota}\left|Y_i^*\right|\right)\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right]
$$

and

$$
\operatorname{ber}^{\iota}\left(\sum_{i=1}^{n}\iota_i(X_i+Y_i)\right)\le 2^{\iota-2}\left\|\sum_{i=1}^{n}\iota_i\left(f^{2\iota}\left|X_i\right|+f^{2\iota}\left|Y_i\right|+g^{2\iota}\left|X_i^*\right|+g^{2\iota}\left|Y_i^*\right|\right)\right\|_{\mathrm{ber}},
$$

which proves the theorem. □

In particular, if we take $X_i=Y_i=0,\iota_1=1,\iota_i=0$ for $i=2,3,...,n$ in Theorem 3.1, then we obtain the following extension of (1.1).

**Corollary 3.2.** *Let $X,Y$ be operators in $\mathcal{B}(\mathcal{H})$. Then we have*

$$
\operatorname{ber}^{\iota}(X+Y)\le\frac{1}{2^{2-\iota}}\left\|f^{2\iota}\left|X\right|+f^{2\iota}\left|Y\right|+g^{2\iota}\left|X^*\right|+g^{2\iota}\left|Y^*\right|\right\|_{\mathrm{ber}}, \qquad (3.4)
$$

*for $\iota\ge 2$.*

By taking $X = Y$ and $f(\varpi) = g(\varpi) = \sqrt{\varpi}$ in (3.4), the outcome is as follows:

$$\mathrm{ber}^{\iota}(X) \leq \frac{1}{2} \left\| |X|^{\iota} + |X^*|^{\iota} \right\|_{\mathrm{ber}}. \tag{3.5}$$

We retrieve inequality (1.1) by putting $\iota = 2$ in the above inequality.

Another generalisation of inequality (1.1) is the following Berezin radius inequality.

**Theorem 3.3.** *Assume $X, Y \in \mathcal{B}(\mathcal{H})$. Then, for $\iota \geq 2$, we get*

$$\mathrm{ber}^{\iota}(X+Y) \leq \frac{1}{2^{3-\iota}} \left\| f^{2\iota}(|X+Y|) + f^{2\iota}(|X-Y|) + g^{2\iota}(|(X+Y)^*|) + g^{2\iota}(|(X-Y)^*|) \right\|_{\mathrm{ber}}. \tag{3.6}$$

*Proof.* Using Lemma 2.1, Proposition 2.3, inequality (2.3) and AM-GM inequality, respectively, we obtain

$$\begin{aligned}
&\left| \left\langle (X+Y)\widehat{k}_{\xi}, \widehat{k}_{\xi} \right\rangle \right|^{\iota} \\
&= \left| \left\langle X\widehat{k}_{\xi}, \widehat{k}_{\xi} \right\rangle + \left\langle Y\widehat{k}_{\xi}, \widehat{k}_{\xi} \right\rangle \right|^{\iota} \\
&\leq 2^{\iota-1} \left( \left| \left\langle X\widehat{k}_{\xi}, \widehat{k}_{\xi} \right\rangle \right|^{\iota} + \left| \left\langle Y\widehat{k}_{\xi}, \widehat{k}_{\xi} \right\rangle \right|^{\iota} \right) \\
&\leq 2^{\iota-2} \left( \left| \left\langle (X+Y)\widehat{k}_{\xi}, \widehat{k}_{\xi} \right\rangle \right|^{\iota} + \left| \left\langle (X-Y)\widehat{k}_{\xi}, \widehat{k}_{\xi} \right\rangle \right|^{\iota} \right) \\
&\leq 2^{\iota-2} \left[ \left\langle f^{\iota}(|X+Y|)\widehat{k}_{\xi}, \widehat{k}_{\xi} \right\rangle \left\langle g^{\iota}(|(X+Y)^*|)\widehat{k}_{\xi}, \widehat{k}_{\xi} \right\rangle \right. \\
&\left. + \left\langle f^{\iota}(|X-Y|)\widehat{k}_{\xi}, \widehat{k}_{\xi} \right\rangle \left\langle g^{\iota}(|(X-Y)^*|)\widehat{k}_{\xi}, \widehat{k}_{\xi} \right\rangle \right] \\
&\leq 2^{\iota-3} \left[ \left\langle f^{2\iota}(|X+Y|)\widehat{k}_{\xi}, \widehat{k}_{\xi} \right\rangle + \left\langle g^{2\iota}(|(X+Y)^*|)\widehat{k}_{\xi}, \widehat{k}_{\xi} \right\rangle \right. \\
&\left. + \left\langle f^{2\iota}(|X-Y|)\widehat{k}_{\xi}, \widehat{k}_{\xi} \right\rangle + \left\langle g^{2\iota}(|(X-Y)^*|)\widehat{k}_{\xi}, \widehat{k}_{\xi} \right\rangle \right] \\
&= 2^{\iota-3} \left\langle f^{2\iota}(|X+Y|) + f^{2\iota}(|X-Y|) + g^{2\iota}(|(X+Y)^*|) + g^{2\iota}(|(X-Y)^*|)\widehat{k}_{\xi}, \widehat{k}_{\xi} \right\rangle.
\end{aligned}$$

and

$$\begin{aligned}
&\left| \left\langle (X+Y)\widehat{k}_{\xi}, \widehat{k}_{\xi} \right\rangle \right|^{\iota} \\
&\leq \frac{1}{2^{3-\iota}} \left\langle f^{2\iota}(|X+Y|) + f^{2\iota}(|X-Y|) + g^{2\iota}(|(X+Y)^*|) + g^{2\iota}(|(X-Y)^*|)\widehat{k}_{\xi}, \widehat{k}_{\xi} \right\rangle.
\end{aligned}$$

Now taking the supremum over all $\xi \in \digamma$ in the above inequality, we get

$$\begin{aligned}
&\sup_{\xi \in \digamma} \left| \left\langle (X+Y)\widehat{k}_{\xi}, \widehat{k}_{\xi} \right\rangle \right|^{\iota} \\
&\leq \frac{1}{2^{3-\iota}} \sup_{\xi \in \digamma} \left\langle f^{2\iota}(|X+Y|) + f^{2\iota}(|X-Y|) + g^{2\iota}(|(X+Y)^*|) + g^{2\iota}(|(X-Y)^*|)\widehat{k}_{\xi}, \widehat{k}_{\xi} \right\rangle
\end{aligned}$$

which is equivalent to

$$\mathrm{ber}^{\iota}(X+Y) \leq \frac{1}{2^{3-\iota}} \left\| f^{2\iota}(|X+Y|) + f^{2\iota}(|X-Y|) + g^{2\iota}(|(X+Y)^*|) + g^{2\iota}(|(X-Y)^*|) \right\|_{\mathrm{ber}},$$

and completes the theorem's proof. $\square$

The consequence that follows is a particular instance of Theorem 3.3. Replacing $Y$ by $X$ in inequality (3.6), we obtain inequality (3.7).

**Corollary 3.4.** *Let $X \in \mathcal{B}(\mathcal{H})$ and $f(\varpi) = g(\varpi) = \sqrt{\varpi}$ and $\iota \geq 2$. Then*

$$(\operatorname{ber}(X))^{\iota} \leq \frac{1}{2^{3-\iota}} \left\| |X|^{\iota} + |X^*|^{\iota} \right\|_{\text{ber}}. \tag{3.7}$$

*Remark* 3.5. According to the inequality in [16, Theorem 3.1],

$$(\operatorname{ber}(X))^{2} \leq \frac{1}{2} \left\| X^*X + XX^* \right\|_{\text{ber}}$$

for $\iota \geq 2$. This inequality follows from inequality (3.7) by replacing $\iota$ by 2.Also, it is clear that inequality (3.5) is sharper than inequality (3.7) for $\iota > 2$.

Another use of Theorem 3.3 is the inequality that follows.

**Theorem 3.6.** *If $X$ and $Y$ are self adjoint operators in $\mathcal{B}(\mathcal{H})$ and $\iota \geq 2$, then we get*

$$(\operatorname{ber}(X+Y))^{\iota} \leq \frac{1}{2^{2-\iota}} \left\| |X+Y|^{\iota} + |X-Y|^{\iota} \right\|_{\text{ber}}. \tag{3.8}$$

*Proof.* Let $f(\varpi) = g(\varpi) = \sqrt{\varpi}$ in inequality (3.6), we obtain

$$\begin{aligned}
(\operatorname{ber}(X+Y))^{\iota} &\leq \frac{1}{2^{3-\iota}} \left\| |X+Y|^{\iota} + |X-Y|^{\iota} + |(X+Y)^*|^{\iota} + |(X-Y)^*|^{\iota} \right\|_{\text{ber}} \\
&= \frac{1}{2^{3-\iota}} \left\| 2\,|X+Y|^{\iota} + 2\,|X-Y|^{\iota} \right\|_{\text{ber}} \\
&= \frac{1}{2^{2-\iota}} \left\| |X+Y|^{\iota} + |X-Y|^{\iota} \right\|_{\text{ber}},
\end{aligned}$$

which is the desired inequality (3.8). □

We get the following result by replacing $Y$ by $X$ in (3.8).

**Corollary 3.7.** *In $\mathcal{B}(\mathcal{H})$, let $X$ be the self adjoint operator and $\iota \geq 2$. Next, we have*

$$(\operatorname{ber}(X))^{\iota} \leq \frac{1}{2^{2-\iota}} \left\| |X|^{\iota} \right\|_{\text{ber}} \tag{3.9}$$

*and, in particular,*

$$(\operatorname{ber}(X))^{2} \leq \left\| |X|^{2} \right\|_{\text{ber}} = \left\| X^{2} \right\|_{\text{ber}}. \tag{3.10}$$

The Berezin radius inequality for products of two operators in $\mathcal{B}(\mathcal{H})$ is the theorem we discuss here.

**Theorem 3.8.** *In $\mathcal{B}(\mathcal{H})$, let $X, Y$ be the self adjoint operators. Next, we have*

$$\operatorname{ber}(XY) \leq \frac{1}{2} \left\| f^{2}\,|XY| + g^{2}\left(|(XY)^*|\right) \right\|_{\text{ber}}. \tag{3.11}$$

*Proof.* From the inequality (2.7) and AM-GM inequality, we get

$$\begin{aligned}\left|\left\langle XY\widehat{k}_\xi, \widehat{k}_\xi\right\rangle\right| &\le \left\langle f^2\left|XY\right|\widehat{k}_\xi, \widehat{k}_\xi\right\rangle^{1/2}\left\langle g^2\left(\left|(XY)^*\right|\right)\widehat{k}_\xi, \widehat{k}_\xi\right\rangle^{1/2}\\ &\le \frac{1}{2}\left[\left\langle f^2\left|XY\right|\widehat{k}_\xi, \widehat{k}_\xi\right\rangle + \left\langle g^2\left(\left|(XY)^*\right|\right)\widehat{k}_\xi, \widehat{k}_\xi\right\rangle\right]\\ &= \frac{1}{2}\left\langle\left(f^2\left|XY\right| + g^2\left(\left|(XY)^*\right|\right)\right)\widehat{k}_\xi, \widehat{k}_\xi\right\rangle\end{aligned}$$

and

$$\sup_{\xi\in F}\left|\left\langle XY\widehat{k}_\xi, \widehat{k}_\xi\right\rangle\right| \le \frac{1}{2}\sup_{\xi\in F}\left\langle\left(f^2\left|XY\right| + g^2\left(\left|(XY)^*\right|\right)\right)\widehat{k}_\xi, \widehat{k}_\xi\right\rangle.$$

Thus, we obtain

$$\operatorname{ber}(XY) \le \frac{1}{2}\left\|f^2\left|XY\right| + g^2\left(\left|(XY)^*\right|\right)\right\|_{\operatorname{ber}}.$$

Hence, we get the required inequality. ☐

Another extension of inequality (1.1) is the Berezin radius inequality for products of operators theorem.

**Theorem 3.9.** *If $X, Y \in \mathcal{B}(\mathcal{H})$, then we get*

$$(\operatorname{ber}(Y^*X))^\iota \le \frac{1}{2}\left\||X|^\iota + (Y^*\left|X^*\right|Y)^\iota\right\|_{\operatorname{ber}} \tag{3.12}$$

*for $\iota \ge 1$.*

*Proof.* From the inequalities (2.6), (2.3) and AM-GM, we have

$$\begin{aligned}\left|\left\langle Y^*X\widehat{k}_\xi, \widehat{k}_\xi\right\rangle\right|^\iota &= \left|\left\langle X\widehat{k}_\xi, Y\widehat{k}_\xi\right\rangle\right|^\iota\\ &\le \left\langle |X|\widehat{k}_\xi, \widehat{k}_\xi\right\rangle^{\iota/2}\left\langle |X^*|Y\widehat{k}_\xi, Y\widehat{k}_\xi\right\rangle^{\iota/2}\\ &= \left\langle |X|\widehat{k}_\xi, \widehat{k}_\xi\right\rangle^{\iota/2}\left\langle Y^*\left|X^*\right|Y\widehat{k}_\xi, \widehat{k}_\xi\right\rangle^{\iota/2}\\ &\le \left\langle |X|^\iota\widehat{k}_\xi, \widehat{k}_\xi\right\rangle^{1/2}\left\langle (Y^*\left|X^*\right|Y)^\iota\widehat{k}_\xi, \widehat{k}_\xi\right\rangle^{1/2}\\ &\le \frac{1}{2}\left[\left\langle |X|^\iota\widehat{k}_\xi, \widehat{k}_\xi\right\rangle + \left\langle (Y^*\left|X^*\right|Y)^\iota\widehat{k}_\xi, \widehat{k}_\xi\right\rangle\right]\\ &= \frac{1}{2}\left\langle\left[|X|^\iota + (Y^*\left|X^*\right|Y)^\iota\right]\widehat{k}_\xi, \widehat{k}_\xi\right\rangle.\end{aligned}$$

Now taking the supremum over every $\xi \in F$, we deduce that

$$(\operatorname{ber}(Y^*X))^\iota \le \frac{1}{2}\left\||X|^\iota + (Y^*\left|X^*\right|Y)^\iota\right\|_{\operatorname{ber}}.$$

☐

Given $\iota = 2$ and $Y = I$, we obtain inequality (1.1) as a particular instance of inequality (3.12).

## 4. Conclusion

In this paper various inequalities have been obtained which generalize (1.1) in various forms. Theorem (3.1) provides a generalization for $n$-tuples of operators which as a consequence provides a special case of (1.1). Another generalization is (3.6) which provides another variation of (3.1) for $\iota = 2$ and $g(\varpi) = f(\varpi) = \sqrt{\varpi}$. Theorems (3.11) and (3.12) concern themselves with product variations of (1.1), where the first one is a generalization with respect to the functions $f$ and $g$ and the second one concerns itself with standard bounded operators which for specific parameters provide (1.1). Application of the said results can be seen in the study of the Toeplitz Corona problem in the Bergman space [1].

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