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MODIFIED DOUBLE LAPLACE TRANSFORM OF PARTIAL DERIVATIVES AND ITS APPLICATIONS

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Abstract. This paper deals with modified double Laplace transforms of partial derivatives and their applications. Starting with standard results on modified double Laplace transform, we developed theorems on modified double Laplace transform of first and second-order partial derivatives. Further, we applied our results to solve homogeneous and non-homogeneous partial differential equations, D' Alembert's wave equation, and Klein-Gordon equation.

1. Introduction and preliminaries

The theory of partial differential equations plays the most important role in mathematics and other fields of science and engineering [25]. The integral transform is one of the most important method to solve partial differential equations [3, 4, 5, 7, 8, 10, 11, 13, 14, 15]. The function of two or more variables and its double Laplace transform plays an important role in solving many problems like wave equations, Laplace equations, heat equations, telegraph equation, and integrodifferential equations [1, 2, 8, 12, 16, 17, 18, 19].

In this paper, we have discussed the properties of the modified double Laplace transform along with its partial derivatives. We also applied our results to solve the wave equation, Klein-Gordon equation, and Cauchy's differential equation.

2. Notations, Definitions and Basic Results

Definition 2.1. (Modified Laplace Transform) The Modified Laplace transform of a function $u(x)$ which is piece-wise continuous and exponential order [1]

$$L_{1,a}[u(x)] = \overline{u}(p) = \int_0^\infty a^{-px} u(x)\, dx,\ (\mathcal{R}(p) > 0,\ a \,\epsilon\, (0,\infty) \tag{2.1}$$

and $L_{1,a}[u(y)] = \overline{u}(q)$, provided that the integral exists [22]. and the corresponding inverse transform is

$$L_{1,a}^{-1}[u(x)] = \frac{1}{2\pi i}\int_{m-i\infty}^{m+i\infty} a^{px}\ u(p,a)\ dx,\ (m \geq 0).$$

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Definition 2.2. (Double Laplace Transform) [7], the Double Laplace transform of a function $u(x,y)$ is given by

$$L_2[u(x,y)] = L[L[u(x;y);x\to p];y\to q] = \int_0^\infty\int_0^\infty e^{-(px+qy)}u(x,y)\,dx\,dy. \tag{2.2}$$

and the corresponding inverse transform is $L_2^{-1}[\overline{\overline{u}}(p,q)] = u(x,y)$ is defined by

$$u(x,y) = L_2^{-1}[\overline{\overline{u}}(p,q)] = \frac{1}{2\pi i}\int_{m-i\infty}^{m+i\infty} e^{px}dp\,\frac{1}{2\pi i}\int_{n-i\infty}^{n+i\infty} e^{qy}\,\overline{\overline{u}}(p,q)\,dq,\ (m,n\geq 0).$$

Definition 2.3. (Modified Double Laplace Transform) [6], the Modified Double Laplace transform of a function $u(x,y)$ is defined by

$$\overline{\overline{u}}(p,q) = L_{2,a}[u(x,y)] = L_a[L_a[u(x,y);\ x\to p];\ y\to q] = L_{2,a}[\overline{u}(p,y);\ y\to q]$$

and

$$\overline{\overline{u}}(p,q) = \int_0^\infty\int_0^\infty a^{-(px+qy)}u(x,y)\,dx\,dy,\quad (a>0). \tag{2.3}$$

The modified double Laplace transform of $u(x,y)$ exists for all p and q, where Re $(p)>m$ and Re $(q)>n$ and $u(x,y)$ is a piece-wise continuous and of exponential order defined in finite intervals $(X,0)$ and $(0,Y)$.
The corresponding inverse transform is

$$L_{2,a}^{-1}[\overline{\overline{u}}(p,q)] = \frac{1}{2\pi i}\int_{m-i\infty}^{m+i\infty} a^{px}\,dp\,\frac{1}{2\pi i}\int_{n-i\infty}^{n+i\infty} a^{qy}\,\overline{\overline{u}}(p,q)\,dq,\quad (m,n\geq 0,\ a>0).$$

Definition 2.4. (Heaviside unit step function) The $H(x,y)$ is Heaviside unit step function given by

$$\begin{aligned} H(x-h,y-k) &= 1,\ if\ x\geqslant h,\ y\geqslant k,\\ &= 0,\ if\ x<h,\ y<k. \end{aligned} \tag{2.4}$$

We use the following notations

(i) $\overline{\overline{u}}(p,q) = L_{2,a}[u(x,y)]$ (ii) $L_{1,a}[u_x(x,0)] = p\ L_{1,a}[u(x,0)] - u(0,0)$
(iii) $u_1(q) = L_{1,a}[u(0,y)]$ (iv) $u_2(p) = L_{1,a}[u(x,0)]$
(v) $u_3(q) = L_{1,a}[u_x(0,y)]$ (vi) $u_4(p) = L_{1,a}[u_y(x,0)]$.

3. Theorems on Modified Double Laplace Transforms

In this section, we recall some of the results and properties of modified double Laplace transforms (MDLT) [6].

Theorem 3.1 (Convolution Theorem)**.** If $L_{2,a}[u(x,y)] = \overline{\overline{u}}(p,q)$ and $L_{2,a}[v(x,y)] = \overline{\overline{v}}(p,q)$, then

$$L_{2,a}[(u**v)(x,y)] = L_{2,a}[u(x,y)]\ L_{2,a}[v(x,y)] = \overline{\overline{u}}(p,q)\ \overline{\overline{v}}(p,q). \tag{3.1}$$

Theorem 3.2.

$$L_{2,a}[u(x)] = \frac{1}{q\ log\ a}[\overline{u}(p)] \tag{3.2}$$

and

$$L_{2,a}[u(y)] = \frac{1}{p\ log\ a}[\overline{u}(q)]. \tag{3.3}$$

Theorem 3.3.

$$L_{2,a}[u(x+y)] = \frac{1}{(p-q)\ log\ a}[\overline{u}(q) - \overline{u}(p)]. \tag{3.4}$$

Theorem 3.4. *(Modified Double Laplace Transform of Derivative) If $u(x,y)$ piece-wise continuous on every finite interval $(X,0)$ and $(0,Y)$, it satisfies the condition of exponential order and u_x exists, then its modified double Laplace transform*

$$L_{2,a}\{u_x\} = p\ log\ a\ \overline{\overline{u}}(p,q) - L_{2,a}\{u(0,y)\}.$$

4. Modified Double Laplace Transforms of Partial Derivatives

Here we developed some results on the modified double Laplace transform of partial derivatives which are included in this section.We use the notation $H(x,y)$ defined by definition (2.4) in this section. We assume $u(x,y)$ as a piece-wise continuous function and exponential order.

Theorem 4.1.

$$L_{2,a}[u_x] = p\ log\ a\ \overline{\overline{u}}(p,q) - L_{1,a}[u(0,y)]. \tag{4.1}$$

Proof. By definition 2.3, we have

$$\begin{aligned} L_{2,a}[u_x] &= \int_0^\infty \int_0^\infty a^{-(px+qy)}\ u_x(x,y)\ dx\ dy \\ &= \int_0^\infty \Big[a^{-(px+qy)}\ u(x,y)\Big]_0^\infty dy - (-p\ log\ a) \int_0^\infty \int_0^\infty a^{-(px+qy)}\ u(x,y)\ dx\ dy \\ &= p\ log\ a\ \overline{\overline{u}}(p,q) - L_{1,a}[u(0,y)]. \end{aligned}$$

□

Theorem 4.2.

$$L_{2,a}[u_y] = q\ log\ a\ \overline{\overline{u}}(p,q) - L_{1,a}[u(x,0)]. \tag{4.2}$$

Proof. By definition 2.3, we have

$$\begin{aligned} L_{2,a}[u_y] &= \int_0^\infty \int_0^\infty a^{-(px+qy)}\ u_y(x,y)\ dx\ dy \\ &= \int_0^\infty \Big[a^{-(px+qy)}\ u(x,y)\Big]_0^\infty dx - (-q\ log\ a) \int_0^\infty \int_0^\infty a^{-(px+qy)}\ u(x,y)\ dx\ dy \\ &= q\ log\ a\ \overline{\overline{u}}(p,q) - L_{1,a}[u(x,0)]. \end{aligned}$$

□

Theorem 4.3.

$$L_{2,a}[u_{xx}] = (p\ log\ a)^2\ \overline{\overline{u}}(p,q) - (p\ log\ a)\ L_{1,a}[u(0,y)] - L_{1,a}[u_x(0,y)]. \tag{4.3}$$

Proof. By definition 2.3, we have

$$\begin{aligned}
L_{2,a}[u_{xx}] &= \int_0^\infty \int_0^\infty a^{-(px+qy)}\ u_{xx}(x,y)\ dx\ dy \\
&= \int_0^\infty \left[a^{-(px+qy)}\ u_x(x,y)\right]_0^\infty dy - (-p\ log\ a) \int_0^\infty \int_0^\infty a^{-(px+qy)}\ u_x(x,y)\ dx\ dy \\
&= p\ log\ a\ [p\ log\ a\ \overline{\overline{u}}(p,q)] - (p\ log\ a)\ L_{1,a}[u(0,y)] - L_{1,a}[u_x(0,y)] \\
&= (p\ log\ a)^2\ \overline{\overline{u}}(p,q) - (p\ log\ a)\ L_{1,a}[u(0,y)] - L_{1,a}[u_x(0,y)].
\end{aligned}$$

□

Theorem 4.4.

$$L_{2,a}[u_{yy}] = (q\ log\ a)^2\ \overline{\overline{u}}(p,q) - (q\ log\ a)\ L_{1,a}[u(x,0)] - L_{1,a}[u_y(x,0)]. \tag{4.4}$$

Proof. By definition 2.3, we have

$$\begin{aligned}
L_{2,a}[u_{yy}] &= \int_0^\infty \int_0^\infty a^{-(px+qy)}\ u_{yy}(x,y)\ dx\ dy \\
&= \int_0^\infty \left[a^{-(px+qy)}\ u_y(x,y)\right]_0^\infty dx - (-q\ log\ a) \int_0^\infty \int_0^\infty a^{-(px+qy)}\ u_y(x,y)\ dx\ dy \\
&= q\ log\ a\ [q\ log\ a\ \overline{\overline{u}}(p,q)] - q\ log\ a\ L_{1,a}[u(x,0)] - L_{1,a}[u_y(x,0)] \\
&= (q\ log\ a)^2\ \overline{\overline{u}}(p,q) - (q\ log\ a)\ L_{1,a}[u(x,0)] - L_{1,a}[u_y(x,0)].
\end{aligned}$$

□

Theorem 4.5.

$$L_{2,a}[u_{xy}] = pq\ (log\ a)^2\ \overline{\overline{u}}(p,q) - (q\ log\ a)\ L_{1,a}[u(0,y)] - (p\ log\ a)\ L_{1,a}[u(x,0)] + u(0,0), \tag{4.5}$$

Proof. By definition 2.3, we have

$$\begin{aligned}
L_{2,a}[u_{xy}] &= \int_0^\infty \int_0^\infty a^{-(px+qy)}\ u_{xy}(x,y)\ dx\ dy \\
&= \int_0^\infty \left[a^{-(px+qy)}\ u_x(x,y)\right]_0^\infty dx - (-q\ log\ a) \int_0^\infty \int_0^\infty a^{-(px+qy)}\ u_x(x,y)\ dx\ dy \\
&= q\ log\ a\ \left[\int_0^\infty \int_0^\infty a^{-(px+qy)}\ u_x(x,y)\ dx\ dy\right] - L_{1,a}[u_x(x,0)]. \qquad (4.6)
\end{aligned}$$

From Theorem (3.3), we have

$$L_{2,a}[u_{xy}] = pq\ (log\ a)^2\ \overline{\overline{u}}(p,q) - (q\ log\ a)\ L_{1,a}[u(0,y)] - (p\ log\ a)\ L_{1,a}[u(x,0)] + u(0,0). \tag{4.7}$$

□

We use these properties of modified double Laplace transforms for further correspondence in this work.

Remark 4.6. Results of [7], equations (44) to (48) are special cases of our Theorems 4.1 to 4.5 with $a = e$.

5. Applications of MDLT to Solve Partial Differential Equations

In this section, we have used the modified double Laplace transform (MDLT) to solve linear partial differential equations. Partial differential equations play an important role in medical imaging [20], electrical signaling of nerves, proper distribution of oxygen to the healing tissues, pharmaceutics, and more [22]. The heat conduction equation is used to determine the temperature distributions induced by heat conduction in solids [11, 21]. It is also used in solid and fluid mechanics, hydrodynamic problems, mechanical vibrations [15], diffusion processes [8], and acoustic behavior of materials [10]. Partial differential equations are used in optimization problems to find the optimal solution for a given system used in the simulation modeling, and optimization of complex systems in engineering [21]. It is used in modeling the variation of a physical quantity, such as pressure, temperature, velocity, displacement, strain, stress, voltage, current, or concentration of a pollutant, with the change of time or location, or both [7, 24].

5.1. **Cauchy's problem for first order partial differential equation.** It is used to describe many physical phenomena, such as heat, sound, electrostatics, and fluid flow, and to explain the propagation of waves and heat conduction. Consider Cauchy's first-order partial differential equation [7]

$$\alpha u_x + \beta u_y = 0, \tag{5.1}$$

with given conditions
$u(x,0) = z(x),\ x > 0,\ u(0,y) = 0,\ y > 0$, and α, β are constants.
Apply Modified Double Laplace transform, which gives

$$\alpha\ L_{2,a}(u_x) + \beta\ L_{2,a}(u_y) = 0.$$

Using Theorems 4.1 and 4.2

$$\alpha\ [p\ log\ a\ \overline{\overline{u}}(p,q) - L_{2,a}[u(0,y)]] + \beta\ [q\ log\ a\ \overline{\overline{u}}(p,q) - L_{2,a}[u(x,0)]] = 0.$$

$$(\alpha p + \beta q)\ log\ a\ \overline{\overline{u}}(p,q) = \beta\ \overline{z}(p)$$

$$\overline{\overline{u}}(p,q) = \frac{\overline{z}(p)}{(q + \frac{\alpha}{\beta}\ p)\ log\ a}.$$

Taking inverse transform with respect to q gives,

$$\overline{u}(p,y) = \frac{\overline{z}(p)}{log\ a}\ a^{-\frac{\alpha p}{\beta} y}.$$

Apply inverse transform with respect to p gives,

$$u(x,y) = L_{2,a}^{-1}\left[\frac{\overline{z}(p)}{log\ a}\ a^{-\frac{\alpha p}{\beta} y}\right] = z(x) * \omega\left(x - \frac{\alpha y}{\beta}\right).$$

By Convolution Theorem 3.3, we have

$$u(x,y) = \int_0^x z(x-\zeta)\ \omega\left(\zeta - \frac{\alpha y}{\beta}\right) d\zeta = z\left(x - \frac{\alpha y}{\beta}\right). \tag{5.2}$$

5.2. **D' Alembert's Wave Equation.** Such wave equations are used in solving problems involving the string hypothesis, gravitational waves, light waves, and sound waves [15], it help us in deciding the development of strings and liquid surfaces, like water waves. Also, with mechanical progression in computer sciences and preliminary methods, there are huge purposes of wave equations as guyed links for the end goal of hiding, vibrations in transmission lines which risers toward the ocean, guyed ranges, etc [14].
We consider D' Alembert's Wave Equation [7].

$$k^2 \, \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}, \; x \geq 0, \; t > 0. \tag{5.3}$$

with given conditions

$$y(x,0) = f(x), \; y_t(x,0) = g(x), \; x > 0,$$

$$y(0,t) = 0, \; y_x(0,t) = 0.$$

Apply Modified Double Laplace transform in equation (5.3), which gives

$$k^2 \; L_{2,a}[y_{xx}(x,t)] = L_{2,a}[y_{tt}(x,t)].$$

Using Theorems 4.3 and 4.4

$$k^2 \; [(p \; log \; a)^2 \; \overline{\overline{y}}(p,q) - (p \; log \; a) \; L_{1,a}\{y(0,t)\} - L_{1,a}\{y_x(0,t)\}]$$

$$\begin{aligned} &= [(q \; log \; a)^2 \; \overline{\overline{y}}(p,q) - (q \; log \; a) \; L_{1,a}\{y(x,0)\} \\ &- L_{1,a}\{y_t(x,0)\}](k^2 \; p^2 - q^2) \; (log \; a) \; \overline{\overline{y}}(p,q) \\ &= -q \; [(log \; a) \; \overline{f}(p) + \overline{g}(p)] \end{aligned}$$

$$\overline{\overline{y}}(p,q) = \frac{q \; log \; a \; \overline{f}(p) + \overline{g}(p)}{(q^2 - k^2 \; p^2) \; log \; a}. \tag{5.4}$$

Inverse Modified Double Laplace transform, gives

$$\begin{aligned} y(x,t) &= \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} a^{px} \left[\overline{f}(p) \; cosh(kpt) + \frac{\overline{g}(p)}{kp} \; sinh(kpt) \right] dp \\ &= \frac{1}{2}[f(x+kt) + f(x-kt)] + \frac{1}{2k} \left[L_{1,a}^{-1} \left\{ \frac{\overline{g}(p)}{p \; log \; a} \; a^{kpt} \right\} L_{1,a}^{-1} \left\{ \frac{\overline{g}(p)}{p \; log \; a} \; a^{-kpt} \right\} \right] \\ &= \frac{1}{2}[f(x+kt) + f(x-kt)] + \frac{1}{2k} \left[\int_0^{x+kt} g(\zeta) d\zeta - \int_0^{x-kt} g(\zeta) d\zeta \right]. \end{aligned}$$

$$y(x,t) = \frac{1}{2}[f(x+kt) + f(x-kt)] + \frac{1}{2k} \int_{x-kt}^{x+kt} g(\zeta) d\zeta, \tag{5.5}$$

where $L_{1,a}^{-1} \left[a^{pa} \; \overline{f}(p) \right] = f(x+a)$ and $L_{1,a}^{-1} \left[\frac{\overline{g}(p)}{p} \right] = \int_0^t g(\zeta) d\zeta$.

5.3. **Homogeneous Partial Differential Equations.** Homogeneous partial differential equations are utilized in electric circuits, where flow and voltage are connected by homogeneous equations [21], wave propagation, vibrations, heat movement [16].
Consider the homogeneous partial differential equation

$$u_{xx} + u_{yy} - 4u = 0.$$

with initial conditions,

$$u(x,0) = x,\ u_y = e^{-2x}$$

and boundary conditions,

$$u(0,y) = y,\ u_x(0,y) = -2y.$$

Apply MDLT on both sides, and we get

$$\begin{aligned} &p^2\ (log\ a)^2\ \ \overline{\overline{u}}(p,q)\ -\ p\ log\ a\ \overline{u}_1(q)\ -\ \overline{u}_3(q)\ +\ q^2\ (log\ a)^2\ \overline{\overline{u}}(p,q) \\ &-\ q\ log\ a\ \overline{u}_2(p)\ -\ \overline{u}_4(p)\ -4\ \overline{\overline{u}}(p,q) = 0. \end{aligned}$$

Using Theorems 4.3 and 4.4,

$$\begin{aligned} &p^2\ (log\ a)^2\ \overline{\overline{u}}(p,q)\ -\ p\ log\ a\ \overline{u}_1(q)\ -\ \overline{u}_3(q)\ +\ q^2\ (log\ a)^2\ \overline{\overline{u}}(p,q) \\ &-q\ log\ a\ \overline{u}_2(p)\ -\ \overline{u}_4(p)\ -\ 4\ \overline{\overline{u}}(p,q) = 0 \\ &[(p^2+q^2)\ (log\ a)^2 - 4]\ \overline{\overline{u}}(p,q)\ -\ p\ log\ a\left(\frac{1}{pq^2\ (log\ a)^3}\right)\ +\ \frac{2}{pq^2\ (log\ a)^3} \\ &-q\ log\ a\left(\frac{1}{p^2q\ (log\ a)^3}\right)-\left(\frac{1}{q\ log\ a\ (p\ log\ a+2)}\right) = 0 \end{aligned}$$

$$[(p^2+q^2)\ (log\ a)^2-4]\ \overline{\overline{u}}(p,q) = \frac{1}{p^2\ (log\ a)^2} + \frac{1}{q^2\ (log\ a)^2} + \frac{1}{q\ log\ a\ (p\ log\ a+2)} - \frac{2}{pq^2\ (log\ a)^3}$$

$$\overline{\overline{u}}(p,q) = \frac{1}{[(p^2+q^2)\ (log\ a)^2-4]}\left[\frac{1}{p^2\ (log\ a)^2} + \frac{1}{q^2\ (log\ a)^2} + \frac{1}{q\ log\ a\ (p\ log\ a+2)} - \frac{2}{pq^2\ (log\ a)^3}\right]$$

$$\overline{\overline{u}}(p,q) = \frac{1}{q^2\ (log\ a)^2\ (p\ log\ a+2)}.$$

Apply inverse MDLT, we get a solution

$$u(x,y) = y\ e^{-2x}.$$

5.4. **Non-Homogeneous Partial Differential Equations.** Heat and diffusion equations are used in modeling heat movement in solids and fluids and the diffusion of particles in gases and wave equations are used in modeling wave propagation in solids, fluids, and gases [16].
Consider a non-homogeneous partial differential equation of the form

$$u_{xy} = -k\ sin(ky),\ y > 0,$$

with initial condition $u(x,0) = x$, and
boundary condition $u(0,y) = 0,\ u(0,0) = 0$, where k is constant.

We have $u_{xy} = -k\ sin(ky)$,
apply MDLT on both sides, and we get

$$L_{2,a}[u_{xy}] = -k\ L_{2,a}[sin(ky)]$$

by Theorem 3.5, we have

$$pq\ (log\ a)^2\ \overline{\overline{u}}(p,q) - q\ log\ a\ \overline{u}_1 - p\ log\ a\ \overline{u}_2(x,0) + u(0,0) = -k\ \frac{[pk\ log\ a]}{[p^2\ (log\ a)^2][q^2\ (log\ a)^2 + k^2]}$$

$$pq\ (log\ a)^2\ \overline{\overline{u}}(p,q)\ -\ p\ log\ a\left(\frac{1}{p^2\ (log\ a)^2}\right) = -\frac{k^2 p\ log\ a}{[p^2\ (log\ a)^2][q^2\ (log\ a)^2 + k^2]},$$

$$pq\ (log\ a)^2\ \overline{\overline{u}}(p,q) = -\frac{k^2}{[p\ log\ a][q^2\ (log\ a)^2 + k^2]} + \frac{1}{p\ log\ a}$$

$$\overline{\overline{u}}(p,q) = -\frac{q}{[p^2\ log\ a][q^2\ (log\ a)^2 + k^2]}.$$

Apply inverse MDLT, we get a solution

$$u(x,y) = x\ cos(ky)$$

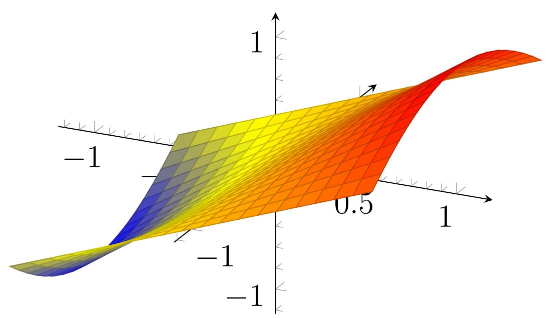

Fig. 1. Graph of $u(x,y) = x\ cos(ky)$, $\forall\ x \in [-1,1]$ and $\forall\ y \in [-1,1]$.

5.5. **Klein-Gordon equation.** We consider the Klein-Gordon equation which play an important role in the study of the condensed matter used in physics, quantum mechanics, and relativistic physics [10].
We consider following Klein-Gordon equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} - u(x,y)\ ;\ (x,y) \in \mathbb{R}^+. \tag{5.6}$$

with initial and boundary conditions

$$u(x,0) = 1 + sinx,\ \frac{\partial u}{\partial y}(x,0) = 0,\ u(0,y) = coshy,\ \frac{\partial u}{\partial y}(0,y) = 1\ ;\ x,y \in \mathbb{R}_+.$$

Apply MDLT on both sides, and we get

$$L_{2,a}\left[\frac{\partial^2 u}{\partial x^2}\right] = L_{2,a}\left[\frac{\partial^2 u}{\partial y^2}\right] - L_{2,a}[u(x,y)]$$

Using Theorems 4.3 and 4.4,

$$p^2\ (log\ a)^2\ \overline{\overline{u}}(p,q) - p\ log\ a\ \overline{u}_1(q)\ -\ \overline{u}_3(q) \tag{5.7}$$
$$= q^2\ (log\ a)^2\ \overline{\overline{u}}(p,q)\ -\ q\ log\ a\ \overline{u}_2(p)\ -\ \overline{u}_4(p)$$
$$-\ \overline{\overline{u}}(p,q)p^2\ (log\ a)^2\ \overline{\overline{u}}(p,q)\ -\ p\ log\ a\left(\frac{q}{p\ (q^2\ (log\ a)^2-1)}\right)-\left(\frac{1}{q\ log\ a}\right)$$
$$= q^2\ (log\ a)^2\ \overline{\overline{u}}(p,q) - q\ log\ a\left(\frac{1}{p\ log\ a}+\frac{1}{p^2\ (log\ a)^2+1}\right)-0-\overline{\overline{u}}(p,q)$$

$$[(p^2-q^2)\ (log\ a)^2+1]\ \overline{\overline{u}}(p,q) = \frac{q\ log\ a}{q^2\ (log\ a)^2-1}\ +\ \frac{1}{q\ log\ a}-\frac{q}{p}-\frac{q}{p^2\ (log\ a)^2+1}$$

$$\overline{\overline{u}}(p,q) = \frac{1}{[(p^2-q^2)\ (log\ a)^2+1]}\left[\frac{q\ log\ a}{q^2\ (log\ a)^2-1}+\frac{1}{q}-\frac{q}{p\ log\ a}-\frac{q}{p^2(log\ a)^2+1}\right]$$

$$\overline{\overline{u}}(p,q) = \frac{1}{q\ log\ a\ [p^2\ (log\ a)^2+1]}+\frac{q}{p\ [q^2\ (log\ a)^2-1]}.$$

Apply inverse MDLT, we get the solution

$$u(x,y) = sinx + coshy. \tag{5.8}$$

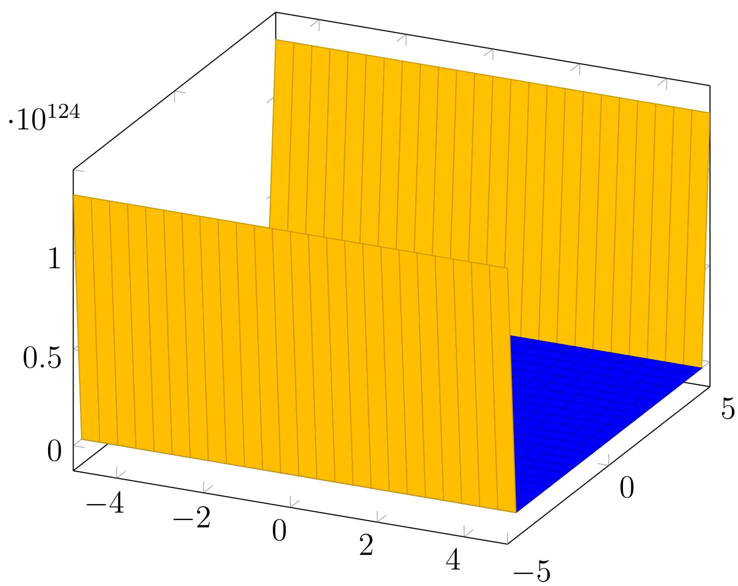

Fig. 2. Graph of $u(x,y) = sinx + coshy$, $\forall\ x\ \epsilon\ [-4,4]$ and $\forall\ y\ \epsilon\ [-5,5]$.

6. Concluding Remark

In this paper, we developed new results on the modified double Laplace transform of partial derivatives. Further, we successfully applied it to solve various partial differential equations. But, there is further scope for developing this theory and its applications.

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