RIGGED NULL HYPERSURFACES IN ALMOST PARACONTACT METRIC MANIFOLDS

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Abstract. Given an almost paracontact metric manifold \((\overline{M}, \overline{\phi}, \eta, \xi, g)\), we study lightlike hypersurfaces \(M\) of the semi-Riemannian manifold \((\overline{M}, \overline{g})\) transversal to the structure vector field \(\xi\). The latter is then a rigging for \(M\) and defines a null section \(E\) of the radical distribution of \(M\) and a screen distribution which turns out to be always semi-invariant. We show that leaves of an integrable screen distribution in such hypersurfaces \(M\) are almost paracontact metric manifolds too. When the ambient space \((\overline{M}, \overline{\phi}, \eta, \xi, g)\) is para-Sasakian, we show that the screen distribution cannot be conformal and that \(E\) is a geodesic vector field in \((M, \nabla)\), where \(\nabla\) is the connection induced on \(M\) by Levi-Civita connection of \(g\) and the local null rigging \(N = \xi - \frac{1}{2}E\). We also find necessary and sufficient conditions for the leaves of the screen distribution to be para-Sasakian too and finally we investigate integrability conditions for some additional distributions induced on \(M\) by the structure \((\overline{M}, \overline{\phi}, \eta, \xi, g)\).

1. Introduction.

Lightlike submanifolds (or null submanifolds) of semi-Riemannian manifolds were introduced by K.L. Duggal and A. Bejancu [6, 7]. Since this pioneering work, several authors have studied lightlike hypersurfaces of semi-Riemannian manifolds and particularly those of paracontact and para-Sasakian manifolds tangent to the structure vector fields (see [1, 2, 3, 4, 11, 12, 7, 13] and references therein). It is well known that the usual technique to study a lightlike hypersurface is to fix on it a geometric data formed by a lightlike section and a screen distribution. Both of them are fixed arbitrarily and independently. So, all geometric objects derived from them depend on these choices. In some recent works [9, 10], M. Gutiérrez and B. Olea used the rigging technique to fix a section of the null distribution and the screen distribution in a natural way and studied the geometry of lightlike hypersurfaces of Lorentzian manifolds. The same method has been used in other works [3, 8]. This technique consists for a given lightlike hypersurface \(\overline{M}\) in a Lorentzian manifold \(\overline{M}\), to choose a vector field \(\xi\) on \(\overline{M}\) which is not tangent to \(\overline{M}\) and construct a lightlike section and the screen distribution.

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The aim of this paper is to use the same technique to study the lightlike geometry of hypersurfaces in almost paracontact metric manifolds and in para-Sasakian manifolds.

The paper is organized as follows. The present section is labeled as Introduction. The next Section 2 is devoted to basic facts on Null hypersurfaces in semi-Riemannian manifolds and the rigging technique following Reference [9]. We also recall definitions and some properties of almost para-contact metric manifolds and of Para-Sasakian manifolds. In Section 3 we introduce rigged Null hypersurfaces in almost paracotact metric manifolds and it appears that they are screen semi-invariant. We give an explicit example of such Null hypersurfaces. Section 4 is devoted to the reduction of the almost paracontact structure. We show that leaves of an integrable screen distribution of a normalized lightlike hypersurface in an almost paracontact manifold are also almost paracontact manifolds. In Section 5, we consider the case where the ambient manifold $M$ is a para-Sasakian manifold and investigate properties of a normalized lightlike hypersurface in $M$. We show that in this case the shape operator of the hypersurface is expressed in term of the structure tensor $\phi$ and we prove that the hypersurface cannot be screen conformal. We give a necessary and sufficient condition for a given integrable screen distribution to be para-Sasakian too. Finally, in Section 6 we investigate integrability conditions for some additional distributions induced on the null hypersurface by the structure of the ambient para-Sasakian manifold.

### 2. Lightlike Hypersurfaces and Paracontact Manifolds.

We start by recalling basic notions on lightlike hypersurfaces, the rigging technique for such hypersurfaces and basic facts on para-Sasakian manifolds. We follow the presentation and some notations from references [7] [5][13] and [9].

#### 2.1. Lightlike hypersurfaces.

Given a $n$-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ and a hypersurface $M$ of $\overline{M}$, we say that $M$ is a lightlike hypersurface if the induced metric $g = i^*\overline{g}$ on the hypersurface $M$ is degenerate, where $i: M \hookrightarrow \overline{M}$ is the canonical immersion. Its means that there exists a lightlike vector field $E \in \Gamma(TM)$ such that

$$g(X, E) = 0, \ \forall X \in \Gamma(TM).$$

(2.1)

We denote by $\text{Rad}(TM)$ the radical distribution which is spanned by $E$. A screen distribution $S(TM)$ is a complementary distribution to $\text{Rad}(TM)$ in $TM$ and the transversal distribution $\text{ltr}(TM)$ is the unique lightlike one-dimensional distribution orthogonal to $S(TM)$ not contained in $TM$.

**Theorem 2.1.** [6] For any nonzero section $E$ of $\text{Rad}(TM)$ on a coordinate neighborhood $U \subset M$, there exists a unique section $N$ of $\text{ltr}(TM)$ on $U$ satisfying

$$\overline{g}(N, E) = 1, \ \overline{g}(N, N) = \overline{g}(N, W) = 0,$$

(2.2)

for all $W \in \Gamma(S(TM)/_{\overline{U}})$.

Since the complementary vector bundle $S(TM)$ of $\text{Rad}(TM)$ in $TM$ is non-degenerate, we can consider the subbundle $S(TM)^\perp$ which is a complementary
orthogonal of $S(TM)$ in $T\bar{M}$ and which is called screen transversal subbundle. One can then consider the following decompositions:

$$TM = S(TM) \perp \text{Rad}(TM)$$  \hspace{1cm} (2.3)

$$T\bar{M} = S(TM) \perp S(TM)^\perp$$  \hspace{1cm} (2.4)

$$S(TM)^\perp = \text{Rad}(TM) \oplus \text{ltr}(TM)$$  \hspace{1cm} (2.5)

$$T\bar{M} = S(TM) \perp \{\text{Rad}(TM) \oplus \text{ltr}(TM)\}$$  \hspace{1cm} (2.6)

From (2.6) we can write

$$\nabla^Y_X = \nabla^Y_X + h(X,Y)$$  \hspace{1cm} (2.7)

$$\nabla^N_X = -A_N(X) + \nabla^t_X N,$$  \hspace{1cm} (2.8)

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(\text{ltr}(TM))$. $\nabla$ and $\nabla^t$ are called the induced connection on $M$ and $\text{ltr}(TM)$ respectively, $h$ and $A_N$ are called the second fundamental form and shape operator. The above equations are the Gauss and Weingarten equations.

Locally, let $E, N$ and $U$ be as in the Theorem 2.1 above. Then for any $X, Y \in \Gamma(TM|_U)$, setting

$$B(X,Y) = \bar{g}(h(X,Y), E), \quad \tau(X) = \bar{g}(\nabla^t_X N, E),$$  \hspace{1cm} (2.9)

we can write

$$\nabla^Y_X = \nabla^Y_X + B(X,Y)N$$  \hspace{1cm} (2.10)

$$\nabla^N_X = -A_N(X) + \tau(X)N,$$  \hspace{1cm} (2.11)

$B$ is called the local second fundamental form of $M$. $B$ is a symmetric tensor that satisfies $B(X,Y) = -\bar{g}(\nabla_X E, Y)$. Moreover, $B(E,.) = 0$ and $E$ is a pregeodesic vector field, in fact $\nabla_E E = -\tau(E)E$. The notion of totally geodesic or umbilic hypersurface also has sense in the degenerate case. Indeed, $M$ is totally geodesic if $B = 0$ and totally umbilic if $B = \rho g$ for a certain $\rho \in C^\infty(M)$.

The decomposition (2.3) allows to define a canonical projection $P : \Gamma(TM) \rightarrow \Gamma(S(TM))$. For each $X \in \Gamma(TM)$, we may write

$$X = PX + \theta(X)E.$$  \hspace{1cm} (2.12)

where

$$\theta(X) = \bar{g}(X,N).$$  \hspace{1cm} (2.13)

We have

$$(\nabla^g_X)(Y,Z) = B(X,Y)\theta(Z) + B(X,Z)\theta(Y)$$  \hspace{1cm} (2.14)

which implies that the induced connection $\nabla$ is a nonmetric connection on $M$.

Given $X \in \Gamma(TM)$, the vector field $\nabla_X E$ belongs to $\Gamma(M)$, so it can be decomposed as

$$\nabla^E_X = -\tau(X)E - A^*_E(X),$$
where $A_E^*(X) \in \Gamma(S(TM))$. The endomorphism $A_E^*$ is called the shape operator of $S(TM)$ and it satisfies $B(X, PY) = g(A_E^*(X), PY)$ and
\[
B(A_E^*(X), Y) = B(X, A_E^*(Y)). \tag{2.15}
\]

The trace of $A_E^*$ is the lightlike mean curvature of $M$, explicitly given by
\[
H_p = \sum_{i=3}^{n} g(A^*(e_i), e_i) = \sum_{i=3}^{n} B(e_i, e_i),
\]
where $\{e_3, \cdots, e_n\}$ is an orthonormal basis of $S(T_p M)$. On the other hand, given $X \in \Gamma(TM)$ and $PY \in S(TM)$, locally, we decompose
\[
\nabla^*_X = \nabla^*_X PY + C(X, PY)E. \tag{2.16}
\]

The tensor $C$ holds $C(X, PY) = g(A_N(X), PY)$ and
\[
C(PX, PY) - C(PY, PX) = g(N, [PX, PY]). \tag{2.17}
\]

When the screen distribution $S(TM)$ is integrable, $\nabla^*$ is the induced Levi-Civita connection from $(M, g)$ and Equations (2.4) and (2.5) show that its second fundamental form is
\[
\Theta(PX, PY) = C(PX, PY)E + B(PX, PY)N \tag{2.18}
\]
where $PX, PY \in S(TM)$. The curvature tensor of $\nabla$ is defined as $R_{XY}Z = \nabla_X\nabla^*_Y - \nabla^*_Y\nabla_X - \nabla^*_Z$ and it satisfies
\[
R_{XY}E = \overline{R}_{XY}E, \tag{2.19}
\]
where $X, Y \in \Gamma(TM)$ and the so called Gauss-Codazzi equations are
\[
g(\overline{R}_{XY}Z, U) = g(R_{XY}Z, U) + B(X, Z)g(A_N(Y), U) - B(Y, Z)g(A_N(X), U), \tag{2.20}
\]
\[
g(\overline{R}_{XY}Z, E) = (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) - \tau(Y)B(X, Z) \tag{2.21}
\]
\[
g(\overline{R}_{XY}Z, N) = g(R_{XY}Z, N), \tag{2.22}
\]
where $X, Y, Z \in \Gamma(TM)$ and $U \in S(TM)$. From these equations it can be deduced the following ones,
\[
g(R_{XY}U, N) = (\nabla^*_X C)(Y, U) - (\nabla^*_Y C)(X, U) + \tau(Y)C(X, U) - \tau(X)C(Y, U) \tag{2.23}
\]
\[
g(R_{XY}E, N) = C(Y, A_E^*(X)) - C(X, A_E^*(Y)) - d\tau(X, Y), \tag{2.24}
\]
where $\nabla^*_C$ is defined as
\[
(\nabla^*_X C)(Y, PZ) = X(C(Y, PZ)) - C(\nabla_X Y, PZ) - C(Y, \nabla_X PZ). \tag{2.25}
\]

Using equation (2.8), we can compute the lightlike sectional curvature with respect to $E$ of a lightlike plane $\Pi = \text{span}(X, E)$ where $X \in S(TM)$ is unitary:
\[
\mathcal{K}_E(\Pi) = (\nabla_E B)(X, X) - (\nabla_X B)(E, X) + \tau(E)B(X, X). \tag{2.26}
\]
2.2. The rigging and rigged vector fields on a lightlike hypersurface.

Take $\xi$ a vector field defined in some open set in $(\overline{M}, \overline{g})$ containing $M$ and denote by $\eta$ the 1-form metrically equivalent to $\xi$. Take $\eta = i^*\overline{\eta}$, where $i : M \hookrightarrow \overline{M}$ is the canonical inclusion and consider the tensors $\hat{g} = \overline{g} + \eta \otimes \eta$ and $\tilde{g} = i^*\hat{g}$ [9]. One shows that the associated metric $\tilde{g}$ is a non-degenerate metric on the hypersurface, [8].

**Definition 2.2.** Let $M$ be a lightlike hypersurface of a semi-riemannian manifold $(\overline{M}, \overline{g})$. A rigging for $M$ is a vector field $\xi$ defined on some open set containing $M$ such that $\xi_p \not\in T_pM$ for each $p \in M$.

From now on we fix $\xi$ a rigging for $M$. The rigging $\xi$ fixes a lightlike vector field $E$ in $M$, which we call rigged vector field.

**Definition 2.3.** The rigged vector field of $\xi$ is the $\tilde{g}$-metrically equivalent vector field to the 1-form $\eta$ and it is denoted $E$.

**Proposition 2.4.** [9] The rigged vector field of $\xi$ is the unique lightlike vector field $E$ in $M$ such that $\overline{g}(\xi, E) = 1$. Moreover, $E$ is $\tilde{g}$-unitary.

We can consider the screen distribution given by $TM \cap \xi^\perp$, which we denote by $S(TM)$. Observe that $S(TM)$ is the $\tilde{g}$-orthogonal subspace to $E$ and the local lightlike transverse vector field to $S(TM)$ is given by

$$N = \xi - \frac{1}{2} \overline{g}(\xi, \xi)E.$$

2.3. Para-Sasakian Manifolds. A differentiable manifold of dimension $(2n + 1)$ is called almost paracontact manifold with the almost paracontact structure $(\phi, \xi, \eta)$ if it admits a tensor field $\phi$ of type $(1, 1)$, a vector field $\xi$ and a 1-form $\eta$ satisfying the following conditions [5, 14]:

$$\phi^2 = I - \eta \otimes \xi \quad (2.27)$$

$$\eta(\xi) = 1 \quad (2.28)$$

$$\phi(\xi) = 0 \quad (2.29)$$

$$\eta \circ \phi = 0, \quad (2.30)$$

where $I$ denotes the identity transformation. If a $(2n + 1)$-dimensional almost paracontact manifold $(\overline{M}, \overline{\phi}, \xi, \overline{\eta})$ admits a pseudo-Riemannian metric $\overline{g}$ such that

$$\overline{g}(\overline{\phi}X, \overline{\phi}Y) = -\overline{g}(X, Y) + \overline{\eta}(X)\overline{\eta}(Y), \quad X, Y \in \Gamma(T\overline{M}), \quad (2.31)$$

then we say that $\overline{M}$ is an almost paracontact metric manifold with an almost paracontact metric structure $(\overline{\phi}, \xi, \overline{\eta}, \overline{g})$ and such metric $\overline{g}$ is called compatible metric. From (2.31) it is easy to see that

$$\overline{g}(\overline{\phi}X, Y) = -\overline{g}(X, \overline{\phi}Y) \quad (2.32)$$

$$\overline{g}(X, \xi) = \overline{\eta}(X), \quad (2.33)$$

for any $X, Y \in \Gamma(T\overline{M})$.

Any compatible metric $\overline{g}$ is necessarily of signature $(n + 1, n)$.

An almost paracontact metric manifold is paracontact if $d\overline{\eta} = \omega$, where we have
set $\omega(X,Y) = \tilde{g}(X,\tilde{\phi}Y)$ and $d\tilde{g}(X,Y) = \frac{1}{2}(X.\tilde{\eta}(Y)−Y.\tilde{\eta}(X)−\tilde{\eta}([X,Y]))$, $\forall X,Y \in \Gamma(TM)$. 

A paracontact metric manifold is a para-Sasakian manifold if 

$$
(\nabla_X \phi)Y = -\tilde{g}(X,Y)\xi + \tilde{\eta}(Y)X, \forall X,Y \in \Gamma(TM).
$$

(2.34)

3. Normalized lightlike hypersurfaces in almost paracontact metric manifolds.

Let $(\bar{M}, \Phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost paracontact metric manifold and $M$ a lightlike hypersurface of $\bar{M}$ such that the structure vector field $\xi$ is a rigging vector field for $M$. It means that at any point of $M$ the rigging $\xi$ is transversal to $M$. We call such a manifold normalized lightlike hypersurface of $\bar{M}$. Let $E$ and $N$ be respectively, the rigged and null transversal vector field associated to $\xi$ such that 

$$
\bar{g}(E,\xi) = 1 \text{ and } \bar{g}(E,N) = 1.
$$

We recall that we locally have 

$$
\xi = N + \frac{1}{2}E.
$$

(3.1)

Then we easily get the following equations 

$$
\tilde{\eta}(E) = 1, \tilde{\eta}(N) = \frac{1}{2},
$$

(3.2)

$$
\tilde{\phi}^2(E) = \frac{1}{2}E - N, \tilde{\phi}^2(N) = \frac{1}{2}N - \frac{1}{4}E.
$$

(3.3)

$$
\tilde{\phi}(N) = -\frac{1}{2}\tilde{\phi}(E).
$$

(3.4)

$$
\bar{g}(\tilde{\phi} E, E) = 0, \bar{g}(\tilde{\phi} E, N) = 0
$$

(3.5)

$$
\bar{g}(\tilde{\phi} N, N) = 0, \bar{g}(\tilde{\phi} N, E) = 0.
$$

(3.6)

$$
\tilde{\phi} (N) = -\frac{1}{2}\tilde{\phi} E \in \Gamma(S(TM)).
$$

(3.7)

As recalled in [5], the normalized hypersurface $(M, g)$ is said to be screen semi-invariant if both $\tilde{\phi}(N)$ and $\tilde{\phi}(E)$ belong to the screen distribution. Hence, Equations (3.3) and (3.7) lead to the following proposition.

**Proposition 3.1.** A normalized lightlike hypersurface of almost para-contact metric manifold $\bar{M}$ is rather a screen semi-invariant lightlike hypersurface of $\bar{M}$.

For $X \in \Gamma(TM)$, we set 

$$
\tilde{\phi}(X) = \phi(X) + u(X)N
$$

(3.8)

and $\eta = i^*\bar{\eta}$ where $i : M \hookrightarrow \bar{M}$ is the canonical immersion. Then we have
**Proposition 3.2.** For all $X, Y \in \Gamma(TM)$

$$g(\phi X, \phi Y) = -g(X,Y) + \eta(X)\eta(Y) + u(X)u(Y). \quad (3.9)$$

**Proof.** Using (2.31) and (3.8), we have for all $X, Y \in \Gamma(TM)$,

$$\overline{g}(\phi X + u(X)N, \phi Y + u(Y)N) = -g(X,Y) + \eta(X)\eta(Y) \quad (3.10)$$

then

$$g(\phi X, \phi Y) = -g(X,Y) + \eta(X)\eta(Y) - u(Y)\overline{g}(\phi X, N) \quad (3.11)$$

On the other hand

$$\overline{g}(\phi X, N) = \overline{g}(\phi X - u(X)N, N) = -\frac{1}{2}u(X). \quad (3.12)$$

Using (3.12) in (3.11) gives (3.9) as expected. □

**Proposition 3.3.** Setting $U = \phi N$ and $e = \frac{1}{2} E$, we have

$$\phi^2 = I - \eta \otimes e - u \otimes U \quad (3.13)$$

$$u \circ \phi = -\eta = -\overline{g}(\cdot, N) \quad (3.14)$$

$$\eta \circ \phi = -\frac{1}{2} u \quad (3.15)$$

$$\phi(e) = -U, \ \phi(U) = -\frac{1}{2} e. \quad (3.16)$$

From the preceding proposition we deduce

$$\eta(U) = 0, \ \eta(e) = \frac{1}{2} \quad (3.17)$$

$$u(e) = 0, \ u(U) = \frac{1}{2} \quad (3.18)$$

which one can get by direct computations.

**Proposition 3.4.** Let $(M, g, N)$ be a normalized lightlike hypersurface of $(\overline{M}, \overline{g})$ and $\phi, u$ and $\eta$ defined as above, then we have

$$g(\phi X, Y) = -g(X, \phi Y) - \eta(X)u(Y) - \eta(Y)u(X), \ \forall X, Y \in \Gamma(TM). \quad (3.19)$$

From this proposition, one deduces that

$$\overline{g}(\phi X, Y) = -\overline{g}(\phi X, \phi Y), \ \forall X, Y \in \Gamma(TM). \quad (3.20)$$

**Proof.** The result follows by direct computations using Proposition 3.2. □

**Proposition 3.5.** Let $(M, g, N)$ be a normalized lightlike hypersurface in $(\overline{M}, \overline{\phi}, \xi, \gamma, \overline{g})$

Then there exists, locally a $\overline{\phi}$-basis $\{e_i, \overline{\phi} e_i, \overline{\phi} E, \overline{\phi}^2 E, \xi\}$ of $T\overline{M}$ where $\{e_i, \overline{\phi} e_i, \overline{\phi} E\}$ is pseudo-orthonormal basis of $S(TM)$.

**Proof.** On a coordinate neighborhood on $\overline{M}$, we set

$$H = (\overline{\phi}(\text{Rad}(TM)))^\perp \cap S(TM). \quad (3.21)$$

We choose a $\overline{g}$-unit vector field $e_1 \in H$. Hence by definition of $H$, we have $e_1 \perp \xi$ and $e_1 \perp \overline{\phi} E$. It follows that $\overline{\phi}(e_1) \perp \xi$ and $\overline{\phi}(e_1) \perp \overline{\phi}(E)$ and furthermore,
we have \( g(\varphi(e_1), \varphi(e_1)) = -1 \).

Next choose another vector field \( e_2 \in H \) such that \( e_2 \perp e_1 \). It follows again that \( \varphi(e_2) \perp \xi \) and we have \( e_2 \perp \xi \) and \( \varphi e_2 \perp e_1 \), \( \varphi e_2 \perp \xi \) and \( g(\varphi(e_2), \varphi(e_2)) = -1 \).

Continuing in this way, we obtain an orthonormal basis \((e_i, \varphi e_i, \varphi E)\) of \( S(TM) \).

We just add the two other orthogonal vector fields in the complementary and orthogonal space of \( S(TM) \) in \( TM \). Namely \( \varphi^2 E = \frac{1}{2} E - N \) and \( \xi \) to obtain the quasi orthonormal \( \varphi \)-basis of \( TM \) as stated. □

**Example 3.6.** Let \( (M = \mathbb{R}^5, g = dx_1 \otimes dx_1 + dx_2 \otimes dx_2 - dx_3 \otimes dx_3 - dx_4 \otimes dx_4 + dx_5 \otimes dx_5) \) be the 5-dimensional flat semi-riemannian manifold with a coordinate system \((x_1, x_2, x_3, x_4, x_5)\). This is an almost paracontact metric manifold with the structure \((\varphi, \xi, \eta)\) given by

\[
\varphi(\frac{\partial}{\partial x_5}) = 0, \quad \varphi(\frac{\partial}{\partial x_1}) = \frac{\partial}{\partial x_3}, \quad \varphi(\frac{\partial}{\partial x_3}) = \frac{\partial}{\partial x_1}, \tag{3.22}
\]

\[
\varphi(\frac{\partial}{\partial x_2}) = \frac{\partial}{\partial x_4}, \quad \varphi(\frac{\partial}{\partial x_4}) = \frac{\partial}{\partial x_2}, \quad \eta = dx_5, \tag{3.23}
\]

\[
\xi = \frac{\partial}{\partial x_5}. \tag{3.24}
\]

\( M = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5/x_5 = 2x_1 - x_3 - 2x_4\} \) is a normalized lightlike hypersurface of \( M \).

Indeed the tangent bundle \( TM \) of \( M \) is spanned by

\[
\left\{ V_1 = \frac{\partial}{\partial x_1} + 2 \frac{\partial}{\partial x_5}, V_2 = \frac{\partial}{\partial x_2}, V_3 = \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_5}, V_4 = \frac{\partial}{\partial x_4} - 2 \frac{\partial}{\partial x_5} \right\}. \]

The rigged vector field \( E \) and transversal vector field are given by

\[
E = -2 \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3} - 2 \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5}, \tag{3.25}
\]

\[
N = \frac{\partial}{\partial x_1} + \frac{1}{2} \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} + \frac{1}{2} \frac{\partial}{\partial x_5}. \tag{3.26}
\]

The screen distribution \( S(TM) = Span\{W_1, W_2, W_3\} \) where

\[
W_1 = V_2, W_2 = V_1 + 2V_3, W_3 = -2V_3 + V_4.
\]

We also have

\[
\phi E = -2 \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_1} - 2 \frac{\partial}{\partial x_2} = -V_1 - 2V_2 - 2V_3 \tag{3.27}
\]

From which we get \( \phi(N) = -\frac{1}{2} \phi(E) = W_1 + \frac{1}{2} W_2 \in S(TM) \). This means that \( M \) is screen semi-invariant as expected.
4. Reduction of the almost paracontact structure to the screen distribution.

In this section, we show that leaves of an integrable screen distribution of a normalized lightlike hypersurface in a paracontact manifold are almost paracontact manifolds.

Let $(\overrightarrow{M}, \overrightarrow{\phi}, \xi, \overrightarrow{\eta}, \overrightarrow{g})$ be an almost paracontact metric manifold and $(M, g, S(TM) = \ker \eta)$ be a normalized lightlike hypersurface with $\xi$ transversal to $M$ as above. Let $\phi$ be a $(1, 1)$–tensor field on $M$ as above i.e.

$$\overrightarrow{\phi}X = \phi X + u(X)N, \ \forall \ X \in \Gamma(TM). \quad (4.1)$$

We set

$$\phi X = \phi_s X + u_s(X)E, \ \forall \ X \in \Gamma(S(TM)) \quad (4.2)$$

where we have used the decomposition (2.3). Then we have the following result.

**Theorem 4.1.** If the screen distribution $S(TM)$ is integrable, then any leaf $M_s$ of $S(TM)$ is an almost paracontact metric manifold with the structure $(M_s, \phi_s, \xi_s, \eta_s, g_s)$ such that

$$\eta_s = u_s|_{S(TM)}, \xi_s = 2U = 2\overrightarrow{\phi}N, g_s = g |_{M_s}.$$ 

**Proof.** We first observe that, $\forall X \in \Gamma(S(TM))$

$$u_s(X) = g(\phi X, N) = g(\overrightarrow{\phi}X, \xi - \frac{1}{2}E) = -\frac{1}{2}u(X). \quad (4.3)$$

On the other hand, using $\phi^2(X) = X - \eta(X)e - u(X)U$, we have for any $X \in \Gamma(S(TM))$

$$\phi(\phi_s(X) + u_s(X)E) = X - u(X)U \quad (4.4)$$

which in turn gives

$$\phi_s^2(X) + u_s(\phi_s X)E + u_s(X)\phi_s(E) + u_s(X)u_s(E) = X - u(X)U \quad (4.5)$$

equating the screen and radical parts in (4.5) gives

$$\phi_s^2(X) = X - u(X)U - u_s(X)\phi_s(E) \quad (4.6)$$

$$u_s(\phi_s(X)) = u_s(X)u_s(E) = 0 \quad (4.7)$$

Now using the fact that $\phi E = \overrightarrow{\phi}E$ is in $S(TM)$, we deduce that $\phi(E) = \phi_s(E) + u_s(E)E$ implies

$$\phi_s(E) = \phi(E) = \overrightarrow{\phi}(E) = -2\overrightarrow{\phi}N = -2U \quad (4.8)$$

and that

$$u_s(E) = 0. \quad (4.9)$$

Equation (4.9) justifies why in (4.7), we have $u_s(X) \cdot u_s(E) = 0$. Using (4.8) and (4.3) in (4.6) gives

$$\phi_s^2(X) = X - u(X)(2U), \forall X \in \Gamma(S(TM)). \quad (4.10)$$
Hence the expected structure vector field of $M_s$ is $2U$ providing that $\phi_s(2U) = 0$ and $\eta(2U) = 1$ and $\eta_s \circ \phi_s = 0$. Indeed we have

$$\eta_s(\xi_s) = u(2U) = 1$$ (4.11)

since $u(U) = \frac{1}{2}$. Now using (4.7) we have $\eta_s \circ \phi_s = u \circ \phi_s = -2u_s \circ \phi_s = 0$. Next we show that $\phi_s(\xi_s) = 0$, we use

$$\phi(U) = -\frac{1}{4} E$$ (4.12)

from which we deduce that $\phi_s(U) = 0$, hence

$$\phi_s(\xi_s) = \phi_s(2U) = 0.$$ (4.13)

We now show that the structure $(M_s, \phi_s, \xi_s, \eta_s, g_s)$ is metric. Using

$$g(\phi_s(X), \phi_s(Y)) = -g(X,Y) + \eta(X)\eta(Y) + u(X)u(Y)$$

and $\eta(X) = 0$ for any $X \in \Gamma(S(TM))$, we have

$$g(\phi_s(X) + u_s(X)E, \phi_s(Y) + u_s(Y)E) = -g(X,Y) + u(X)u(Y)$$

which in turn gives

$$g(\phi_s(X), \phi_s(Y)) = -g(X,Y) + u(X)u(Y) \forall X, Y \in \Gamma(S(TM))$$

as expected.

Example 4.2. The Example 3.6 above gives a case of a foliated hypersurface $M = \{(u_1, u_2, u_3, u_4, 2u_1 - u_3 - 2u_4), u_i \in \mathbb{R}\}$ for which one finds that $\eta = 2du_1 - du_3 - 2du_4$. Hence the paracontact structure reduces to leaves of its screen distribution.

5. Normalized lightlike hypersurfaces in para-Sasakian manifolds.

We now consider the case where $\overline{M}$ is para-Sasakian. We show that the shape operator is expressed in term of the structure tensor field $\phi$. We also give necessary and sufficient conditions for $M$ to be totally geodesic and for leaves of an integrable screen distribution to be Para-Sasakian too.

Proposition 5.1. Let $(\overline{M}, \overline{\phi}, \overline{\eta}, \xi, \overline{g})$ be a para-Sasakian manifold and $(M, g)$ a normalized lightlike hypersurface with $\xi$ as rigging vector field. Then for $X, Y \in \Gamma(TM)$, we have

$$(\nabla_X \phi)(Y) = -g(X,Y)e + \eta(Y)X + B(X,Y)U + u(Y)A_N(X)$$ (5.1)

$$(\nabla_X u)(Y) = -g(X,Y) - B(X,\phi Y) - \tau(X)u(Y),$$ (5.2)

where $e = \frac{E}{2}$, $U = \overline{\phi}(N)$.

Proof. The proof is straightforward computation using the following equations verified by $\overline{M}$.

$$(\nabla_X \overline{\phi})(Y) = -\overline{g}(X,Y)\xi + \overline{\eta}(Y)X$$ (5.3)

$$\nabla_X \xi = -\overline{\phi}(X).$$ (5.4)
Proposition 5.2. Under the same hypothesis as in the preceding proposition, we have
\[ \forall X \in \Gamma(TM), \]
\[ \overline{\phi}(X) = A_N(X) + \frac{1}{2} A^*_E(X) + \frac{1}{2} \tau(X)E - \tau(X)N \quad (5.5) \]
\[ \overline{\phi}(N) = -\frac{1}{2} A_N(E). \quad (5.6) \]

Proof. For any \( X \in \Gamma(TM) \), by (5.4), we have
\[ \nabla_X N + \frac{1}{2} \nabla_X E = -\phi(X) - u(X)N \]
then
\[ -A_N(X) + \tau(X)N + \frac{1}{2} (\nabla_X E + B(X, E)N) = -\phi(X) - u(X)N, \]
and finally
\[ -A_N(X) + \tau(X)N + \frac{1}{2} (-A^*_E(X) - \tau(X)E) = -\phi(X) - u(X)N. \]
Equating the tangent and transversal parts gives
\[ u(X) = -\tau(X) \quad (5.8) \]
\[ \phi(X) = A_N(X) + \frac{1}{2} A^*_E(X) + \frac{1}{2} \tau(X)E. \quad (5.9) \]
(5.8) and (5.9) gives (5.5).

On the other hand (3.7) gives
\[ \overline{\phi}(N) = -\frac{1}{2} \overline{\phi}(E) \in \Gamma(S(TM)). \]
But \( \overline{\phi}(E) = \phi(E) = A_N(E) \) by using (5.9). Hence \( \overline{\phi}(N) = -\frac{1}{2} A_N(E) \) which is (5.6) as expected. \( \square \)

Corollary 5.3. Let \((\overline{M}, \overline{\phi}, \overline{\eta}, \xi, \overline{g})\) be a para-Sasakian manifold and \((M, g)\) a normalized lightlike hypersurface with \( \xi \) as rigging vector field.

(I) The null section \( E \) is geodesic in \((\overline{M}, \nabla)\), i.e. \( \nabla^B_E N = 0 \).

(II) The screen distribution \( ST(M) \) cannot be conformal.

(III) The induced connection \( \nabla \) does not coincide with any of the \( '\alpha\)-connection \( \nabla^\alpha \) being the Riemannian connection of the metric \( g + \alpha \eta \otimes \eta \) as defined in [12].

Proof. (I): From (5.9) it follows that \( \phi(E) = A_N(E) + \frac{1}{2} \tau(E)E \) which gives \( \tau(E) = 0 \), since \( \phi(E) = \overline{\phi}(E) = A_N(E) \).

(II): If the screen distribution is conformal then there exists \( \alpha \in C^\infty(M) \) such that \( A_N = \alpha A_E^* \). Then we have \( \phi(E) = A_N(E) = \alpha A_E^*(E) = 0 \). Which is a contradiction since we have \( g(\overline{\phi}E, \overline{\phi}E) = 1 \).

(III): Using the similar reasoning as in the proof of Theorem 4.1 in [4] one shows that \( \nabla = \nabla^\alpha \) iff \( A_E^* = \alpha A_N \) and \( d\alpha + 2\alpha \tau = 0 \). This would also lead to the same contradiction as above, namely \( \phi(E) = 0 \). \( \square \)

The distribution \( \overline{\phi}(Rad(TM)) \) is nondegenerate since \( g(\overline{\phi}E, \overline{\phi}E) = 1 \), then we can define the unique nondegenerate distribution \( D_0 \) by
Definition 5.4.

\[ S(TM) = D_0 \perp \bar{\phi}(\text{Rad}(TM)). \]  

(5.10)

This leads to the following decomposition

\[ TM = \{ \bar{\phi}(\text{Rad}(TM) \perp D_0) \} \perp \text{Rad}(TM). \]  

(5.11)

\[ T\overline{M} = \bar{\phi}(\text{Rad}(TM)) \perp D_0 \perp \{ \text{Rad}(TM) \oplus \text{ltr}(TM) \}. \]  

(5.12)

It is easy to prove the following

Proposition 5.5. \( D_0 \) is \( \bar{\phi} \)-invariant.

Now we set

\[ D = \text{Rad}(TM) \perp D_0 \text{ and } D' = \bar{\phi}(\text{Rad}(TM)). \]  

(5.13)

and it follows that

\[ T(M) = D \oplus D' \]  

(5.14)

Theorem 5.6. Let \((\overline{M}, \overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})\) be a \((2n+1)\)-para-Sasakian manifold and \((M, \phi, U, \eta, g)\) be a normalized lightlike hypersurface of \((\overline{M}, \overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})\).

Then \( M \) is totally geodesic if and only if \( \forall X \in \Gamma(TM), \forall Y \in \Gamma(D), \)

\[ (\nabla^\phi_X Y) = -g(X, Y)e + \eta(Y)X \]  

(5.15)

\[ \frac{1}{2}A_N(X) = g(X, U)e - \frac{1}{2}\nabla_X^e - \phi(\nabla_X^U). \]  

(5.16)

Proof. Assume that \( M \) is totally geodesic. By (5.1), we have for \( X \in \Gamma(TM) \) and \( Y \in \Gamma(D) \)

\[ (\nabla^\phi_X Y) = -g(X, Y)e + \eta(Y)X, \]

since \( u(Y) = \overline{g}(\overline{\phi}(Y), E) = -\overline{g}(Y, \overline{\phi}E) = 0 \). On the other hand

\[ (\nabla^\phi_X U) = -g(X, U)e + \eta(U)X + u(U)A_N(X), \]

then

\[ (\nabla^\phi_X(U) - \phi(\nabla_X^U) = -g(X, U)e + \frac{1}{2}A_N(X), \]

since \( u(U) = \frac{1}{2}, \eta(U) = 0 \). Then finally

\[ \frac{1}{2}A_N(X) = g(X, U)e - \frac{1}{2}\nabla_X^e - \phi(\nabla_X^U). \]

Conversely, let \( X, Y \in \Gamma(TM) \). By (5.15), \( \exists \alpha \in C^\infty(M) \) such that \( Y = Y_D + \alpha U \).

We have

\[ B(X, Y) = B(X, Y_D) + \alpha B(X, U). \]  

(5.17)

By (5.1) and for \( Y = Y_D \), we have

\[ (\nabla^\phi_X)Y_D = -g(X, Y_D)e + \eta(Y_D)X + B(X, Y_D)U + u(Y_D)A_N(X). \]  

(5.18)

By (5.15), we have

\[ (\nabla^\phi_X)Y_D = -g(X, Y_D)e + \eta(Y_D)X. \]  

(5.19)
Equating (5.18) and (5.19) gives
\[ B(X, Y_D)U + u(Y_D)A_N(X) = 0, \]
which gives \( B(X, Y_D) = 0 \) since \( u(Y_D) = 0 \).
On the other hand, by (5.1), we have
\[
\nabla^g_{\phi} U_X = -g(X, U)e + B(X, U)U + \frac{1}{2} A_N(X),
\]
and by using (5.16), we have
\[
\nabla^g_{\phi} U_X - \phi(\nabla^g_{\phi} X) = -g(X, U)e + B(X, U)U + \frac{1}{2} A_N(X) - g(\nabla^g_{\phi} U_X),
\]
which gives \( B(X, U) = 0 \).  

The following theorem gives a condition for the leaves of integrable screen distribution to be para-Sasakian.

**Theorem 5.7.** Let \((\overline{M}, \overline{\phi}, \xi, \overline{\eta}, \overline{g})\) be a \((2n+1)\)-para-Sasakian manifold and \((M, \phi, U, \eta, g)\) be a normalized lightlike hypersurface of \((\overline{M}, \overline{\phi}, \xi, \overline{\eta}, \overline{g})\). Assume that the screen distribution \(S(TM)\) is integrable. Then the leaves \(M_s\) of \(S(TM)\) have a para-Sasakian structure if and only if
\[
A^*_E(PX) = \phi_s(PX) - PX, \forall X \in \Gamma(TM)
\]
(5.20)

**Proof.** Since \(\overline{M}\) is para-Sasakian manifold, we have for all \(X, Y \in \Gamma(TM)\)
\[
(\nabla^g_{PX})PY = -\overline{g}(PX, PY)\xi
= -\overline{g}(PX, PY)e - \overline{g}(PX, PY)N.
\]
(5.21)

On the other hand, we have
\[
(\nabla^g_{PX})PY = \nabla^g_{PX} - \phi(\nabla^g_{PX}),
\]
(5.22)

A straightforward computation gives
\[
\nabla^g_{PX} = \nabla^g_{\phi(PY) + u(PY)N}
\]
\[
= \nabla^g_{\phi(PY) + B(PX, \phi(PY))N + PX(u(PY))N + u(PY)\nabla^N_{PX}}
\]
\[
= \nabla^{\phi_s(PY)} + C(PX, \phi_s(PY))E + PX(u_s(PY))E
- u_s(PY)A^*_E(PX) - u_s(PY)\tau(PX)E
+ B(PX, \phi_s(PY))N + PX(u(PY))N
- u(PY)A_N(PX) + u(PY)\tau(PX)N,
\]
(5.23)

\[
\phi(\nabla^g_{PX}) = \phi(\nabla^g_{PX} + B(PX, PY)N)
\]
\[
= \phi_s(\nabla^{\phi_s(PY)} + u_s(\nabla^{\phi_s(PY)})E + C(PX, PY)\phi E
+ u(\nabla^{\phi_s(PY)}N + B(PX, PY)\phi N).
\]
(5.24)
Equations (5.23) and (5.24) in (5.22) give

\[(\nabla_{\phi}^\phi)(PY) = (\nabla_{\phi}^\phi)(PY) + \{(\nabla_{\phi}^\phi)(PY) + C(PX, PY)\phi E - u_s(PY)\tau(X)\}E
+ C\{(\nabla_{\phi}^\phi)(PY) + B(PX, \phi_s(PY)) + u(PY)\tau(PX)\}N
- B(PX, PY)\phi N - C(PX, PY)\phi E
- A_N(PX)u(PY) - u_s(PY)A_E^*(PX).\]  
(5.25)

Equating (5.25) and (5.21) give

\[(\nabla_{\phi}^\phi)(PY) = -g(PX, PY) - B(PX, \phi_s(PY)) - u(PY)\tau(PX) \]  
(5.26)

Equation (5.26) is relation (5.2). By (5.28), we have

\[(\nabla_{\phi}^\phi)(PY) = B(PX, PY)\phi E + C(PX, PY)\phi E
+ A_N(PX)u(PY) + u_s(PY)A_E^*(PX) \]  
(5.27)

Equation (5.26) is relation (5.2). By (5.28), we have

\[(\nabla_{\phi}^\phi)(PY) = B(PX, PY)\phi E + C(PX, PY)\phi E
+ u_s(PY)A_E^*(PX) \]  
(5.29)

If \(A^*_E(PX) = \phi_s(PX) - PX\) then \(A_N(PX) - \frac{1}{2}A_E^*(PX) = PX\) and by (5.29) we have

\[\{\nabla_{\phi}^\phi\}(PY) = -g_s(PX, PY)(2U) + u(PY)PX,\]

then \((M_s, \phi_s, \xi_s, \eta_s, g_s)\) is a para-Sasakian manifold.
Conversely if \((M_s, \phi_s, \xi_s, \eta_s, g_s)\) is para-Sasakian i.e.

\[(\nabla_{\phi}^\phi)(PY) = -g_s(PX, PY)(2U) + u(PY)PX \]  
(5.30)

and then equating (5.29) and (5.30) gives

\[A_N(PX) - \frac{1}{2}A_E^*(PX) = PX.\]
Example 5.8. Let $\mathcal{M} = \mathbb{R}^5$ endowed with the metric $\mathcal{g}$ given by its following matrix in the cartesian coordinate system $(x_1, x_2, x_3, x_4, x_5)$:

$$[\mathcal{g}] = \frac{1}{4} \begin{pmatrix}
1 + x_3^2 & -x_3x_4 & 0 & 0 & -x_3 \\
-x_3x_4 & -1 + x_4^2 & 0 & 0 & x_4 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
-x_3 & x_4 & 0 & 0 & 1
\end{pmatrix} \quad (5.31)$$

Then the Christoffel coefficients of the Levi-Civita connection of $\mathcal{g}$ are given by:

$$\Gamma^1_{13} = \Gamma^2_{14} = \Gamma^4_{12} = -\Gamma^5_{35} = \frac{1}{2} \Gamma^3_{11} = \frac{1}{2} x_3$$
$$\Gamma^1_{13} = \Gamma^2_{24} = \Gamma^3_{12} = -\Gamma^5_{45} = \frac{1}{2} \Gamma^4_{22} = -\frac{1}{2} x_4$$
$$\Gamma^5_{13} = -\frac{1}{2} (x_3^2 - 1), \Gamma^5_{14} = \Gamma^5_{23} = -\frac{1}{2} x_3 x_4, \Gamma^5_{24} = \frac{1}{2} (x_4^2 + 1),$$

and $\Gamma^5_{ij} = 0$ for any other triple $(i, j, k)$.

We define an almost paracontact structure $(\tilde{\phi}, \xi, \eta)$ on $\mathcal{M}$ by:

$$\tilde{\phi}(\frac{\partial}{\partial x_3}) = 0, \tilde{\phi}(\frac{\partial}{\partial x_2}) = -\frac{\partial}{\partial x_3}, \tilde{\phi}(\frac{\partial}{\partial x_2}) = -\frac{\partial}{\partial x_1} - x_3 \frac{\partial}{\partial x_5}, \tilde{\phi}(\frac{\partial}{\partial x_4}) = -\frac{\partial}{\partial x_4}, \tilde{\phi}(\frac{\partial}{\partial x_4}) =$$
$$-\frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3}, \eta = \frac{1}{2} (x_3 dx_1 - x_4 dx_2 - dx_5), \xi = -2 \frac{\partial}{\partial x_5}.$$ Then one shows that $(\mathcal{M} = \mathbb{R}^5, \tilde{\phi}, \xi, \eta, \mathcal{g})$ is a para-Sasakian manifold.

Let us consider the hypersurface $M = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5/x_5 = \frac{1}{2} (arcsin(x_3) + x_3 \sqrt{1 + x_3^2 + x_3^2}) \}$ of $\mathcal{M}$.

Around any point $m = (u_1, u_2, u_3, u_4, \frac{1}{2} (arcsin(u_3) + u_3 \sqrt{1 + u_3^2 + u_3^2})) \in M$, we consider the local coordinates system $(u_1, u_2, u_3, u_4)$ and set: $V_1 = \frac{\partial}{\partial u_1}, V_2 = \frac{\partial}{\partial u_2}, V_3 = \frac{\partial}{\partial u_3}, V_4 = \frac{\partial}{\partial u_4}.$

The tangent bundle $TM$ at $m$ is spanned by $V_1, V_2, V_3, V_4$ and one has

$$\left\{ V_1 = \frac{\partial}{\partial x_1}, V_2 = \frac{\partial}{\partial x_2}, V_3 = \frac{\partial}{\partial x_3} + \sqrt{1 + x_3^2} \frac{\partial}{\partial x_5}, V_4 = \frac{\partial}{\partial x_4} + x_4 \frac{\partial}{\partial x_5} \right\}. \quad (5.32)$$

It is easy the check that $\xi$ is transversal vector field to $M$ and that the latter is a lightlike hypersurface with the radical distribution $\text{Rad}(TM)$ and the lightlike transversal distribution $\text{ltr}(TM)$ given by

$$\text{Rad}(TM) = \text{span}\left\{ E = -2u_3 V_1 - 2u_4 V_2 - 2 \sqrt{1 + u_3^2} V_3 + 2u_4 V_4 \right\}$$

$$\text{ltr}(TM) = \text{span}\left\{ N = x_3 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_2} + \sqrt{1 + x_3^2} \frac{\partial}{\partial x_3} - x_4 \frac{\partial}{\partial x_4} + (x_3^2 - x_4^2 - 1) \frac{\partial}{\partial x_5} \right\}.$$
The 1-form $\eta$ induced on $M$ by $\eta$ is given by

$$\eta = \frac{1}{2} \left( u_3 du_1 - u_4 du_2 - \sqrt{1 + u_3^2} du_3 - u_4 du_4 \right)$$

(5.33)

From which one gets the screen distribution $S(TM)$ which is not integrable and spanned by

$$W_1 = V_1 + \frac{u_3}{\sqrt{1 + u_3^2}} V_3, \quad W_2 = V_2 - \frac{u_4}{\sqrt{1 + u_3^2}} V_3, \quad W_3 = V_4 - \frac{u_4}{\sqrt{1 + u_3^2}} V_3.$$

We also compute

$$\bar{\phi}(E) = 2\alpha W_1 - 2u_4 W_2 + 2u_3 W_3, \quad \alpha = \sqrt{1 + u_3^2}$$

and deduce that the distribution $D_0$ is spanned by

$$(\alpha u_3 u_4 + u_4 \alpha^2 + u_3^3 + u_3 u_4^2)W_1 - (\alpha - u_3 u_4^2 - u_4 u_3^2)W_2 \quad \text{and}$$

$$(\alpha u_3 u_4 + u_4 \alpha^2 + u_3 u_4^2)W_1 - (\alpha - u_3 u_4^2 - u_4 u_3^2)W_3.$$

6. Integrability of some distributions of $TM$.

In this section we assume that $M$ is para-Sasakian and investigate the conditions under which the subdistributions $D_0$ and $D$ of $TM$ introduced above are integrable distributions.

6.1. Distribution $D_0$. We consider the distribution $D_0$ defined in (5.10) as orthogonal complementary of $\{\phi(Rad(TM))\}$ in $S(TM)$ and we set

$$\beta = \bar{\phi}(Rad(TM)) \perp Rad(TM).$$

(6.1)

For all $X \in \Gamma(TM), Y \in \Gamma(D_0)$ and $Z \in \Gamma(\beta)$, following the decomposition (5.11) we have

$$\nabla_Y^X = \nabla_Y^X + h(X,Y)$$

(6.2)

$$\nabla_Z^X = -A_Z(X) + \nabla_X^\beta Z$$

(6.3)

where $\nabla$ is a linear connection on $D_0$, $h : \Gamma(TM) \times \Gamma(D_0) \rightarrow \Gamma(\beta)$ is an $C^\infty(M)$-bilinear map, $A_Z$ is $\Gamma(D_0)$-valued $C^\infty(M)$-linear operator on $\Gamma(TM)$ and $\nabla^\beta$ is a linear connection on $\beta$.

**Lemma 6.1.** Let $(M, g)$ be a normalized lightlike hypersurface of a para-Sasakian manifold $(\overline{M}, \bar{\phi}, \xi, \eta, \overline{g})$, then

$$\overline{g}(\nabla^\beta_X Y, E) = -\overline{g}(X, Y), \quad \forall X, Y \in \Gamma(TM).$$

(6.4)

**Proof.**

$$\overline{g}(\nabla^\beta_X Y, E) = \overline{g}(-\overline{g}(X, Y)\xi + \eta(Y)X, E)$$

$$= -\overline{g}(X, Y)\overline{g}(\xi, E) + \eta(Y)\overline{g}(X, E)$$

$$= -\overline{g}(X, Y).$$

$\square$
According to decomposition (6.1), we set
\[ \hat{h}(X,Y) = \alpha_1(X,Y)\phi E + \alpha_2(X,Y)E, \]  
(6.5)
for all \( X \in \Gamma(TM) \), \( Y \in \Gamma(D_0) \), where we have
\[ \alpha_1(X,Y) = \bar{g}(\hat{h}(X,Y), \phi E) \]
\[ \alpha_2(X,Y) = \bar{g}(\hat{h}(X,Y), N). \]  
(6.6)
By (6.2), one has
\[ \nabla^Y_X = \hat{\nabla}^Y_X + \alpha_1(X,Y)\phi E + \alpha_2(X,Y)E, \]  
(6.7)
then
\[ g(\nabla^Y_X, \phi E) = \bar{g}(\hat{\nabla}^Y_X + \alpha_1(X,Y)\phi E + \alpha_2(X,Y)E, \phi E) = \alpha_1(X,Y), \]  
(6.8)
since \( g(\phi E, \phi E) = 1 \). On the other hand we can express \( \alpha_1 \) as follows:
\[ \alpha_1(X,Y) = g(\nabla^Y_X, \phi E) = -\bar{g}(\hat{\phi} \nabla^Y_X, E) = -\bar{g}(\hat{\phi} \nabla^Y_X + \nabla^Y_X, E) = -\bar{g}(X,Y) + \bar{g}(\phi Y, \nabla^E_X), \text{ since } (\nabla^0 = 0) \]
\[ = -\bar{g}(X,Y) + \bar{g}(X,Y) - A^*_E(X) - \tau(X)E \]
\[ = -\bar{g}(X,Y) - B(X, \phi Y) \]
since \( g(A^*_E(X), Y) = B(X,Y) \) and \( D_0 \) is \( \phi \)-invariant. We also compute
\[ \alpha_2(X,Y) = g(\nabla^Y_X, N) = g(\nabla^Y_X + C(X,Y)E, N) = C(X,Y). \]
Then
\[ \hat{h}(X,Y) = -(g(X,Y) + B(X, \phi Y))\phi E + C(X,Y)E \]  
(6.9)
and
\[ \nabla^Y_X = \hat{\nabla}^Y_X - (g(X,Y) + B(X, \phi Y))\phi E + C(X,Y)E. \]  
(6.10)

**Theorem 6.2.** Let \((M, g)\) be a normalized lightlike hypersurface of a para-Sasakian manifold \((\bar{M}, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})\).

Then the distribution \( D_0 \) is integrable if and only if
\[ C(X,Y) = C(Y,X), B(X, \phi Y) = B(\phi X, Y), \forall X,Y \in \Gamma(D_0). \]  
(6.11)

**Proof.** Let \( X, Y \in \Gamma(D_0) \). Since \( \nabla \) is a torsion free connection, we have
\[ [X,Y] = \nabla^Y_X - \nabla^X_Y \]
\[ = \hat{\nabla}^Y_X - \hat{\nabla}^Y_X + B(\bar{\phi} X, Y) - B(X, \phi Y)\phi E \]
\[ + (C(X,Y) - C(Y,X))E. \]  
(6.12)
from which the statement in the theorem follows. \( \square \)

**Corollary 6.3.** Let \((\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})\) be a para-Sasakian manifold and \((M, g)\) a normalized lightlike hypersurface of \((\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})\). Then \( \hat{h} \) is symmetric on \( D_0 \) if and only if \( D_0 \) is integrable.
Theorem 6.4. Let \((M, g)\) be a normalized lightlike hypersurface of a para-Sasakian manifold \((\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})\) such that \(D_0\) is integrable and
\[
C(X, Y) = C(\overline{\phi}X, \overline{\phi}Y),
\]
for any \(X, Y \in \Gamma(D_0)\). Then \(\text{trace}(\hat{h}) = -2(n - 1)\phi E\)

Proof. Using decomposition (5.11), we find that \(\text{rank } D_0 = 2n - 2\). Now we consider an orthonormal \(\overline{\phi}-\)basis \(\{e_i, \overline{\phi}e_i\}\) of \(D_0\), \(i = 1, 2 \cdots n - 1\). We have
\[
\text{trace}(\hat{h}) = \sum_{i=1}^{n-1} \hat{h}(e_i, e_i) - \sum_{i=1}^{n-1} \hat{h}(\overline{\phi}e_i, \overline{\phi}e_i).
\]
Hence, using (6.9) we get
\[
\hat{h}(e_i, e_i) - \hat{h}(\overline{\phi}e_i, \overline{\phi}e_i) = -2\phi E - (C(\overline{\phi}e_i, \overline{\phi}e_i) - C(e_i, e_i))E = -2\phi E
\]
which leads to the result as stated \(\Box\)

Now using decomposition (5.12) and putting \(\nu = \overline{\phi}(\text{Rad}(TM)) \perp \{\text{Rad}(TM) \oplus ltr(TM)\}\) for any \(X \in \Gamma(TM), Y \in D_0\) and \(Z \in \Gamma(\nu)\), we have
\[
\hat{\nabla}^Y_X = \hat{\nabla}^Y_X + \hat{h}(X, Y)
\]
\[
\hat{\nabla}^Z_X = -\hat{A}_Z(X) + \nabla^\nu_X Z
\]
where \(\hat{\nabla}\) is a linear connection on \(D_0\), \(\hat{h} : \Gamma(TM) \times \Gamma(D_0) \rightarrow \Gamma(\nu)\) is an \(C^\infty(M)\)-bilinear map, \(\hat{A}_Z\) is an \(C^\infty(M)\)-linear operator on \(\Gamma(TM)\) with values in \(D_0\) and \(\nabla^\nu\) is a linear connection on \(\nu\).

We set
\[
\hat{h}(X, Y) = \beta_1(X, Y)\phi E + \beta_2(X, Y)E + \beta_3(X, Y)N,
\]
for any \(X, Y \in \Gamma(D_0)\). Then we easily check that
\[
\beta_1(X, Y) = g(\hat{h}(X, Y), \phi E) = -(g(X, Y) + B(X, \phi Y))\phi E
\]
\[
\beta_2(X, Y) = g(\hat{h}(X, Y), N) = C(X, Y)
\]
\[
\beta_3(X, Y) = g(\hat{h}(X, Y), E) = B(X, Y).
\]
\[
\hat{h}(X, Y) = -\left(g(X, Y) + B(X, \phi Y)\right)\phi E + C(X, Y)E + B(X, Y)N,
\]
\[
\hat{\nabla}^Y_X = \hat{\nabla}^Y_X - \left(g(X, Y) + B(X, \phi Y)\right)\phi E + C(X, Y)E + B(X, Y)N.
\]

Theorem 6.5. Let \((M, g)\) be a normalized lightlike hypersurface of a para-Sasakian manifold \((\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})\) as above. Assume that the distribution \(D_0\) is integrable and
\[
C(X, Y) = C(\overline{\phi}X, \overline{\phi}Y)
\]
\[
B(X, Y) = B(\overline{\phi}X, \overline{\phi}Y)
\]
for any $X, Y \in \Gamma(D_0)$. Then on $D_0$ we have
\[
\text{trace}(\hat{h}) = -2(n-1)\bar{\phi}E.
\]

Proof. We consider an orthonormal $\bar{\phi}$–basis \{$e_i, \bar{\phi}e_i$\} of $D_0$, $i = 1, 2 \cdots n-1$. We easily see that
\[
\text{trace}(\hat{h}) = \text{trace}(\hat{\bar{h}}) + \sum_{i=1}^{n-1} \left( -B(\bar{\phi}e_i, \bar{\phi}e_i) + B(e_i, e_i) \right) N
\]
which in turn gives the result as stated $\Box$

6.2. Distribution D. We consider the distribution $D$, defined by
\[
D = \text{Rad}(TM) \perp D_0. \tag{6.14}
\]
and recall
\[
D' = \bar{\phi}(\text{Rad}(TM)), \quad TM = (D_0 \perp D') \perp \text{Rad}(TM) = D \perp D'.
\]
For $X \in \Gamma(TM), Y \in \Gamma(D)$ and $Z \in \Gamma(D')$, we have
\[
\nabla^Y_X = \tilde{\nabla}^Y_X + \bar{h}(X,Y) \tag{6.15}
\]
\[
\nabla^Z_X = -\tilde{A}_Z(X) + \nabla^{D'}_X Z \tag{6.16}
\]
where $\tilde{\nabla}$ is a linear connection on $D$, $\bar{h} : \Gamma(TM) \times \Gamma(D) \rightarrow \Gamma(D')$ is an $C^\infty(M)$-bilinear map, $\tilde{A}_Z$ is an $C^\infty(M)$-linear operator with values in $D$ and $\nabla^{D'}$ is a linear connection on $D'$. We set
\[
\tilde{h}(X,Y) = \sigma_1(X,Y)\bar{\phi}E. \tag{6.17}
\]
Then we have
\[
\sigma_1(X,Y) = \bar{g}(\nabla^Y_X, \bar{\phi}E) = \bar{g}(\nabla^Y_X - B(X,Y)N, \bar{\phi}E) = -\bar{g}(\bar{\phi}(\nabla^Y_X), E) = \bar{g}(-\bar{g}(X,Y)\xi + \bar{\eta}(Y)X, E) - \bar{g}(\nabla^{\bar{\phi}(Y)}_X, E) = -(g(X,Y) + B(X, \bar{\phi}Y)) \tag{6.18}
\]
and
\[
\nabla^Y_X = \tilde{\nabla}^Y_X - (g(X,Y) + B(X, \bar{\phi}Y))\bar{\phi}E. \tag{6.19}
\]
Hence we have the following result:

Proposition 6.6. Let $(M, g)$ be a normalized lightlike hypersurface of a para-Sasakian manifold $(\overline{M}, \overline{\phi}, \xi, \eta, \bar{g})$ as above. Then the distribution $D$ is integrable if and only if
\[
B(X, \bar{\phi}Y) = B(\bar{\phi}X, Y), \forall X,Y \in \Gamma(D) \tag{6.20}
\]

Proof. The result stated follows from Frobenius criterium for integrability and the decomposition $TM = D \perp D'$ since for all $X,Y \in \Gamma(D)$, we have
\[
[X,Y] = \tilde{\nabla}^Y_X - \tilde{\nabla}^X_Y + (B(\bar{\phi}X, Y) - B(X, \bar{\phi}Y))\bar{\phi}E.
\] $\square$
References


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