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ON THE RELATIONSHIP BETWEEN THE LOGARITHMIC LOWER ORDER OF COEFFICIENTS AND THE GROWTH OF SOLUTIONS OF COMPLEX LINEAR DIFFERENTIAL EQUATIONS IN $\overline{\mathbb{C}} \setminus \{z_0\}$

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Abstract. In this article, we study the growth of solutions of the homogeneous complex linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0,$$

where the coefficients $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ are analytic or meromorphic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$. Under the sufficient condition that there exists one dominant coefficient by its logarithmic lower order or by its logarithmic lower type. We extend some precedent results due to Liu, Long and Zeng and others.

1. Introduction and Main Results

There are numerous valuable results showcasing the applications of Nevanlinna theory in the field of complex differential equations (see e.g. [20, 23, 24]), as well as in areas such as value sharing of meromorphic functions (see e.g. [1, 25, 26, 28]). Nevanlinna theory has been applied to get insight into the properties of solutions of complex differential equations, particularly their growth. In this context, researchers have explored domains beyond the complex plane. In the year 2016, Fettouch and Hamouda in [16] firstly investigated the growth of solutions of linear differential equations where the coefficients are analytic or meromorphic functions in the extended complex plane except a finite singular point $\overline{\mathbb{C}} \setminus \{z_0\}$. Subsequently, by adopting the same idea, further research has yielded significant findings regarding the growth of solutions to linear differential equations (see e.g. [7, 8, 11, 12, 13, 17, 18, 21, 22]). The concepts of logarithmic order and logarithmic type of Chern, [9, 10] have always been used to study the growth of solutions to linear differential equations and linear difference equations for that case when the coefficients are entire or meromorphic in $\mathbb{C}$ of zero order (see e.g. [3, 4, 5, 6, 15]). Among these studies Ferraoun and Belaïdi in [15] compared the logarithmic lower order of coefficients with the growth of solutions of higher order linear differential equations. Inspired by this work, in this article we introduce the logarithmic lower order and the logarithmic lower type of meromorphic function in $\overline{\mathbb{C}} \setminus \{z_0\}$ and

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use them to investigate the growth of solutions to the same higher order linear differential equation and its relation with the growth of coefficients which are analytic or meromorphic in $\overline{\mathbb{C}}\setminus\{z_0\}$. An earlier version of this work was posted on arXiv [14]. Here we use some fundamental results and some standard notations of the Nevanlinna distribution theory of meromorphic functions (see [19, 20, 28]). At first, let's recall some notations and definitions which can be found in [11, 16, 22], for all $R \in (0,\infty)$ and $p \geq 1$, we define $\exp_1 R = e^R, \exp_{p+1} R = \exp(\exp_p R)$, $\log_1 R = \log R$ and $\log_{p+1} R = \log(\log_p R)$. Let f be a meromorphic function in $\overline{\mathbb{C}}-\{z_0\}$ and let $n(t,f)$ be the number of poles of f in $\{z \in \mathbb{C} : t \leq |z-z_0|\}\cup\{\infty\}$ (counting multiplicities), where $\overline{n}(t,f)$ is the number of distinct poles of f. Subsequently, the counting function of f near z_0 is defined by

$$N_{z_0}(r,f) = -\int_{\infty}^{r} \frac{n(t,f)-n(\infty,f)}{t}dt - n(\infty,f)\log r$$

and

$$\overline{N}_{z_0}(r,f) = -\int_{\infty}^{r} \frac{\overline{n}(t,f)-\overline{n}(\infty,f)}{t}dt - \overline{n}(\infty,f)\log r,$$

where the proximity function of f near z_0 is defined by

$$m_{z_0}(r,f) = \frac{1}{2\pi}\int_{o}^{2\pi} \log^+ |f(z_0 - re^{i\phi})|d\phi.$$

The characteristic function of f near z_0 is defined by

$$T_{z_0}(r,f) = m_{z_0}(r,f) + N_{z_0}(r,f).$$

Definition 1 ([22]). Let f be a meromorphic function in $\overline{\mathbb{C}}-\{z_0\}$, p and q be two integers with $p \geq q \geq 1$. The $[p,q]$-order and the $[p,q]$-type of f near z_0 are respectively defined by

$$\sigma_{[p,q]}(f,z_0) = \limsup_{r\longrightarrow 0} \frac{\log_p^+ T_{z_0}(r,f)}{\log_q \frac{1}{r}}, \quad \tau_{[p,q]}(f,z_0) = \limsup_{r\longrightarrow 0} \frac{\log_{p-1}^+ T_{z_0}(r,f)}{(\log_{q-1}\frac{1}{r})^{\sigma_{[p,q]}(f,z_0)}}$$

if $\sigma_{[p,q]}(f,z_0) \in (0,+\infty)$. If f is an analytic function in $\overline{\mathbb{C}}-\{z_0\}$, then

$$\sigma_{[p,q]}(f,z_0) = \limsup_{r\longrightarrow 0} \frac{\log_{p+1}^+ M_{z_0}(r,f)}{\log_q \frac{1}{r}}, \quad \tau_{[p,q],M}(f,z_0) = \limsup_{r\longrightarrow 0} \frac{\log_p^+ M_{z_0}(r,f)}{(\log_{q-1}\frac{1}{r})^{\sigma_{[p,q]}(f,z_0)}}$$

if $\sigma_{[p,q]}(f,z_0) \in (0,+\infty)$, where $M_{z_0}(r,f) = \max\{|f(z)| : |z-z_0| = r\}$.

Definition 2 ([11]). Let f be a meromorphic function in $\overline{\mathbb{C}}-\{z_0\}$, p and q be two integers with $p \geq q \geq 1$. The lower $[p,q]$-order and the lower $[p,q]$-type of f near z_0 are respectively defined by

$$\mu_{[p,q]}(f,z_0) = \liminf_{r\longrightarrow 0} \frac{\log_p^+ T_{z_0}(r,f)}{\log_q \frac{1}{r}}, \quad \underline{\tau}_{[p,q]}(f,z_0) = \liminf_{r\longrightarrow 0} \frac{\log_{p-1}^+ T_{z_0}(r,f)}{(\log_{q-1}\frac{1}{r})^{\mu_{[p,q]}(f,z_0)}}$$

if $\mu_{[p,q]}(f,z_0) \in (0,+\infty)$. If f is an analytic function in $\overline{\mathbb{C}}-\{z_0\}$, then

$$\mu_{[p,q]}(f,z_0) = \liminf_{r\longrightarrow 0} \frac{\log_{p+1}^+ M_{z_0}(r,f)}{\log_q \frac{1}{r}}, \quad \underline{\tau}_{[p,q],M}(f,z_0) = \liminf_{r\longrightarrow 0} \frac{\log_p^+ M_{z_0}(r,f)}{(\log_{q-1}\frac{1}{r})^{\mu_{[p,q]}(f,z_0)}}$$

if $\mu_{[p,q]}(f, z_0) \in (0, +\infty)$.

Now we introduce the definitions of the logarithmic order, the logarithmic type, the logarithmic lower order and the logarithmic lower type of a meromorphic function f in $\overline{\mathbb{C}} - \{z_0\}$ as follow.

Definition 3 ([13])**.** Let f be a meromorphic function in $\overline{\mathbb{C}}-\{z_0\}$, the logarithmic order of f near z_0 is given by

$$\sigma_{[1,2]}(f, z_0) = \sigma_{\log}(f, z_0) = \limsup_{r \longrightarrow 0} \frac{\log^+ T_{z_0}(r, f)}{\log \log \frac{1}{r}}$$

and if $\sigma_{\log}(f, z_0) \in [1, +\infty)$, then the logarithmic type of f near z_0 is defined by

$$\tau_{[1,2]}(f, z_0) = \tau_{\log}(f, z_0) = \limsup_{r \longrightarrow 0} \frac{T_{z_0}(r, f)}{(\log \frac{1}{r})^{\sigma_{\log}(f,z_0)}}.$$

If f is an analytic function in $\overline{\mathbb{C}} - \{z_0\}$, then

$$\sigma_{[1,2]}(f, z_0) = \sigma_{\log}(f, z_0) = \limsup_{r \longrightarrow 0} \frac{\log^+ \log^+ M_{z_0}(r, f)}{\log \log \frac{1}{r}},$$

and if $\sigma_{\log}(f, z_0) \in [1, +\infty)$, then the logarithmic type of f near z_0 is defined by

$$\tau_{[1,2],M}(f, z_0) = \tau_{\log,M}(f, z_0) = \limsup_{r \longrightarrow 0} \frac{\log^+ M_{z_0}(r, f)}{(\log \frac{1}{r})^{\sigma_{\log}(f,z_0)}}.$$

Definition 4. Let f be a meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$, the logarithmic lower order of f near z_0 is given by

$$\mu_{[1,2]}(f, z_0) = \mu_{\log}(f, z_0) = \liminf_{r \longrightarrow 0} \frac{\log^+ T_{z_0}(r, f)}{\log \log \frac{1}{r}},$$

and if $\mu_{\log}(f, z_0) \in [1, +\infty)$, then the logarithmic lower type of f near z_0 is given by

$$\underline{\tau}_{[1,2]}(f, z_0) = \underline{\tau}_{\log}(f, z_0) = \liminf_{r \longrightarrow 0} \frac{T_{z_0}(r, f)}{(\log \frac{1}{r})^{\mu_{\log}(f,z_0)}}.$$

If f is an analytic function in $\overline{\mathbb{C}} - \{z_0\}$, then

$$\mu_{[1,2]}(f, z_0) = \mu_{\log}(f, z_0) = \liminf_{r \longrightarrow 0} \frac{\log^+ \log^+ M_{z_0}(r, f)}{\log \log \frac{1}{r}},$$

and if $\mu_{\log}(f, z_0) \in [1, +\infty)$, then the logarithmic lower type of f near z_0 is given by

$$\underline{\tau}_{[1,2],M}(f, z_0) = \underline{\tau}_{\log,M}(f, z_0) = \liminf_{r \longrightarrow 0} \frac{\log^+ M_{z_0}(r, f)}{(\log \frac{1}{r})^{\mu_{\log}(f,z_0)}}.$$

Remark 1. According to (Lemma 2.2, [16]) if f is a non constant meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$, then $g(\omega) = f(z_0 - \frac{1}{\omega})$ is meromorphic in $\mathbb{C}$ and they satisfy

$$T(R, g) = T_{z_0}(\frac{1}{R}, f).$$

Consequently all the properties of the logarithmic order for the meromorphic functions in $\mathbb{C}$ are hold, such as all the non-constant rational functions which are analytic in $\overline{\mathbb{C}}-\{z_0\}$ are of logarithmic order equals one, where there is no transcendental meromorphic function in $\overline{\mathbb{C}}-\{z_0\}$ of logarithmic order less than one, further, constant functions have zero logarithmic order and there are no meromorphic functions in $\overline{\mathbb{C}}-\{z_0\}$ of logarithmic order between zero and one (see, [9, 10]).

Definition 5. The $[p,q]$ exponent of convergence of the sequence of zeros and distinct zeros of a meromorphic function f in $\overline{\mathbb{C}}-\{z_0\}$ are respectively defined by

$$\lambda_{[p,q]}(f,z_0)=\limsup_{r\longrightarrow 0}\frac{\log_p^+ N_{z_0}(r,\frac{1}{f})}{\log_q\frac{1}{r}},\qquad \overline{\lambda}_{[p,q]}(f,z_0)=\limsup_{r\longrightarrow 0}\frac{\log_p^+ \overline{N}_{z_0}(r,\frac{1}{f})}{\log_q\frac{1}{r}}.$$

Definition 6. The logarithmic exponent of convergence of the sequence of poles and distinct poles of a meromorphic function f in $\overline{\mathbb{C}}-\{z_0\}$ are respectively defined by

$$\lambda_{\log}\big(\frac{1}{f},z_0\big)=\limsup_{r\longrightarrow 0}\frac{\log^+ N_{z_0}(r,f)}{\log\log\frac{1}{r}}-1,\qquad \overline{\lambda}_{\log}\big(\frac{1}{f},z_0\big)=\limsup_{r\longrightarrow 0}\frac{\log^+ \overline{N}_{z_0}(r,f)}{\log\log\frac{1}{r}}-1.$$

Before we start stating our results, we may mention some of the previous results obtained on the growth of solutions of linear differential equations in the extended complex plane except a finite singular point $\overline{\mathbb{C}}\setminus\{z_0\}$ and we focus particularly on those involving the lower order. In [21], Liu and his co-authors considered the following complex homogeneous second order linear differential equation

$$f''+A(z)f'+B(z)f=0, \tag{1}$$

where $A(z)$ and $B(z)$ are both analytic functions in $\overline{\mathbb{C}}\setminus\{z_0\}$, and obtained the following theorems.

Theorem A ([21])**.** *Let $A(z)$ and $B(z)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$ satisfying $\mu(A,z_0)<\mu(B,z_0)<\infty$. Then, every non trivial solution $f(z)$ of (1) that is analytic in $\overline{\mathbb{C}}\backslash\{z_0\}$, satisfies $\sigma_2(f,z_0)\geq\mu(B,z_0)$.*

Theorem B ([21])**.** *Let $A(z)$ and $B(z)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$ satisfying $\mu(A,z_0)=\mu(B,z_0)$ with $\underline{\tau}_M(A,z_0)<\underline{\tau}_M(B,z_0)$. Then, every non trivial solution $f(z)$ of (1) that is analytic in $\overline{\mathbb{C}}\backslash\{z_0\}$, satisfies $\sigma_2(f,z_0)\geq\mu(B,z_0)$.*

For more general case, in a recent paper [11], we considered the following higher order linear differential equation

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=0, \tag{2}$$

where $A_j(z)$ $(j=0,1,\ldots,k-1)$ are analytic functions $\overline{\mathbb{C}}-\{z_0\}$, and we obtained the following results.

Theorem C ([11])**.** *Let $A_0(z),...,A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$. Assume that*

$$\max\left\{\sigma_{[p,q]}(A_j,z_0):j\neq 0\right\}<\mu_{[p,q]}(A_0,z_0)\leq\sigma_{[p,q]}(A_0,z_0)<+\infty.$$

Then, every analytic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}\backslash\{z_0\}$ of (2) satisfies

$$\mu_{[p,q]}(A_0,z_0)=\mu_{[p+1,q]}(f,z_0)\leq\sigma_{[p+1,q]}(f,z_0)=\sigma_{[p,q]}(A_0,z_0).$$

Theorem D ([11])**.** *Let $A_0(z), \ldots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}\setminus\{z_0\}$. Assume that*

$$\max\left\{\sigma_{[p,q]}(A_j, z_0) : j \neq 0\right\} \leq \mu_{[p,q]}(A_0, z_0) \leq \sigma_{[p,q]}(A_0, z_0) < +\infty$$

and

$$\max\left\{\tau_{[p,q],M}(A_j, z_0) : \sigma_{[p,q]}(A_j, z_0) = \mu_{[p,q]}(A_0, z_0) > 0\right\} < \underline{\tau}_{[p,q],M}(A_0, z_0) < +\infty.$$

Then, every analytic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}\setminus\{z_0\}$ of (2) satisfies $\mu_{[p,q]}(A_0, z_0) = \mu_{[p+1,q]}(f, z_0) \leq \sigma_{[p+1,q]}(f, z_0) = \sigma_{[p,q]}(A_0, z_0)$.

As we mentioned before, in this article you may find several similarities between the results obtained in [15] for complex plane $\mathbb{C}$ and the present results which we also consider them as extensions to the above theorems and other results to the case that all the coefficients are zero order analytic or meromorphic functions in $\overline{\mathbb{C}}\setminus\{z_0\}$. The first five theorems are for the special case that the coefficients of (2) are analytic or meromorphic in $\overline{\mathbb{C}}\setminus\{z_0\}$ and $A_0(z)$ dominates them, the rest of the results are for the case when the dominant coefficient is an arbitrary $A_s(z)$. In fact, we obtain the following results.

Theorem 1. *Let $A_0(z), \ldots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}\setminus\{z_0\}$ of finite logarithmic order such that $\mu_{\log}(A_0, z_0) > 1$ with $\max\left\{\sigma_{\log}(A_j, z_0) : j \neq 0\right\} \leq \mu_{\log}(A_0, z_0) \leq \sigma_{\log}(A_0, z_0) < +\infty$ and*

$$\sum_{\sigma_{\log}(A_j,z_0)=\mu_{\log}(A_0,z_0),j\neq 0} \tau_{\log}(A_j, z_0) < \underline{\tau}_{\log}(A_0, z_0) < +\infty.$$

Then, every analytic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}\setminus\{z_0\}$ of (2) satisfies $1 < \mu_{\log}(A_0, z_0) = \mu_{[2,2]}(f, z_0) \leq \sigma_{[2,2]}(f, z_0) = \sigma_{\log}(A_0, z_0) = \overline{\lambda}_{[2,2]}(f-\varphi, z_0) = \lambda_{[2,2]}(f-\varphi, z_0)$, where $\varphi(z)(\not\equiv 0)$ is an analytic function in $\overline{\mathbb{C}}\setminus\{z_0\}$ satisfying $\sigma_{[2,2]}(\varphi, z_0) < \mu_{\log}(A_0, z_0)$.

Theorem 2. *Let $A_0(z), \ldots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}\setminus\{z_0\}$ of finite logarithmic order such that $\mu_{\log}(A_0, z_0) > 1$ with $\max\left\{\sigma_{\log}(A_j, z_0) : j \neq 0\right\} \leq \mu_{\log}(A_0, z_0) \leq \sigma_{\log}(A_0, z_0) < +\infty$ and*

$$\limsup_{r\longrightarrow 0} \frac{\sum_{j\neq 0} m_{z_0}(r, A_j)}{m_{z_0}(r, A_0)} < 1.$$

Then, every analytic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}\setminus\{z_0\}$ of (2) satisfies $1 < \mu_{\log}(A_0, z_0) = \mu_{[2,2]}(f, z_0) \leq \sigma_{[2,2]}(f, z_0) = \sigma_{\log}(A_0, z_0) = \overline{\lambda}_{[2,2]}(f-\varphi, z_0) = \lambda_{[2,2]}(f-\varphi, z_0)$, where $\varphi(z)(\not\equiv 0)$ is an analytic function in $\overline{\mathbb{C}}\setminus\{z_0\}$ satisfying $\sigma_{[2,2]}(\varphi, z_0) < \mu_{\log}(A_0, z_0)$.

Theorem 3. *Let $A_0(z), \ldots, A_{k-1}(z)$ be meromorphic functions in $\overline{\mathbb{C}}\setminus\{z_0\}$ of finite logarithmic order such that $\mu_{\log}(A_0, z_0) > 1$ with*

$$\liminf_{r\longrightarrow 0} \frac{m_{z_0}(r, A_0)}{T_{z_0}(r, A_0)} = \delta > 0,$$

$$\max\left\{\sigma_{\log}(A_j, z_0) : j \neq 0\right\} \leq \mu_{\log}(A_0, z_0) \leq \sigma_{\log}(A_0, z_0) < +\infty$$

and

$$\sum_{\sigma_{\log}(A_j,z_0)=\mu_{\log}(A_0,z_0),j\neq 0} \tau_{\log}(A_j, z_0) < \delta\underline{\tau}_{\log}(A_0, z_0) < +\infty.$$

Then, every meromorphic solution $f(z)(\not\equiv 0)$ *in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of (2) satisfies* $1 < \mu_{\log}(A_0, z_0) \leq \mu_{[2,2]}(f, z_0)$.

Theorem 4. *Let* $A_0(z), ..., A_{k-1}(z)$ *be meromorphic functions in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of finite logarithmic order such that* $\mu_{\log}(A_0, z_0) > 1$ *with*

$$\liminf_{r\longrightarrow 0} \frac{m_{z_0}(r, A_0)}{T_{z_0}(r, A_0)} = \delta > 0 \quad \textit{and} \quad \limsup_{r\longrightarrow 0} \frac{\sum_{j\neq 0} m_{z_0}(r, A_j)}{m_{z_0}(r, A_0)} < 1.$$

Then, every meromorphic solution $f(z)(\not\equiv 0)$ *in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of (2) satisfies* $1 < \mu_{\log}(A_0, z_0) \leq \mu_{[2,2]}(f, z_0)$.

Theorem 5. *Let* $A_0(z), ..., A_{k-1}(z)$ *be meromorphic functions in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of finite logarithmic order with* $\lambda_{\log}(\frac{1}{A_0}, z_0)+1 < \mu_{\log}(A_0, z_0)$, $\max\left\{\sigma_{\log}(A_j, z_0) : j \neq 0\right\} \leq \mu_{\log}(A_0, z_0) \leq \sigma_{\log}(A_0, z_0) < +\infty$ *and*

$$\sum_{\sigma_{\log}(A_j,z_0)=\mu_{\log}(A_0,z_0), j\neq 0} \tau_{\log}(A_j, z_0) < \underline{\tau}_{\log}(A_0, z_0) < +\infty.$$

Then, every meromorphic solution $f(z)(\not\equiv 0)$ *in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of (2) satisfies* $1 < \mu_{\log}(A_0, z_0) \leq \mu_{[2,2]}(f, z_0)$.

Theorem 6. *Let* $A_0(z), ..., A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of finite logarithmic order. Suppose there exists an integer* s $(0 \leq s \leq k-1)$ *such that* $A_s(z)$ *satisfies* $\mu_{\log}(A_s, z_0) > 1$ *with* $\max\left\{\sigma_{\log}(A_j, z_0) : j \neq s\right\} \leq \mu_{\log}(A_s, z_0) < +\infty$ *and*

$$\sum_{\sigma_{\log}(A_j,z_0)=\mu_{\log}(A_s,z_0), j\neq s} \tau_{\log}(A_j, z_0) < \underline{\tau}_{\log}(A_s, z_0) < +\infty.$$

Then, every meromorphic solution $f(z)(\not\equiv 0)$ *in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of (2) satisfies* $\mu_{[2,2]}(f, z_0) \leq \mu_{\log}(A_s, z_0) \leq \mu_{\log}(f, z_0)$.

Remark 2. As for the case when the dominant coefficient is $A_0(z)$, we can replace the condition

$$\sum_{\sigma_{\log}(A_j,z_0)=\mu_{\log}(A_s,z_0), j\neq s} \tau_{\log}(A_j, z_0) < \underline{\tau}_{\log}(A_s, z_0) < +\infty$$

in Theorem 6 by $\limsup_{r\longrightarrow 0} \frac{\sum_{j\neq s} m_{z_0}(r,A_j)}{m_{z_0}(r,A_s)} < 1$.

Theorem 7. *Let* $A_0(z), ..., A_{k-1}(z)$ *be meromorphic functions in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of finite logarithmic order. Suppose there exists an integer* s $(0 \leq s \leq k-1)$ *such that* $A_s(z)$ *satisfies* $\mu_{\log}(A_s, z_0) > 1$ *with* $\liminf_{r\longrightarrow 0} \frac{m_{z_0}(r,A_s)}{T_{z_0}(r,A_s)} = \delta > 0$, $\max\left\{\sigma_{\log}(A_j, z_0) : j \neq s\right\} \leq \mu_{\log}(A_s, z_0) < +\infty$ *and*

$$\sum_{\sigma_{\log}(A_j,z_0)=\mu_{\log}(A_s,z_0), j\neq s} \tau_{\log}(A_j, z_0) < \delta\underline{\tau}_{\log}(A_s, z_0) < +\infty.$$

Then, every meromorphic solution $f(z)(\not\equiv 0)$ *in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of (2) satisfies* $1 < \mu_{\log}(A_s, z_0) \leq \mu_{\log}(f, z_0)$.

Remark 3. We can also replace the conditions $\max\left\{\sigma_{\log}(A_j,z_0) : j \neq s\right\} \leq \mu_{\log}(A_s,z_0) < +\infty$ and

$$\sum_{\sigma_{\log}(A_j,z_0)=\mu_{\log}(A_s,z_0),j\neq s} \tau_{\log}(A_j,z_0) < \delta\underline{\tau}_{\log}(A_s,z_0) < +\infty$$

in Theorem 7 by $\limsup_{r\longrightarrow 0} \frac{\sum_{j\neq s} m_{z_0}(r,A_j)}{m_{z_0}(r,A_s)} < 1$ or we replace the condition $\liminf_{r\longrightarrow 0} \frac{m_{z_0}(r,A_s)}{T_{z_0}(r,A_s)} = \delta > 0$ by $\lambda_{\log}(\frac{1}{A_s},z_0) + 1 < \mu_{\log}(A_s,z_0)$, which clearly includes the assumption that $\mu_{\log}(A_s,z_0) > 1$.

Remark 4. The results in Theorem 6 and Theorem 7 may be understood as an extension respectively of Theorem 6 and Theorem 2 in [12] when an arbitrary coefficient A_s dominating the others coefficients by its lower logarithmic order and lower logarithmic type.

Some lemmas

We need the following lemmas to prove our results. First, we define the logarithmic measure of a set $E \subset (0,1)$ by $m_l(E) = \int_E \frac{dr}{r}$.

Lemma 1 ([11, 22]). *Let $f(z)$ be a non-constant analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$ and let $V_{z_0,}(r,f)$ be a central index of $f(z)$ near the singular point z_0. Then*

$$\sigma_{[p,q]}(f,z_0) = \limsup_{r\longrightarrow 0} \frac{\log_p^+ V_{z_0}(r,f)}{\log_q \frac{1}{r}}, \qquad \mu_{[p,q]}(f,z_0) = \liminf_{r\longrightarrow 0} \frac{\log_p^+ V_{z_0}(r,f)}{\log_q \frac{1}{r}}.$$

Lemma 2 ([18]). *Let f be a non-constant meromorphic function in $\overline{\mathbb{C}}\backslash\{z_0\}$. Then there exists a set E_1 of $(0,1)$ that has finite logarithmic measure such that for all $j = 0,...,k$, we have*

$$\frac{f^{(j)}(z_r)}{f(z_r)} = \left(\frac{V_{z_0}(r,f)}{z_0 - z_r}\right)^j (1 + o(1)),$$

as $r \to 0$, $r \notin E_1$, where z_r is a point in the circle $|z - z_0| = r$ that satisfies $|f(z_r)| = \max_{|z-z_0|=r} |f(z)|$.

Lemma 3 ([11, 22]). *Let f be a non-constant analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$ with $\mu_{[p,q]}(f,z_0) \leq \sigma_{[p,q]}(f,z_0) < \infty$. Then*

(i) there exists a set $E_2 \subset (0,1)$ that has infinite logarithmic measure such that for all $|z - z_0| = r \in E_2$, we have

$$\sigma_{[p,q]}(f,z_0) = \lim_{r\longrightarrow 0} \frac{\log_p T_{z_0}(r,f)}{\log_q \frac{1}{r}} = \lim_{r\longrightarrow 0} \frac{\log_{p+1} M_{z_0}(r,f)}{\log_q \frac{1}{r}}.$$

(ii) there exists a set $E_3 \subset (0,1)$ that has infinite logarithmic measure such that for all $|z - z_0| = r \in E_3$, we have

$$\mu_{[p,q]}(f,z_0) = \lim_{r\longrightarrow 0} \frac{\log_p T_{z_0}(r,f)}{\log_q \frac{1}{r}} = \lim_{r\longrightarrow 0} \frac{\log_{p+1} M_{z_0}(r,f)}{\log_q \frac{1}{r}}.$$

By using similar proofs as for Lemma 3, we can prove the following lemma.

Lemma 4. *Let f be a non-constant analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$ with $\mu = \mu_{\log}(f, z_0) \leq \sigma_{\log}(f, z_0) = \sigma$. Then*

(i) there exists a set E_4 of $(0,1)$ that has infinite logarithmic measure such that for all $|z - z_0| = r \in E_4$, we have

$$\lim_{r\longrightarrow 0} \frac{\log\log M_{z_0}(r,f)}{\log\log\frac{1}{r}} = \lim_{r\longrightarrow 0} \frac{\log T_{z_0}(r,f)}{\log\log\frac{1}{r}} = \sigma.$$

(ii) there exists a set E_5 of $(0,1)$ that has infinite logarithmic measure such that for all $|z - z_0| = r \in E_5$, we have

$$\lim_{r\longrightarrow 0} \frac{\log\log M_{z_0}(r,f)}{\log\log\frac{1}{r}} = \lim_{r\longrightarrow 0} \frac{\log T_{z_0}(r,f)}{\log\log\frac{1}{r}} = \mu.$$

Lemma 5. *Let $A_0(z), ..., A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$ of finite logarithmic order. If there exists an integer $s(0 \leq s \leq k-1)$ such that $1 \leq \max\left\{\mu_{\log}(A_s, z_0), \sigma_{\log}(A_j, z_0) : j \neq s\right\} \leq \alpha$, then every analytic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}\backslash\{z_0\}$ of (2) satisfies $\mu_{[2,2]}(f, z_0) \leq \alpha$.*

Proof. Suppose that $f(z)(\not\equiv 0)$ is an analytic solution of (2) in $\overline{\mathbb{C}}\backslash\{z_0\}$. By (2), we have

$$\begin{aligned}\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq & \left|A_{k-1}(z)\right|\left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \cdots + \left|A_s(z)\right|\left|\frac{f^{(s)}(z)}{f(z)}\right| \\ & + \cdots + \left|A_1(z)\right|\left|\frac{f'(z)}{f(z)}\right| + \left|A_0(z)\right|.\end{aligned} \tag{3}$$

By Lemma 2, there exists a set $E_1 \subset (0,1)$ that has finite logarithmic measure such that, for all $r \notin E_1$ and $r \to 0$, we get

$$\frac{f^{(j)}(z_r)}{f(z_r)} = \left(\frac{V_{z_0}(r,f)}{z_0 - z_r}\right)^j (1 + o(1)), \quad j = 0, ..., k. \tag{4}$$

Since $\max\left\{\sigma_{\log}(A_j, z_0) : j = 0, ..., k-1, j \neq s\right\} \leq \alpha < +\infty$, then for any given $\varepsilon > 0$, there exists $r_1 \in (0,1)$ such that for all $|z - z_0| = r \in (0, r_1)$, we have

$$\left|A_j(z)\right| \leq M_{z_0}(r, A_j) \leq \exp\left\{\left(\log\frac{1}{r}\right)^{\sigma_{\log}(A_j,z_0)+\varepsilon}\right\} \leq \exp\left\{\left(\log\frac{1}{r}\right)^{\alpha+\varepsilon}\right\}, \quad j \neq s. \tag{5}$$

By Lemma 4, there exists a set $E_5 \subset (0,1)$ that has infinite logarithmic measure such that, for any given $\varepsilon > 0$ and for all $r \in E_5$ we get

$$\left|A_s(z)\right| \leq M_{z_0}(r, A_s) \leq \exp\left\{\left(\log\frac{1}{r}\right)^{\mu_{\log}(A_s,z_0)+\varepsilon}\right\} \leq \exp\left\{\left(\log\frac{1}{r}\right)^{\alpha+\varepsilon}\right\}. \tag{6}$$

Substituting (4) -(6) into (3), for any given $\varepsilon > 0$ and for all $r \in E_5 \cap (0, r_1) \setminus E_1$, we obtain

$$V_{z_0}(r,f) \leq kr\exp\left\{\left(\log\frac{1}{r}\right)^{\alpha+\varepsilon}\right\}\Big|1 + o(1)\Big|. \tag{7}$$

It follows by (7) and Lemma 1 that, $\mu_{[2,2]}(f, z_0) \leq \alpha$. $\square$

Lemma 6 ([13]). *Let $A_0(z),...,A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$ of finite logarithmic order with* $\max\{\sigma_{\log}(A_j,z_0) : j=0,...,k-1\} \leq \beta < +\infty$. *Then, every analytic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}\backslash\{z_0\}$ of (2) satisfies $\sigma_{[2,2]}(f,z_0) \leq \beta$.*

Lemma 7 ([16]). *Let f be a non-constant meromorphic function in $\overline{\mathbb{C}}\backslash\{z_0\}$, let $\lambda > 0$, $\varepsilon > 0$ be given real constants and $j \in \mathbb{N}$. Then*

(i) there exist a set $E_6 \subset (0,1)$ of finite logarithmic measure and a constant $C > 0$ that depends only on λ and j such that for all $|z-z_0| = r \in (0,1)\backslash E_6$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq C\left[\frac{1}{r^2}T_{z_0}(\lambda r,f)\log T_{z_0}(\lambda r,f)\right]^j. \tag{8}$$

(ii) there exist a set $E_7 \subset [0,2\pi)$ that has a linear measure zero and a constant $C > 0$ that depends on λ and j such that for all $\theta \in [0,2\pi)\backslash E_7$, there exists a constant $r_0 = r_0(\theta) > 0$ such that (8) holds for all z satisfying $\arg(z-z_0) \in [0,2\pi)\backslash E_7$ and $r = |z-z_0| < r_0$.

Lemma 8. *Let $A_0(z),...,A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$ of finite logarithmic order with* $\max\{\sigma_{\log}(A_j,z_0) : j \neq 0\} < \sigma_{\log}(A_0,z_0) = \sigma < +\infty$. *Then, every analytic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}\backslash\{z_0\}$ of (2) satisfies $\sigma_{\log}(A_0,z_0) - 1 \leq \sigma_{[2,2]}(f,z_0) \leq \sigma_{\log}(A_0,z_0)$ and $\sigma_{[2,2]}(f,z_0) = \sigma_{\log}(A_0,z_0)$ if $\sigma_{\log}(A_0,z_0) > 1$.*

Proof. By (2), we have

$$|A_0(z)| \leq \left|\frac{f^{(k)}(z)}{f(z)}\right| + |A_{k-1}(z)|\left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \cdots + |A_1(z)|\left|\frac{f'(z)}{f(z)}\right|. \tag{9}$$

By Lemma 7, there exist a set $E_6 \subset (0,1)$ having finite logarithmic measure and a constant $C > 0$ that depends only on λ, such for all $r \notin E_6$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq C\left[\frac{1}{r}T_{z_0}(\lambda r,f)\right]^{2j}, \qquad j=1,...,k. \tag{10}$$

Setting $\sigma_0 = \max\{\sigma_{\log}(A_j,z_0) : j \neq 0\} < \sigma_{\log}(A_0,z_0) = \sigma < +\infty$. Then for any given $\varepsilon, 0 < 2\varepsilon < \sigma - \sigma_0$, there exists a $r_3 > 0$ such that for all $|z-z_0| = r \in (0,r_3)$, we get

$$|A_j(z)| \leq \exp\left\{\left(\log\frac{1}{r}\right)^{\sigma_0+\varepsilon}\right\}, \qquad j=1,...,k-1. \tag{11}$$

By Lemma 4, there exists a set $E_4 \subset (0,1)$ of infinite logarithmic measure such that, for all $r \in E_4$ and $|A_0(z)| = M_{z_0}(r,A_0)$, we have

$$|A_0(z)| > \exp\left\{\left(\log\frac{1}{r}\right)^{\sigma-\varepsilon}\right\}. \tag{12}$$

Substituting (10), (11) and (12) into (9), for all $r \in E_4 \cap (0,r_3) \setminus E_6$, we obtain

$$\exp\left\{\left(\log\frac{1}{r}\right)^{\sigma-\varepsilon}\right\} \leq kC\left[\frac{1}{r}T_{z_0}(\lambda r,f)\right]^{2k}\exp\left\{\left(\log\frac{1}{r}\right)^{\sigma_0+\varepsilon}\right\}. \tag{13}$$

From (13) and $0 < 2\varepsilon < \sigma - \sigma_0$, we get

$$\sigma_{[2,2]}(f,z_0) \geq \sigma - 1 - \varepsilon \quad if \quad \sigma \geq 1 \quad and \quad \sigma_{[2,2]}(f,z_0) \geq \sigma - \varepsilon \quad if \quad \sigma > 1. \quad (14)$$

Since $0 < 2\varepsilon < \sigma - \sigma_0$, then $\sigma_{[2,2]}(f,z_0) \geq \sigma - 1 \geq 0$ and if $\sigma > 1$, then $\sigma_{[2,2]}(f,z_0) \geq \sigma$. Further, by (14) and Lemma 6, we have $0 \leq \sigma - 1 \leq \sigma_{[2,2]}(f,z_0) \leq \sigma_{\log}(A_0,z_0)$ and $1 < \sigma \leq \sigma_{[2,2]}(f,z_0) \leq \sigma_{\log}(A_0,z_0) = \sigma$. Thus, we obtain $\sigma_{\log}(A_0,z_0) - 1 \leq \sigma_{[2,2]}(f,z_0) \leq \sigma_{\log}(A_0,z_0)$ and $1 < \sigma_{\log}(A_0,z_0) = \sigma_{[2,2]}(f,z_0)$. □

Lemma 9 ([7])**.** *Let f be a non-constant meromorphic function in $\overline{\mathbb{C}} \backslash \{z_0\}$ and let $k \in \mathbb{N}$. Then*

$$m_{z_0}\big(r, \frac{f^{(k)}(z)}{f(z)}\big) = O\big(\log T_{z_0}(r,f) + \log \frac{1}{r}\big),$$

for all $r \in (0,1) \backslash E_8$ with $m_l(E_8) < \infty$. If f is of finite order, then

$$m_{z_0}\big(r, \frac{f^{(k)}(z)}{f(z)}\big) = O\big(\log \frac{1}{r}\big), \qquad r \in (0,1).$$

Lemma 10 ([22])**.** *Let f be a non-constant meromorphic function in $\overline{\mathbb{C}} \backslash \{z_0\}$ and let k and j be two integers such that $k \neq j$. Then*

$$m_{z_0}\big(r, \frac{f^{(k)}(z)}{f^{(j)}(z)}\big) \leq O\big(T_{z_0}(r,f) + \log \frac{1}{r}\big),$$

for all $r \in (0, r_4] \backslash E_9$ with $m_l(E_9) < \infty$.

Lemma 11 ([27])**.** *Assume $f \not\equiv 0$ is a solution of (2), set $g = f - \varphi$. Then g satisfies*

$$g^{(k)} + A_{k-1} g^{(k-1)} + \cdots + A_1 g' + A_0 g = -\Big[\varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \cdots + A_1 \varphi' + A_0 \varphi\Big]. \quad (15)$$

Lemma 12 ([16])**.** *Let f be a non-constant meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$ and set $g(\omega) = f(z_0 - \frac{1}{\omega})$. Then $g(\omega)$ is meromorphic in $\mathbb{C}$ and we have*

$$T(R,g) = T_{z_0}(\frac{1}{R}, f).$$

Lemma 13 ([2])**.** *Let f be a meromorphic function in $\mathbb{C}$ with $p \geq q \geq 1$. Then*

$$\sigma_{[p,q]}(f') = \sigma_{[p,q]}(f).$$

Lemma 14. *Let f be a non-constant meromorphic function in $\overline{\mathbb{C}} \backslash \{z_0\}$ with $p \geq q \geq 1$. Then*

$$\sigma_{[p,q]}(f^{(n)}, z_0) = \sigma_{[p,q]}(f, z_0), \qquad n \in \mathbb{N}.$$

Proof. It is sufficient to prove that $\sigma_{[p,q]}(f', z_0) = \sigma_{[p,q]}(f, z_0)$. By Lemma 12, $g(\omega) = f(z_0 - \frac{1}{\omega})$ is meromorphic in $\mathbb{C}$ and $\sigma_{[p,q]}(g) = \sigma_{[p,q]}(f, z_0)$. By Lemma 13 we have $\sigma_{[p,q]}(g') = \sigma_{[p,q]}(g)$ where $f'(z) = \frac{1}{\omega^2} g'(\omega)$. Set $h(\omega) = \frac{1}{\omega^2} g'(\omega)$. Clearly $\sigma_{[p,q]}(h) = \sigma_{[p,q]}(g')$. In the other hand by Lemma 12, we have $\sigma_{[p,q]}(h) = \sigma_{[p,q]}(f', z_0)$. So, we deduce that $\sigma_{[p,q]}(f, z_0) = \sigma_{[p,q]}(f', z_0)$. □

Lemma 15 ([22]). *Let f be a non-constant meromorphic function in $\overline{\mathbb{C}}-\{z_0\}$. Then*

$$T_{z_0}\big(r, \frac{1}{f}\big) = T_{z_0}(r, f) + O(1).$$

Lemma 16 ([13]). *Let $F(z) \not\equiv 0$, $A_0(z), ..., A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$ and let f be a non-constant analytic solution in $\overline{\mathbb{C}}\backslash\{z_0\}$ of the equation*

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = F(z), \tag{16}$$

such that

$$\max\big\{\sigma_{[2,2]}(F, z_0), \sigma_{[2,2]}(A_j, z_0) : (j = 0, ..., k-1)\big\} < \sigma_{[2,2]}(f, z_0).$$

Then $\overline{\lambda}_{[2,2]}(f, z_0) = \lambda_{[2,2]}(f, z_0) = \sigma_{[2,2]}(f, z_0)$.

Proof of the theorems

Proof of Theorem 1.

Proof. Suppose that $f(z)(\not\equiv 0)$ is an analytic solution of (2) in $\overline{\mathbb{C}}\backslash\{z_0\}$. By (2), we get

$$-A_0(z) = \frac{f^{(k)}(z)}{f(z)} + A_{k-1}(z)\frac{f^{(k-1)}(z)}{f(z)} + \cdots + A_1(z)\frac{f'(z)}{f(z)}. \tag{17}$$

By (17) and Lemma 9, there exists a set $E_8 \subset (0, 1)$, having finite logarithmic measure, such that for all $r \in (0, 1) \setminus E_8$, we have

$$\begin{aligned} m_{z_0}(r, A_0(z)) &\leq \sum_{j=1}^{k} m_{z_0}\big(r, \frac{f^{(j)}(z)}{f(z)}\big) + \sum_{j=1}^{k-1} m_{z_0}(r, A_j(z)) + O(1) \\ &\leq O\big(\log T_{z_0}(r, f(z)) + \log\frac{1}{r}\big) + \sum_{j=1}^{k-1} m_{z_0}(r, A_j(z)), \end{aligned} \tag{18}$$

then we obtain

$$T_{z_0}(r, A_0(z)) = m_{z_0}(r, A_0(z)) \leq O\big(\log T_{z_0}(r, f(z)) + \log\frac{1}{r}\big) + \sum_{j=1}^{k-1} T_{z_0}(r, A_j(z)). \tag{19}$$

First, we assume $\sigma = \max\big\{\sigma_{\log}(A_j, z_0) : j = 1, ..., k-1\big\} < \mu_{\log}(A_0, z_0) = \mu$. Then for any given $0 < 2\varepsilon < \mu - \sigma$, there exists $r_1 \in (0, 1)$ such that for all $|z - z_0| = r \in (0, r_1)$, we get

$$T_{z_0}(r, A_0) \geq \left(\log\frac{1}{r}\right)^{\mu_{\log}(A_0,z_0)-\varepsilon} \tag{20}$$

and

$$T_{z_0}(r, A_j) \leq \left(\log\frac{1}{r}\right)^{\sigma+\varepsilon}, \quad j = 1, ..., k-1. \tag{21}$$

Substituting (20) and (21) into (19), for the above ε and for all $|z-z_0| = r \in (0, r_1) \setminus E_8$, we obtain

$$\left(\log \frac{1}{r}\right)^{\mu-\varepsilon} \leq O\big(\log T_{z_0}(r,f) + \log \frac{1}{r}\big) + (k-1)\left(\log \frac{1}{r}\right)^{\sigma+\varepsilon}. \tag{22}$$

Then, by $0 < 2\varepsilon < \mu - \sigma$, we have

$$\big(1 - o(1)\big)\left(\log \frac{1}{r}\right)^{\mu-\varepsilon} \leq O\big(\log T_{z_0}(r,f) + \log \frac{1}{r}\big). \tag{23}$$

This implies that, $\mu - \varepsilon \leq \mu_{[2,2]}(f, z_0)$ because $\mu_{\log}(A_0, z_0) > 1$. Since $\varepsilon > 0$ is arbitrary, we obtain $1 < \mu_{\log}(A_0, z_0) \leq \mu_{[2,2]}(f, z_0)$. On the other hand, by Lemma 5, we have $\mu_{[2,2]}(f, z_0) \leq \mu_{\log}(A_0, z_0)$. Thus, we get that every analytic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}\backslash\{z_0\}$ of (2) satisfies $1 < \mu_{\log}(A_0, z_0) = \mu_{[2,2]}(f, z_0)$. Now we prove that $\overline{\lambda}_{[2,2]}(f-\varphi, z_0) = \lambda_{[2,2]}(f-\varphi, z_0) = \sigma_{[2,2]}(f, z_0) = \sigma_{\log}(A_0, z_0) > 1$. Set $g = f - \varphi$. By Lemma 8 and since $\varphi(z)(\not\equiv 0)$ satisfies $\sigma_{[2,2]}(\varphi, z_0) < \mu_{\log}(A_0, z_0) \leq \sigma_{\log}(A_0, z_0)$, then we have $\sigma_{[2,2]}(g, z_0) = \sigma_{[2,2]}(f, z_0) = \sigma_{\log}(A_0, z_0) \geq \mu_{\log}(A_0, z_0) > 1$. By Lemma 11 g satisfies (15), Set $G = \varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \cdots + A_1\varphi' + A_0\varphi$. If $G \equiv 0$, then by the first part of the proof (or by Lemma 8), we get $\sigma_{[2,2]}(\varphi, z_0) \geq \mu_{[2,2]}(\varphi, z_0) = \mu_{\log}(A_0, z_0)$, which is a contradiction, thus $G \not\equiv 0$. Since $G \not\equiv 0$, then by Lemma 14, we have $\sigma_{[2,2]}(G, z_0) \leq \sigma_{[2,2]}(\varphi, z_0) < \mu_{\log}(A_0, z_0) \leq \sigma_{\log}(A_0, z_0) = \sigma_{[2,2]}(g, z_0)$. By Lemma 16, we obtain $\overline{\lambda}_{[2,2]}(g, z_0) = \lambda_{[2,2]}(g, z_0) = \sigma_{[2,2]}(g, z_0)$. Then, we deduce that $\overline{\lambda}_{[2,2]}(f-\varphi, z_0) = \lambda_{[2,2]}(f-\varphi, z_0) = \sigma_{[2,2]}(f, z_0) = \sigma_{\log}(A_0, z_0)$.

Now we assume that $\max\big\{\sigma_{\log}(A_j, z_0) : j \neq 0\big\} = \mu_{\log}(A_0, z_0) = \mu$ and

$$\tau_1 = \sum_{\sigma_{\log}(A_j,z_0)=\mu_{\log}(A_0,z_0),\, j\neq 0} \tau_{\log}(A_j, z_0) < \underline{\tau}_{\log}(A_0, z_0) = \underline{\tau}.$$

Then, there exists a set $J \subseteq \{1, ..., k\}$, such that for $j \in J$, we get $\sigma_{\log}(A_j, z_0) = \mu_{\log}(A_0, z_0) = \mu$ with $\tau_1 = \sum_{j\in J} \tau_{\log}(A_j, z_0) < \underline{\tau}_{\log}(A_0, z_0) = \underline{\tau}$, where for $j \in \{1, ..., k\} \setminus J$, we have $\sigma_{\log}(A_j, z_0) < \mu_{\log}(A_0, z_0) = \mu$. Hence, for any given ε $\big(0 < \varepsilon < \frac{\underline{\tau}-\tau_1}{k}\big)$, there exists a constant $r_2 \in (0, 1)$, such that for all $|z-z_0| = r \in (0, r_2)$, we have

$$T_{z_0}(r, A_j) \leq \big(\tau_{\log}(A_j, z_0) + \varepsilon\big)\left(\log \frac{1}{r}\right)^{\mu_{\log}(A_0,z_0)}, \quad j \in J, \tag{24}$$

$$T_{z_0}(r, A_j) \leq \left(\log \frac{1}{r}\right)^{\sigma_0}, \quad j \in \{1, ..., k\} \setminus J, \quad 0 < \sigma_0 < \mu \tag{25}$$

and

$$T_{z_0}(r, A_0) \geq \big(\underline{\tau} - \varepsilon\big)\left(\log \frac{1}{r}\right)^{\mu_{\log}(A_0,z_0)}. \tag{26}$$

By substituting (24)-(26) into (19), for the above ε and for all $|z-z_0| = r \in (0, r_2) \setminus E_8$, we get

$$\begin{aligned}\left(\underline{\tau}-\varepsilon\right)\left(\log\frac{1}{r}\right)^{\mu} &\leq O\left(\log T_{z_0}(r,f)+\log\frac{1}{r}\right)+\sum_{j\in J}\left(\tau_{\log}(A_j,z_0)+\varepsilon\right)\left(\log\frac{1}{r}\right)^{\mu}\\ &\quad+\sum_{j\in\{1,\ldots,k\}\setminus J}\left(\log\frac{1}{r}\right)^{\sigma_0}\\ &\leq O\left(\log T_{z_0}(r,f)+\log\frac{1}{r}\right)+\left(\tau_1+(k-1)\varepsilon\right)\left(\log\frac{1}{r}\right)^{\mu}\\ &\quad+(k-1)\left(\log\frac{1}{r}\right)^{\sigma_0}\end{aligned} \tag{27}$$

and so

$$\left(1-o(1)\right)\left(\underline{\tau}-\tau_1-k\varepsilon\right)\left(\log\frac{1}{r}\right)^{\mu} \leq O\left(\log T_{z_0}(r,f)+\log\frac{1}{r}\right). \tag{28}$$

By (28), it follows that

$$1 < \mu_{\log}(A_0,z_0) \leq \mu_{[2,2]}(f,z_0). \tag{29}$$

From (29) and Lemma 5, we conclude that every analytic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}\setminus\{z_0\}$ of (2) satisfies $1 < \mu_{\log}(A_0,z_0) = \mu_{[2,2]}(f,z_0)$. We prove that $\overline{\lambda}_{[2,2]}(f-\varphi,z_0) = \lambda_{[2,2]}(f-\varphi,z_0) = \sigma_{[2,2]}(f,z_0) = \sigma_{\log}(A_0,z_0) > 1$, similarly as in the proof for the first case. $\square$

Proof of Theorem 2.

Proof. We assume that $\limsup_{r\longrightarrow 0}\frac{\sum_{j=1}^{k-1} m_{z_0}(r,A_j)}{m_{z_0}(r,A_0)} < \beta < 1$. Then for $r \to 0$, we have

$$\sum_{j=1}^{k-1} m_{z_0}(r,A_j) < \beta m_{z_0}(r,A_0). \tag{30}$$

Substituting (30) into (18), for all $r \in (0,1) \setminus E_8$, we obtain

$$(1-\beta)T_{z_0}(r,A_0) = (1-\beta)m_{z_0}(r,A_0) \leq O\left(\log T_{z_0}(r,f)+\log\frac{1}{r}\right). \tag{31}$$

By the definition of $\mu_{\log}(A_0,z_0) = \mu$, for any given $\varepsilon > 0$ there exists $r_3 \in (0,1)$ such that for all $|z-z_0| = r \in (0,r_3)$, (20) holds. Then by substituting (20) into (31), for any given $\varepsilon > 0$ and for all $r \in (0,r_3) \setminus E_8$, we get

$$\left(\log\frac{1}{r}\right)^{\mu-\varepsilon} \leq O\left(\log T_{z_0}(r,f)+\log\frac{1}{r}\right), \tag{32}$$

which implies that, $\mu-\varepsilon \leq \mu_{[2,2]}(f,z_0)$ because $\mu_{\log}(A_0,z_0) > 1$. Since $\varepsilon > 0$ is arbitrary, we obtain

$$1 < \mu_{\log}(A_0,z_0) \leq \mu_{[2,2]}(f,z_0). \tag{33}$$

It follows by (33) and Lemma 5 that $1 < \mu_{\log}(A_0, z_0) = \mu_{[2,2]}(f, z_0)$. Similarly as in the proof of Theorem 1 we prove that $\overline{\lambda}_{[2,2]}(f - \varphi, z_0) = \lambda_{[2,2]}(f - \varphi, z_0) = \sigma_{[2,2]}(f, z_0) = \sigma_{\log}(A_0, z_0) > 1$. □

Proof of Theorem 3.

Proof. Suppose that $f(z)(\not\equiv 0)$ is a meromorphic solution of (2) in $\overline{\mathbb{C}}\backslash\{z_0\}$. As in the proof of Theorem 1, first, if $\sigma = \max\big\{\sigma_{\log}(A_j, z_0) : j = 1, ..., k-1\big\} < \mu_{\log}(A_0, z_0) = \mu$, then for any given $0 < 2\varepsilon < \mu - \sigma$, there exists $r_4 \in (0, 1)$, such that for all $|z - z_0| = r \in (0, r_4)$, (21) holds. By the condition $\liminf_{r\longrightarrow 0} \frac{m_{z_0}(r,A_0)}{T_{z_0}(r,A_0)} = \delta > 0$ and the definition of $\mu_{\log}(A_0, z_0) = \mu$, for the above ε, there exists $r_5 \in (0, 1)$ such that for all $|z - z_0| = r \in (0, r_5)$, we have

$$m_{z_0}(r, A_0) \geq \frac{\delta}{2} T_{z_0}(r, A_0) \geq \frac{\delta}{2}\Big(\log\frac{1}{r}\Big)^{\mu-\frac{\varepsilon}{2}} \geq \Big(\log\frac{1}{r}\Big)^{\mu-\varepsilon}. \tag{34}$$

By substituting (21) and (34) into (19), for any given $0 < 2\varepsilon < \mu - \sigma$, and for all $|z - z_0| = r \in (0, r_4) \cap (0, r_5) \setminus E_8$, we obtain

$$\Big(\log\frac{1}{r}\Big)^{\mu-\varepsilon} \leq O\Big(\log T_{z_0}(r, f) + \log\frac{1}{r}\Big) + (k-1)\Big(\log\frac{1}{r}\Big)^{\sigma+\varepsilon}, \tag{35}$$

that is

$$\big(1 - o(1)\big)\Big(\log\frac{1}{r}\Big)^{\mu-\varepsilon} \leq O\Big(\log T_{z_0}(r, f) + \log\frac{1}{r}\Big). \tag{36}$$

It follows that, $1 < \mu - \varepsilon \leq \mu_{[2,2]}(f, z_0)$. Since $\varepsilon > 0$ is arbitrary, we get $1 < \mu_{\log}(A_0, z_0) \leq \mu_{[2,2]}(f, z_0)$. Next, if $\max\big\{\sigma_{\log}(A_j, z_0) : j \neq 0\big\} = \mu_{\log}(A_0, z_0) = \mu$ and $\tau_1 = \sum_{\sigma_{\log}(A_j,z_0)=\mu_{\log}(A_0,z_0), j\neq 0} \tau_{\log}(A_j, z_0) < \delta\underline{\tau}_{\log}(A_0, z_0) = \delta\underline{\tau}$, then by the condition $\liminf_{r\longrightarrow 0} \frac{m_{z_0}(r,A_0)}{T_{z_0}(r,A_0)} = \delta > 0$ with $\delta \leq 1$ and the definition of $\underline{\tau}_{\log}(A_0, z_0) = \underline{\tau}$, for any given $\varepsilon > 0$, there exists a constant $r_6 \in (0, 1)$ such that for all $|z - z_0| = r \in (0, r_6)$, we have

$$\begin{aligned} m_{z_0}(r, A_0) \geq (\delta - \varepsilon) T_{z_0}(r, A_0) &\geq (\delta - \varepsilon)\big(\underline{\tau} - \varepsilon\big)\Big(\log\frac{1}{r}\Big)^{\mu} \\ &= \big(\delta\underline{\tau} - (\underline{\tau} + \delta)\varepsilon + \varepsilon^2\big)\Big(\log\frac{1}{r}\Big)^{\mu} \\ &\geq \big(\delta\underline{\tau} - (\underline{\tau} + 1)\varepsilon\big)\Big(\log\frac{1}{r}\Big)^{\mu}. \end{aligned} \tag{37}$$

For any given ε $\big(0 < (\underline{\tau} + k)\varepsilon < \delta\underline{\tau} - \tau_1\big)$, there exists $r_7 \in (0, 1)$, such that for all $|z - z_0| = r \in (0, r_7)$, (24) and (25) hold. Then, by substituting (24), (25) and (37) into (19), for the above ε and for all $|z - z_0| = r \in (0, r_6) \cap (0, r_7) \setminus E_8$, we get

$$\begin{aligned} &\big(\delta\underline{\tau}-(\underline{\tau}+1)\varepsilon\big)\bigg(\log\frac{1}{r}\bigg)^{\mu}\\ &\leq O\big(\log T_{z_0}(r,f)+\log\frac{1}{r}\big)+\sum_{j\in J}\big(\tau_{\log}(A_j,z_0)+\varepsilon\big)\bigg(\log\frac{1}{r}\bigg)^{\mu}\\ &\quad+\sum_{j\in\{1,\dots,k\}\setminus J}\bigg(\log\frac{1}{r}\bigg)^{\sigma_0} \qquad (38)\\ &\leq O\big(\log T_{z_0}(r,f)+\log\frac{1}{r}\big)+\big(\tau_1+(k-1)\varepsilon\big)\bigg(\log\frac{1}{r}\bigg)^{\mu}\\ &\quad+(k-1)\bigg(\log\frac{1}{r}\bigg)^{\sigma_0}. \end{aligned}$$

So

$$\big(1-o(1)\big)\big(\delta\underline{\tau}-\tau_1-(\underline{\tau}+k)\varepsilon\big)\bigg(\log\frac{1}{r}\bigg)^{\mu}\leq O\big(\log T_{z_0}(r,f)+\log\frac{1}{r}\big). \qquad (39)$$

By (39), we obtain $1<\mu_{\log}(A_0,z_0)\leq\mu_{[2,2]}(f,z_0)$. □

Proof of Theorem 4.

Proof. Let $f(z)(\not\equiv 0)$ be a meromorphic solution of (2) in $\overline{\mathbb{C}}\backslash\{z_0\}$. For any given $\varepsilon>0$, there exists $r_8\in(0,1)$, such that for all $|z-z_0|=r\in(0,r_8)$, (30) and (34) hold. Then, by substituting (34) into (31), for any given $\varepsilon>0$ and for all $|z-z_0|=r\in(0,r_8)\setminus E_8$, we have

$$\bigg(\log\frac{1}{r}\bigg)^{\mu-\varepsilon}\leq O\big(\log T_{z_0}(r,f)+\log\frac{1}{r}\big). \qquad (40)$$

It follows that $1<\mu_{\log}(A_0,z_0)\leq\mu_{[2,2]}(f,z_0)$. □

Proof of Theorem 5.

Proof. Let $f(z)(\not\equiv 0)$ be a meromorphic meromorphic solution of (2) in $\overline{\mathbb{C}}\backslash\{z_0\}$. By (18), for all $r\in(0,1)\setminus E_8$, we obtain

$$\begin{aligned} T_{z_0}(r,A_0(z))&=m_{z_0}(r,A_0(z))+N_{z_0}(r,A_0(z))\\ &\leq O\big(\log T_{z_0}(r,f)+\log\frac{1}{r}\big)+\sum_{j=1}^{k-1}m_{z_0}(r,A_j)+N_{z_0}(r,A_0(z)). \end{aligned} \qquad (41)$$

Also as in the proof of Theorem 1, first, if $\sigma=\max\big\{\sigma_{\log}(A_j,z_0):j=1,...,k-1\big\}<\mu_{\log}(A_0,z_0)=\mu$, then for any given ε $\big(0<2\varepsilon<\mu-\sigma\big)$, there exists $r_9\in(0,1)$, such that for all $|z-z_0|=r\in(0,r_9)$, (20) and (21) hold. By the definition of $\lambda_{\log}(\frac{1}{A_0},z_0)=\lambda$, for any given ε $\big(0<2\varepsilon<\mu-\lambda-1\big)$, there exists $r_{10}\in(0,1)$, such that for all $|z-z_0|=r\in(0,r_{10})$, we have

$$N_{z_0}(r,A_0)\leq\bigg(\log\frac{1}{r}\bigg)^{\lambda_{\log}(\frac{1}{A_0},z_0)+1+\varepsilon}. \qquad (42)$$

By substituting (20), (21) and (42) into (41), for sufficiently small ε satisfying $0 < 2\varepsilon < \min\{\mu - \sigma, \mu - \lambda - 1\}$ and for all $|z - z_0| = r \in (0, r_9) \cap (0, r_{10}) \setminus E_8$, we get

$$\left(\log \frac{1}{r}\right)^{\mu-\varepsilon} \leq O\big(\log T_{z_0}(r, f) + \log \frac{1}{r}\big) + (k-1)\left(\log \frac{1}{r}\right)^{\sigma+\varepsilon} + \left(\log \frac{1}{r}\right)^{\lambda+1+\varepsilon}. \quad (43)$$

It follows that

$$\big(1 - o(1)\big)\left(\log \frac{1}{r}\right)^{\mu-\varepsilon} \leq O\big(\log T_{z_0}(r, f)\big), \quad (44)$$

that is $1 < \mu_{\log}(A_0, z_0) \leq \mu_{[2,2]}(f, z_0)$. Now, if $\max\{\sigma_{\log}(A_j, z_0) : j \neq 0\} = \mu_{\log}(A_0, z_0) = \mu$ and $\tau_1 = \sum_{\sigma_{\log}(A_j,z_0)=\mu_{\log}(A_0,z_0), j\neq 0} \tau_{\log}(A_j, z_0) < \underline{\tau}_{\log}(A_0, z_0) = \underline{\tau}$, then for any given ε $\left(0 < \varepsilon < \frac{\underline{\tau}-\tau_1}{k}\right)$, there exists a constant $r_{11} \in (0, 1)$, such that for all $|z - z_0| = r \in (0, r_{11})$, (24), (25) and (26) hold. By substituting (24)-(26) and (42) into (41), for sufficiently small ε satisfying $0 < \varepsilon < \min\left\{\frac{\underline{\tau}-\tau_1}{k}, \frac{\mu-\lambda-1}{2}\right\}$ and for all $|z - z_0| = r \in (0, r_{10}) \cup (0, r_{11}) \setminus E_8$, we obtain

$$\begin{aligned}
\big(\underline{\tau} - \varepsilon\big)\left(\log \frac{1}{r}\right)^{\mu} &\leq O\big(\log T_{z_0}(r, f) + \log \frac{1}{r}\big) + \sum_{j\in J} \big(\tau_{\log}(A_j, z_0) + \varepsilon\big)\left(\log \frac{1}{r}\right)^{\mu} \\
&\quad + \sum_{j\in\{1,\dots,k\}\setminus J} \left(\log \frac{1}{r}\right)^{\sigma_0} + \left(\log \frac{1}{r}\right)^{\lambda+1+\varepsilon} \\
&\leq O\big(\log T_{z_0}(r, f) + \log \frac{1}{r}\big) + \big(\tau_1 + (k-1)\varepsilon\big)\left(\log \frac{1}{r}\right)^{\mu} \\
&\quad + (k-1)\left(\log \frac{1}{r}\right)^{\sigma_0} + \left(\log \frac{1}{r}\right)^{\lambda+1+\varepsilon}.
\end{aligned} \quad (45)$$

So

$$\big(1 - o(1)\big)\big(\underline{\tau} - \tau_1 - k\varepsilon\big)\left(\log \frac{1}{r}\right)^{\mu} \leq O\big(\log T_{z_0}(r, f)\big). \quad (46)$$

From (46), we deduce that $1 < \mu_{\log}(A_0, z_0) \leq \mu_{[2,2]}(f, z_0)$. □

Proof of Theorem 6.

Proof. Let $f(z)(\not\equiv 0)$ be an analytic solution in $\overline{\mathbb{C}}\setminus\{z_0\}$ of (2). We suppose that $\mu_{\log}(f, z_0) < \infty$. By (2), we have

$$\begin{aligned}
-A_s(z) =& \frac{f^{(k)}(z)}{f^{(s)}(z)} + A_{k-1}(z)\frac{f^{(k-1)}(z)}{f^{(s)}(z)} + \cdots + A_{s+1}(z)\frac{f^{(s+1)}(z)}{f^{(s)}(z)} \\
&+ A_{s-1}(z)\frac{f^{(s-1)}(z)}{f^{(s)}(z)} + \cdots + A_0(z)\frac{f(z)}{f^{(s)}(z)}.
\end{aligned} \quad (47)$$

By (47) and Lemma 10, there exist a constant $r_{12} \in (0,1)$, and a set $E_9 \subset (0, r_{12}]$, having finite logarithmic measure, such that for all $r \in (0, r_{12}] \setminus E_9$, we get

$$\begin{aligned} m_{z_0}(r, A_s(z)) &\leq \sum_{j=0, j\neq s}^{k} m_{z_0}\big(r, \frac{f^{(j)}(z)}{f^{(s)}(z)}\big) + \sum_{j=0, j\neq s}^{k-1} m_{z_0}(r, A_j(z)) + O(1) \\ &\leq O\big(T_{z_0}(r, f(z)) + \log\frac{1}{r}\big) + \sum_{j=0, j\neq s}^{k-1} m_{z_0}(r, A_j(z)). \end{aligned} \tag{48}$$

Then

$$T_{z_0}(r, A_s(z)) = m_{z_0}(r, A_s(z)) \leq O\big(T_{z_0}(r, f(z)) + \log\frac{1}{r}\big) + \sum_{j=0, j\neq s}^{k-1} T_{z_0}(r, A_j(z)). \tag{49}$$

First, we suppose that $\sigma = \max\big\{\sigma_{\log}(A_j, z_0) : j = 0, ..., k-1, j \neq s\big\} < \mu_{\log}(A_s, z_0) = \mu$. Then for any given $0 < 2\varepsilon < \mu - \sigma$, there exists $r_{13} \in (0,1)$ such that for all $|z - z_0| = r \in (0, r_{13})$, we have

$$T_{z_0}(r, A_s) \geq \bigg(\log\frac{1}{r}\bigg)^{\mu_{\log}(A_s, z_0) - \varepsilon} \tag{50}$$

and

$$T_{z_0}(r, A_j) \leq \bigg(\log\frac{1}{r}\bigg)^{\sigma+\varepsilon}, \quad j = 0, ..., s-1, s+1, ..., k-1. \tag{51}$$

Substituting (50) and (51) into (49), for the above ε and for all $|z - z_0| = r \in (0, r_{12}] \cap (0, r_{13}) \setminus E_9$, we obtain

$$\bigg(\log\frac{1}{r}\bigg)^{\mu-\varepsilon} \leq O\big(T_{z_0}(r, f) + \log\frac{1}{r}\big) + (k-1)\bigg(\log\frac{1}{r}\bigg)^{\sigma+\varepsilon}. \tag{52}$$

Thus,

$$\big(1 - o(1)\big)\bigg(\log\frac{1}{r}\bigg)^{\mu-\varepsilon} \leq O\big(T_{z_0}(r, f) + \log\frac{1}{r}\big), \tag{53}$$

which implies that, $1 < \mu - \varepsilon \leq \mu_{\log}(f, z_0)$. Since $\varepsilon > 0$ is arbitrary, we get $1 < \mu_{\log}(A_s, z_0) \leq \mu_{\log}(f, z_0)$. On the other hand, by Lemma 5, we have $\mu_{[2,2]}(f, z_0) \leq \mu_{\log}(A_s, z_0)$. Hence, $\mu_{[2,2]}(f, z_0) \leq \mu_{\log}(A_s, z_0) \leq \mu_{\log}(f, z_0)$. Now we suppose that $\max\big\{\sigma_{\log}(A_j, z_0) : j = 0, ..., k-1, j \neq s\big\} = \mu_{\log}(A_s, z_0) = \mu$ and $\tau_1 = \sum_{\sigma_{\log}(A_j, z_0) = \mu_{\log}(A_s, z_0), j\neq s} \tau_{\log}(A_j, z_0) < \underline{\tau}_{\log}(A_s, z_0) = \underline{\tau}$. Then, there exists a set $J \subseteq \{0, 1, ..., k\} \setminus \{s\}$, such that for $j \in J$, we have $\sigma_{\log}(A_j, z_0) = \mu_{\log}(A_s, z_0) = \mu$ with $\tau_1 = \sum_{j\in J} \tau_{\log}(A_j, z_0) < \underline{\tau}_{\log}(A_s, z_0) = \underline{\tau}$ and for $j \in \{0, 1, ..., s-1, s+1, ..., k\} \setminus J$, we get $\sigma_{\log}(A_j, z_0) < \mu_{\log}(A_s, z_0) = \mu$. Hence, for any given ε $\big(0 < \varepsilon < \frac{\underline{\tau} - \tau_1}{k}\big)$, there exists $r_{14} \in (0,1)$, such that for all $|z - z_0| = r \in (0, r_{14})$, we have

$$T_{z_0}(r, A_j) \leq \big(\tau_{\log}(A_j, z_0) + \varepsilon\big)\bigg(\log\frac{1}{r}\bigg)^{\mu_{\log}(A_s, z_0)}, \quad j \in J, \tag{54}$$

$$T_{z_0}(r, A_j) \leq \bigg(\log\frac{1}{r}\bigg)^{\sigma_0}, \quad j \in \{0, 1, ..., s-1, s+1, ..., k\} \setminus J, \quad 0 < \sigma_0 < \mu \tag{55}$$

and

$$T_{z_0}(r,A_s) \geq \left(\underline{\tau}-\varepsilon\right)\left(\log\frac{1}{r}\right)^{\mu_{\log}(A_s,z_0)}. \tag{56}$$

By substituting (54)-(56) into (49), for the above ε and for all $|z-z_0|=r \in (0,r_{12}] \cap (0,r_{14}) \setminus E_9$, we get

$$\begin{aligned}\left(\underline{\tau}-\varepsilon\right)\left(\log\frac{1}{r}\right)^{\mu} &\leq O\big(T_{z_0}(r,f)+\log\frac{1}{r}\big)+\sum_{j\in J}\left(\tau_{\log}(A_j,z_0)+\varepsilon\right)\left(\log\frac{1}{r}\right)^{\mu} \\ &\quad + \sum_{j\in\{0,1,\dots,s-1,s+1,\dots,k\}\setminus J}\left(\log\frac{1}{r}\right)^{\sigma_0} \\ &\leq O\big(T_{z_0}(r,f)+\log\frac{1}{r}\big)+\left(\tau_1+(k-1)\varepsilon\right)\left(\log\frac{1}{r}\right)^{\mu} \\ &\quad +(k-1)\left(\log\frac{1}{r}\right)^{\sigma_0}\end{aligned} \tag{57}$$

and so

$$\left(1-o(1)\right)\left(\underline{\tau}-\tau_1-k\varepsilon\right)\left(\log\frac{1}{r}\right)^{\mu} \leq O\big(T_{z_0}(r,f)+\log\frac{1}{r}\big). \tag{58}$$

It follows that

$$1<\mu_{\log}(A_s,z_0) \leq \mu_{\log}(f,z_0). \tag{59}$$

Then, from (59) and Lemma 5, we deduce that every analytic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}\backslash\{z_0\}$ of (2) satisfies $\mu_{[2,2]}(f,z_0) \leq \mu_{\log}(A_s,z_0) \leq \mu_{\log}(f,z_0)$. □

Proof of Theorem 7.

Proof. Here we use a similar discussion as in the proof of Theorem 6. Let $f(z)(\not\equiv 0)$ be a meromorphic solution in $\overline{\mathbb{C}}\backslash\{z_0\}$ of (2) with $\mu_{\log}(f,z_0)<\infty$, otherwise, the result is trivial. First, if $\sigma=\max\left\{\sigma_{\log}(A_j,z_0): j=0,\dots,k-1, j\neq s\right\}<\mu_{\log}(A_s,z_0)=\mu$, then for any given $0<2\varepsilon<\mu-\sigma$, there exists $r_{15}\in(0,1)$ such that for all $|z-z_0|=r\in(0,r_{15})$, (51) holds. By $\liminf_{r\longrightarrow 0}\frac{m_{z_0}(r,A_s)}{T_{z_0}(r,A_s)}=\delta>0$ and the definition of $\mu_{\log}(A_s,z_0)=\mu$, for the above ε, there exists $r_{16}\in(0,1)$ such that for all $|z-z_0|=r\in(0,r_{16})$, we have

$$m_{z_0}(r,A_s) \geq \frac{\delta}{2}T_{z_0}(r,A_s) \geq \frac{\delta}{2}\left(\log\frac{1}{r}\right)^{\mu_{\log}(A_s,z_0)-\frac{\varepsilon}{2}} \geq \left(\log\frac{1}{r}\right)^{\mu_{\log}(A_s,z_0)-\varepsilon}. \tag{60}$$

Substituting (51) and (60) into (49), for the above ε and for all $|z-z_0|=r \in (0,r_{12}]\cap(0,r_{15})\cap(0,r_{16})\setminus E_9$, we get

$$\left(\log\frac{1}{r}\right)^{\mu-\varepsilon} \leq O\big(T_{z_0}(r,f)+\log\frac{1}{r}\big)+(k-1)\left(\log\frac{1}{r}\right)^{\sigma+\varepsilon}. \tag{61}$$

Thus

$$\left(1-o(1)\right)\left(\log\frac{1}{r}\right)^{\mu-\varepsilon} \leq O\big(T_{z_0}(r,f)+\log\frac{1}{r}\big). \tag{62}$$

It follows that, $1 < \mu - \varepsilon \leq \mu_{\log}(f, z_0)$. Since $\varepsilon > 0$ is arbitrary, we obtain $1 < \mu_{\log}(A_s, z_0) \leq \mu_{\log}(f, z_0)$. Now if $\max\{\sigma_{\log}(A_j, z_0) : j = 0, ..., k-1, j \neq s\} = \mu_{\log}(A_s, z_0) = \mu$ and

$$\tau_1 = \sum_{\sigma_{\log}(A_j,z_0)=\mu_{\log}(A_s,z_0),j\neq s} \tau_{\log}(A_j, z_0) < \delta\underline{\tau}_{\log}(A_s, z_0) = \delta\underline{\tau},$$

then by $\liminf_{r\longrightarrow 0} \frac{m_{z_0}(r,A_s)}{T_{z_0}(r,A_s)} = \delta > 0$ with $\delta \leq 1$ and the definition of $\underline{\tau}_{\log}(A_s, z_0) = \underline{\tau}$, for any given $\varepsilon > 0$, there exists $r_{17} \in (0, 1)$ such that for all $|z - z_0| = r \in (0, r_{17})$, we have

$$\begin{aligned} m_{z_0}(r, A_s) \geq (\delta - \varepsilon)T_{z_0}(r, A_s) &\geq (\delta - \varepsilon)(\underline{\tau} - \varepsilon)\left(\log\frac{1}{r}\right)^{\mu_{\log}(A_s,z_0)} \\ &\geq \left(\delta\underline{\tau} - (\underline{\tau} + 1)\varepsilon\right)\left(\log\frac{1}{r}\right)^{\mu_{\log}(A_s,z_0)}. \end{aligned} \tag{63}$$

For any given ε $\left(0 < (\underline{\tau} + k)\varepsilon < \delta\underline{\tau} - \tau_1\right)$, there exists $r_{18} \in (0, 1)$, such that for all $|z - z_0| = r \in (0, r_{18})$, (54) and (55) hold. By substituting (54), (55) and (63) into (49), for the above ε and for all $|z - z_0| = r \in (0, r_{12}] \cap (0, r_{17}) \cap (0, r_{18}) \setminus E_9$, we obtain

$$\begin{aligned} &\left(\delta\underline{\tau} - (\underline{\tau} + 1)\varepsilon\right)\left(\log\frac{1}{r}\right)^{\mu} \\ &\leq O\left(T_{z_0}(r, f) + \log\frac{1}{r}\right) + \sum_{j\in J}\left(\tau_{\log}(A_j, z_0) + \varepsilon\right)\left(\log\frac{1}{r}\right)^{\mu} \\ &\quad + \sum_{j\in\{0,1,...,s-1,s+1,...,k\}\setminus J}\left(\log\frac{1}{r}\right)^{\sigma_0} \\ &\leq O\left(T_{z_0}(r, f) + \log\frac{1}{r}\right) + \left(\tau_1 + (k-1)\varepsilon\right)\left(\log\frac{1}{r}\right)^{\mu} + (k-1)\left(\log\frac{1}{r}\right)^{\sigma_0}, \end{aligned} \tag{64}$$

that is

$$\left(1 - o(1)\right)\left(\delta\underline{\tau} - \tau_1 - (\underline{\tau} + k)\varepsilon\right)\left(\log\frac{1}{r}\right)^{\mu} \leq O\left(T_{z_0}(r, f) + \log\frac{1}{r}\right). \tag{65}$$

This implies that $1 < \mu_{\log}(A_s, z_0) \leq \mu_{\log}(f, z_0)$. □

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