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RIGGED NULL HYPERSURFACES IN ALMOST PARACONTACT METRIC MANIFOLDS

F. NGAKEU[1] *, A.C. DJOMAKO[2] AND BITJONG NDOMBOL[3]

ABSTRACT. Given an almost paracontact metric manifold $(\overline{M}, \overline{\phi}, \overline{\eta}, \xi, \overline{g})$, we study lightlike hypersurfaces M of the semi-Riemannian manifold $(\overline{M}, \overline{g})$ transversal to the structure vector field ξ. The latter is then a rigging for M and defines a null section E of the radical distribution of M and a screen distribution which turns out to be always semi-invariant. We show that leaves of an integrable screen distribution in such hypersurfaces M are almost paracontact metric manifolds too. When the ambient space $(\overline{M}, \overline{\phi}, \overline{\eta}, \xi, \overline{g})$ is para-Sasakian, we show that the screen distribution cannot be conformal and that E is a geodesic vector field in (M, ∇), where ∇ is the connection induced on M by Levi-Civita connection of $\overline{g}$ and the local null rigging $N = \xi - \frac{1}{2}E$. We also find necessary and sufficient conditions for the leaves of the screen distribution to be para-Sasakian too and finally we investigate integrability conditions for some additional distributions induced on M by the structure $(\overline{M}, \overline{\phi}, \overline{\eta}, \xi, \overline{g})$.

1. INTRODUCTION.

Lightlike submanifolds (or null submanifolds) of semi-Riemannian manifolds were introduced by K.L. Duggal and A. Bejancu [6, 7]. Since this pioneering work, several authors have studied lightlike hypersurfaces of semi-Riemannian manifolds and particularly those of paracontact and para-Sasakian manifolds tangent to the structure vector fields (see [1, 2, 3, 4, 11, 12, 7, 13] and references therein). It is well known that the usual technique to study a lightlike hypersurface is to fix on it a geometric data formed by a lightlike section and a screen distribution. Both of them are fixed arbitrarily and independently. So, all geometric objects derived from them depend on these choices. In some recent works [9, 10], M. Gutiérrez and B. Olea used the rigging technique to fix a section of the null distribution and the screen distribution in a natural way and studied the geometry of lightlike hypersurfaces of Lorentzian manifolds. The same method has been used in other works [3, 8]. This technique consists for a given lightlike hypersurface M in a Lorentzian manifold $\overline{M}$, to choose a vector field ξ on $\overline{M}$ which is not tangent to M and construct a lightlike section and the screen distribution.

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* Corresponding author.
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The aim of this paper is to use the same technique to study the lightlike geometry of hypersurfaces in almost paracontact metric manifolds and in para-Sasakian manifolds.
The paper is organized as follows. The present section is labeled as Introduction. The next Section 2 is devoted to basic facts on Null hypersurfaces in semi-Riemannian manifolds and the rigging technique following Reference [9]. We also recall definitions and some properties of almost para-contact metric manifolds and of Para-Sasakian manifolds. In Section 3 we introduce rigged Null hypersurfaces in almost paracontact metric manifolds and it appears that they are screen semi-invariant. We give an explicit example of such Null hypersurfaces. Section 4 is devoted to the reduction of the almost paracontact structure. We show that leaves of an integrable screen distribution of a normalized lightlike hypersurface in an almost paracontact manifold are also almost paracontact manifolds. In Section 5, we consider the case where the ambient manifold $\overline{M}$ is a para-Sasakian manifold and investigate properties of a normalized lightlike hypersurface in $\overline{M}$. We show that in this case the shape operator of the hypersurface is expressed in term of the structure tensor $\overline{\phi}$ and we prove that the hypersurface cannot be screen conformal. We give a necessary and sufficient condition for a given integrable screen distribution to be para-Sasakian too. Finally, in Section 6 we investigate integrability conditions for some additional distributions induced on the null hypersurface by the structure of the ambient para-Sasakian manifold.

2. Lightlike Hypersurfaces and Paracontact Manifolds.

We start by recalling basic notions on lightlike hypersurfaces, the rigging technique for such hypersurfaces and basic facts on para-Sasakian manifolds. We follow the presentation and some notations from references [7] [5][13] and [9].

2.1. **Lightlike hypersurfaces.** Given a n-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ and a hypersurface M of $\overline{M}$, we say that M is a lightlike hypersurface if the induced metric $g = i^*\overline{g}$ on the hypersurface M is degenerate, where $i : M \hookrightarrow \overline{M}$ is the canonical immersion. Its means that there exists a lightlike vector field $E \in \Gamma(TM)$ such that

$$g(X, E) \;=\; 0,\; \forall X \in \Gamma(TM). \tag{2.1}$$

We denote by $Rad(TM)$ the radical distribution which is spanned by E. A screen distribution $S(TM)$ is a complementary distribution to $Rad(TM)$ in TM and the transversal distribution $ltr(TM)$ is the unique lightlike one-dimensional distribution orthogonal to $S(TM)$ not contained in TM.

Theorem 2.1. [6] *For any nonzero section E of Rad(TM) on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique section N of ltr(TM) on $\mathcal{U}$ satisfying*

$$\overline{g}(N, E) = 1, \qquad \overline{g}(N, N) = \overline{g}(N, W) = 0, \tag{2.2}$$

for all $W \in \Gamma(S(TM)/_{\mathcal{U}})$.

Since the complementary vector bundle $S(TM)$ of $Rad(TM)$ in TM is non-degenerate, we can consider the subbundle $S(TM)^{\perp}$ which is a complementary orthogonal of $S(TM)$ in $T\overline{M}$ and which is called *screen transversal subbundle*. One can then consider the following decompositions:

$$TM = S(TM) \perp Rad(TM) \tag{2.3}$$

$$T\overline{M} = S(TM) \perp S(TM)^{\perp} \tag{2.4}$$

$$S(TM)^{\perp} = Rad(TM) \oplus ltr(TM) \tag{2.5}$$

$$\begin{aligned} T\overline{M} &= S(TM) \perp \{Rad(TM) \oplus ltr(TM)\} \\ &= TM \oplus ltr(TM) \end{aligned} \tag{2.6}$$

From (2.6) we can write

$$\overline{\nabla}_X^Y = \nabla_X^Y + h(X,Y) \tag{2.7}$$

$$\overline{\nabla}_X^N = -A_N(X) + \nabla_X^t N, \tag{2.8}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(ltr(TM))$. ∇ and ∇^t are called the *induced connection* on M and $ltr(TM)$ respectively, h and A_N are called the *second fundamental form* and *shape operator*. The above equations are the Gauss and Weingarten equations.

Locally, let E, N and $\mathcal{U}$ be as in the Theorem 2.1 above. Then for any $X, Y \in \Gamma(TM \mid_{\mathcal{U}})$, setting

$$B(X,Y) = \overline{g}(h(X,Y), E), \qquad \tau(X) = \overline{g}(\nabla_X^t N, E), \tag{2.9}$$

we can write

$$\overline{\nabla}_X^Y = \nabla_X^Y + B(X,Y)N \tag{2.10}$$

$$\overline{\nabla}_X^N = -A_N(X) + \tau(X)N, \tag{2.11}$$

B is called the local second fundamental form of M. B is a symmetric tensor that satisfies $B(X,Y) = -g(\nabla_X E, Y)$. Moreover, $B(E,.) = 0$ and E is a pre-geodesic vector field, in fact $\nabla_E E = -\tau(E)E$. The notion of totally geodesic or umbilic hypersurface also has sense in the degenerate case. Indeed, M is totally geodesic if $B = 0$ and totally umbilic if $B = \rho g$ for a certain $\rho \in C^{\infty}(M)$.
The decomposition (2.3) allows to define a canonical projection $P : \Gamma(TM) \longrightarrow \Gamma(S(TM))$. For each $X \in \Gamma(TM)$, we may write

$$X = PX + \theta(X)E. \tag{2.12}$$

where

$$\theta(X) = \overline{g}(X, N). \tag{2.13}$$

We have

$$(\nabla_X^g)(Y,Z) = B(X,Y)\theta(Z) + B(X,Z)\theta(Y) \tag{2.14}$$

which implies that the induced connection ∇ is a nonmetric connection on M. Given $X \in \Gamma(TM)$, the vector field $\nabla_X E$ belongs to $\Gamma(M)$, so it can be decomposed as

$$\nabla_X^E = -\tau(X)E - A_E^*(X),$$

where $A^*_E(X) \in \Gamma(S(TM))$. The endomorphism A^*_E is called *the shape operator* of $S(TM)$ and it satisfies $B(X, PY) = g(A^*_E(X), PY)$ and

$$B(A^*_E(X), Y) \;=\; B(X, A^*_E(Y)). \tag{2.15}$$

The trace of A^*_E is the lightlike mean curvature of M, explicitly given by

$$H_p = \sum_{i=3}^{n} g(A^*(e_i), e_i) = \sum_{i=3}^{n} B(e_i, e_i),$$

where $\{e_3, \cdots, e_n\}$ is an orthonormal basis of $S(T_pM)$. On the other hand, given $X \in \Gamma(TM)$ and $PY \in S(TM)$, locally, we decompose

$$\nabla^{PY}_X \;=\; \nabla^*_X PY + C(X, PY)E. \tag{2.16}$$

The tensor C holds $C(X, PY) = g(A_N(X), PY)$ and

$$C(PX, PY) - C(PY, PX) \;=\; g(N, [PX, PY]). \tag{2.17}$$

When the screen distribution $S(TM)$ is integrable, ∇^* is the induced Levi-Civita connection from (M, g) and Equations (2.4) and (2.5) show that its second fondamental form is

$$\Theta(PX, PY) \;=\; C(PX, PY)E + B(PX, PY)N \tag{2.18}$$

where $PX, PY \in S(TM)$. The curvature tensor of ∇ is defined as $R_{XY}Z = \nabla_X\nabla^Z_Y - \nabla_Y\nabla^Z_X - \nabla^Z_{[X,Y]}$ and it satisfies

$$R_{XY}E \;=\; \overline{R}_{XY}E, \tag{2.19}$$

where $X, Y \in \Gamma(TM)$ and the so called Gauss-Codazzi equations are

$$\begin{aligned} g(\overline{R}_{XY}Z, U) \;&=\; g(R_{XY}Z, U) + B(X, Z)g(A_N(Y), U) \\ &\quad -B(Y, Z)g(A_N(X), U), \end{aligned} \tag{2.20}$$

$$\begin{aligned} g(\overline{R}_{XY}Z, E) \;&=\; (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) \\ &\quad -\tau(Y)B(X, Z) \end{aligned} \tag{2.21}$$

$$g(\overline{R}_{XY}Z, N) \;=\; g(R_{XY}Z, N), \tag{2.22}$$

where $X, Y, Z \in \Gamma(TM)$ and $U \in S(TM)$. From these equations it can be deduced the following ones,

$$\begin{aligned} g(R_{XY}U, N) \;&=\; (\nabla^*_X C)(Y, U) - (\nabla^*_Y C))(X, U) \\ &\quad +\tau(Y)C(X, U) - \tau(X)C(Y, U) \end{aligned} \tag{2.23}$$

$$g(R_{XY}E, N) \;=\; C(Y, A^*_E(X)) - C(X, A^*(Y)) - d\tau(X, Y), \tag{2.24}$$

where $\nabla^*_U C$ is defined as

$$(\nabla^*_X C)(Y, PZ) = X(C(Y, PZ)) - C(\nabla_X Y, PZ) - C(Y, \nabla^*_X PZ). \tag{2.25}$$

Using equation (2.8), we can compute the lightlike sectional curvature with respect to E of a lightlike plane $\Pi = span(X, E)$ where $X \in S(TM)$ is unitary:

$$\mathcal{K}_E(\Pi) = (\nabla_E B)(X, X) - (\nabla_X B)(E, X) + \tau(E)B(X, X). \tag{2.26}$$

2.2. **The rigging and rigged vector fields on a lightlike hypersurface.** Take ξ a vector field defined in some open set in $(\overline{M},\overline{g})$ containing M and denote by $\overline{\eta}$ the 1-form metrically equivalent to ξ. Take $\eta = i^*\overline{\eta}$, where $i : M \hookrightarrow \overline{M}$ is the canonical inclusion and consider the tensors $\widehat{g} = \overline{g} + \overline{\eta}\otimes\overline{\eta}$ and $\widetilde{g} = i^*\widehat{g}$ [9]. One shows that the associated metric $\widetilde{g}$ is a non-degenerate metric on the hypersurface, [8].

Definition 2.2. Let M be a lightlike hypersurface of a semi-riemannian manifold $(\overline{M},\overline{g})$. A rigging for M is a vector field ξ defined on some open set containing M such that $\xi_p \notin T_pM$ for each $p \in M$.

From now on we fix ξ a rigging for M. The rigging ξ fixes a lightlike vector field E in M, which we call rigged vector field.

Definition 2.3. The rigged vector field of ξ is the $\widetilde{g}$-metrically equivalent vector field to the 1-form η and it is denoted E.

Proposition 2.4. [9] *The rigged vector field of ξ is the unique lightlike vector field E in M such that $\overline{g}(\xi,E) = 1$. Moreover, E is $\widetilde{g}$-unitary.*

We can consider the screen distribution given by $TM\cap\xi^{\perp}$, which we denote by $S(TM)$. Observe that $S(TM)$ is the $\widetilde{g}$-orthogonal subspace to E and the local lightlike transverse vector field to $S(TM)$ is given by

$$N = \xi - \frac{1}{2}\overline{g}(\xi,\xi)E.$$

2.3. **Para-Sasakian Manifolds.** A differentiable manifold of dimension $(2n+1)$ is called almost paracontact manifold with the almost paracontact structure $(\overline{\phi},\xi,\overline{\eta})$ if it admits a tensor field $\overline{\phi}$ of type $(1,1)$, a vector field ξ and a 1-form $\overline{\eta}$ satisfying the following conditions [5, 14]:

$$\overline{\phi}^2 = I - \overline{\eta}\otimes\xi \tag{2.27}$$
$$\overline{\eta}(\xi) = 1 \tag{2.28}$$
$$\overline{\phi}(\xi) = 0 \tag{2.29}$$
$$\overline{\eta}\circ\overline{\phi} = 0, \tag{2.30}$$

where I denotes the identity transformation. If a $(2n+1)$-dimensional almost paracontact manifold $(\overline{M},\overline{\phi},\xi,\overline{\eta})$ admits a pseudo-Riemannian metric $\overline{g}$ such that

$$\overline{g}(\overline{\phi}X,\overline{\phi}Y) = -\overline{g}(X,Y) + \overline{\eta}(X)\overline{\eta}(Y),\ X,Y \in \Gamma(T\overline{M}), \tag{2.31}$$

then we say that $\overline{M}$ is an almost paracontact metric manifold with an almost paracontact metric structure $(\overline{\phi},\xi,\overline{\eta},\overline{g})$ and such metric $\overline{g}$ is called compatible metric. From (2.31) it is easy to see that

$$\overline{g}(\overline{\phi}X,Y) = -\overline{g}(X,\overline{\phi}Y) \tag{2.32}$$
$$\overline{g}(X,\xi) = \overline{\eta}(X), \tag{2.33}$$

for any $X,Y \in \Gamma(T\overline{M})$.

Any compatible metric $\overline{g}$ is necessarily of signature $(n+1,n)$.
An almost paracontact metric manifold is paracontact if $d\overline{\eta} = \omega$, where we have

set $\omega(X,Y)=\bar{g}(X,\bar{\phi}Y)$ and $d\bar{\eta}(X,Y)=\frac{1}{2}(X.\bar{\eta}(Y)-Y.\bar{\eta}(X)-\bar{\eta}([X,Y])), \quad \forall X,Y\in\Gamma(T\overline{M})$.
A paracontact metric manifold is a para-Sasakian manifold if

$$(\overline{\nabla}_X\overline{\phi})Y \;=\; -\overline{g}(X,Y)\xi+\overline{\eta}(Y)X,\ \forall X,Y\in\Gamma(T\overline{M}). \tag{2.34}$$

3. Normalized lightlike hypersurfaces in almost paracontact metric manifolds.

Let $(\overline{M},\overline{\Phi},\xi,\overline{\eta},\overline{g})$ be a $(2n+1)$-dimensional almost paracontact metric manifold and M a lightlike hypersurface of $\overline{M}$ such that the structure vector field ξ is a rigging vector field for M. It means that at any point of M the rigging ξ is transversal to M. We call such a manifold *normalized lightlike hypersurface* of $\overline{M}$. Let E and N be respectively, the rigged and null transversal vector field associated to ξ such that

$$\overline{g}(E,\xi)=1 \textit{ and } \overline{g}(E,N)=1.$$

We recall that we locally have

$$\xi \;=\; N+\frac{1}{2}E. \tag{3.1}$$

Then we easily get the following equations

$$\overline{\eta}(E)=1,\ \overline{\eta}(N)=\frac{1}{2}, \tag{3.2}$$

$$\overline{\phi}^2(E)=\frac{1}{2}E-N,\ \overline{\phi}^2(N)=\frac{1}{2}N-\frac{1}{4}E. \tag{3.3}$$

$$\overline{\phi}(N)=-\frac{1}{2}\overline{\phi}(E). \tag{3.4}$$

$$\overline{g}(\overline{\phi}\ E,E)=0,\ \overline{g}(\overline{\phi}\ E,N)=0 \tag{3.5}$$

$$\overline{g}(\overline{\phi}\ N,N)=0,\ \overline{g}(\overline{\phi}\ N,E)=0. \tag{3.6}$$

$$\overline{\phi}\ (N)=-\frac{1}{2}\overline{\phi}\ E\in\Gamma(S(TM)). \tag{3.7}$$

As recalled in [5], the normalized hypersurface (M,g) is said to be screen semi-invariant if both $\overline{\phi}\ (N)$ and $\overline{\phi}\ (E)$ belong to the screen distribution. Hence, Equations (3.3) and (3.7)lead to the following proposition.

Proposition 3.1. *A normalized lightlike hypersurface of almost para-contact metric manifold $\overline{M}$ is rather a screen semi-invariant lightlike hypersurface of $\overline{M}$.*

For $X\in\Gamma(TM)$, we set

$$\overline{\phi}(X)=\phi(X)+u(X)N \tag{3.8}$$

and $\eta=i^*\overline{\eta}$ where $i:M\hookrightarrow\overline{M}$ is the canonical immersion. Then we have

Proposition 3.2. *For all* $X, Y \in \Gamma(TM)$

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y) + u(X)u(Y). \tag{3.9}$$

Proof. Using (2.31) and (3.8), we have for all $X, Y \in \Gamma(TM)$,

$$\overline{g}(\phi X + u(X)N, \phi Y + u(Y)N) = -g(X, Y) + \eta(X)\eta(Y) \tag{3.10}$$

then

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y) - u(Y)\overline{g}(\phi X, N) - u(X)\overline{g}(N, \phi Y) \tag{3.11}$$

On the other hand

$$\overline{g}(\phi X, N) = \overline{g}(\overline{\phi} X - u(X)N, N) = -\frac{1}{2}u(X). \tag{3.12}$$

Using (3.12) in (3.11) gives (3.9) as expected. □

Proposition 3.3. *Setting* $U = \overline{\phi}\ N$ *and* $e = \frac{1}{2}E$*, we have*

$$\phi^2 = I - \eta \otimes e - u \otimes U \tag{3.13}$$

$$u \circ \phi = -\eta = -\overline{g}(\cdot, N) \tag{3.14}$$

$$\eta \circ \phi = -\frac{1}{2}u \tag{3.15}$$

$$\phi(e) = -U, \ \phi(U) = -\frac{1}{2}e. \tag{3.16}$$

From the preceding proposition we deduce

$$\eta(U) = 0, \ \eta(e) = \frac{1}{2} \tag{3.17}$$

$$u(e) = 0, \ u(U) = \frac{1}{2} \tag{3.18}$$

which one can get by direct computations.

Proposition 3.4. *Let* (M, g, N) *be a normalized lightlike hypersurface of* $(\overline{M}, \overline{g})$ *and* ϕ, u *and* η *defined as above, then we have*

$$g(\phi X, Y) = -g(X, \phi Y) - \eta(X)u(Y) - \eta(Y)u(X), \ \forall X, Y \in \Gamma(TM) \tag{3.19}$$

From this proposition, one deduces that

$$\overline{g}(\overline{\phi}X, Y) = -\overline{g}(X, \overline{\phi}Y), \forall X, Y \in \Gamma(TM). \tag{3.20}$$

Proof. The result follows by direct computations using Proposition 3.2. □

Proposition 3.5. *Let* (M, g, N) *be a normalized lightlike hypersurface in* $(\overline{M}, \overline{\phi}, \xi, \overline{\eta}, \overline{g})$ *Then there exists, locally a* $\overline{\phi}$*-basis* $\{e_i, \overline{\phi}e_i, \overline{\phi}E, \overline{\phi}^2 E, \xi\}$ *of* $T\overline{M}$ *where* $\{e_i, \overline{\phi}e_i, \overline{\phi}E\}$ *is pseudo-orthonormal basis of* $S(TM)$.

Proof. On a coordinate neighborhood on $\overline{M}$, we set

$$H = (\overline{\phi}(Rad(TM)))^{\perp} \cap S(TM). \tag{3.21}$$

We choose a $\overline{g}$-unit vector field $e_1 \in H$. Hence by definition of H, we have $e_1 \perp \xi$ and $e_1 \perp \overline{\phi}E$. It follows that $\overline{\phi}(e_1) \perp \xi$ and $\overline{\phi}(e_1) \perp \overline{\phi}(E)$ and futhermore,

we have $g(\overline{\phi}(e_1), \overline{\phi}(e_1)) = -1$.
Next choose another vector field $e_2 \in H$ such that $e_2 \perp e_1$. It follows again that $\overline{\phi}(e_2) \perp \xi$ and we have $e_2 \perp \xi$ and $\overline{\phi}e_2 \perp e_1$, $\overline{\phi}e_2 \perp \xi$, $e_2 \perp \overline{\phi}E$, $\overline{\phi}e_2 \perp \overline{\phi}E$ and $g(\overline{\phi}(e_2), \overline{\phi}(e_2)) = -1$.
Continuing in this way, we obtain an orthonormal basis $(e_i, \overline{\phi}e_i, \overline{\phi}E)$ of S(TM).
We just add the two other orthogonal vector fields in the complementary and orthogonal space of S(TM) in $T\overline{M}$. Namely $\overline{\phi}^2 E = \frac{1}{2}E - N$ and ξ to obtain the quasi orthonormal $\overline{\phi}$-basis of $T\overline{M}$ as stated. □

Example 3.6. Let $(\overline{M} = \mathbb{R}^5, \overline{g} = dx_1 \otimes dx_1 + dx_2 \otimes dx_2 - dx_3 \otimes dx_3 - dx_4 \otimes dx_4 + dx_5 \otimes dx_5)$ be the 5-dimensional flat semi-riemannian manifold with a coordinate system $(x_1, x_2, x_3, x_4, x_5)$.This is an almost paracontact metric manifold with the structure $(\overline{\phi}, \xi, \overline{\eta})$ given by

$$\overline{\phi}(\frac{\partial}{\partial x_5}) = 0, \quad \overline{\phi}(\tfrac{\partial}{\partial x_1}) = \tfrac{\partial}{\partial x_3}, \quad \overline{\phi}(\frac{\partial}{\partial x_3}) = \frac{\partial}{\partial x_1}, \tag{3.22}$$

$$\overline{\phi}(\frac{\partial}{\partial x_2}) = \frac{\partial}{\partial x_4}, \quad \overline{\phi}(\tfrac{\partial}{\partial x_4}) = \tfrac{\partial}{\partial x_2}, \quad \overline{\eta} = dx_5, \tag{3.23}$$

$$\xi = \frac{\partial}{\partial x_5}. \tag{3.24}$$

$M = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 / x_5 = 2x_1 - x_3 - 2x_4\}$ is a normalized lightlike hypersurface of $\overline{M}$.
Indeed the tangent bundle TM of M is spanned by

$$\Big\{V_1 = \frac{\partial}{\partial x_1} + 2\frac{\partial}{\partial x_5}, V_2 = \frac{\partial}{\partial x_2}, V_3 = \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_5}, V_4 = \frac{\partial}{\partial x_4} - 2\frac{\partial}{\partial x_5}\Big\}.$$

The rigged vector field E and transversal vector field are given by

$$E = -2\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3} - 2\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} \tag{3.25}$$

$$N = \frac{\partial}{\partial x_1} + \frac{1}{2}\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} + \frac{1}{2}\frac{\partial}{\partial x_5}. \tag{3.26}$$

The screen distribution $S(TM) = Span\Big\{W_1, W_2, W_3\Big\}$ where

$$W_1 = V_2, W_2 = V_1 + 2V_3, W_3 = -2V_3 + V_4.$$

We also have

$$\begin{aligned} \phi E &= -2\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_1} - 2\frac{\partial}{\partial x_2} \\ &= -V_1 - 2V_2 - 2V_3 \end{aligned} \tag{3.27}$$

From which we get $\phi(N) = -\frac{1}{2}\phi(E) = W_1 + \frac{1}{2}W_2 \in S(TM)$. This means that M is screen semi-invariant as expected.

4. Reduction of the almost paracontact structure to the screen distribution.

In this section, we show that leaves of an integrable screen distribution of a normalized lightlike hypersurface in a paracontact manifold are almost paracontact manifolds.
Let $(\overline{M}, \overline{\phi}, \xi, \overline{\eta}, \overline{g})$ be an almost paracontact metric manifold and $(M, g, S(TM) = \ker\eta)$ be a normalized lightlike hypersurface with ξ transversal to M as above.
Let ϕ be a $(1,1)-$tensor field on M as above i.e.

$$\overline{\phi}X = \phi X + u(X)N, \ \forall\ X \ \in \Gamma(TM). \tag{4.1}$$

We set

$$\phi X = \phi_s X + u_s(X)E, \ \forall\ X \ \in \Gamma(S(TM)) \tag{4.2}$$

where we have used the decomposition (2.3). Then we have the following result.

Theorem 4.1. *If the screen distribution S(TM) is integrable, then any leaf M_s of S(TM) is an almost paracontact metric manifold with the structure $(M_s, \phi_s, \xi_s, \eta_s, g_s)$ such that*

$$\eta_s = u\mid_{S(TM)}, \xi_s = 2U = 2\overline{\phi}N, g_s = g\mid_{M_s}.$$

Proof. We first observe that, $\forall X \in \Gamma(S(TM))$

$$\begin{aligned} u_s(X) &= \overline{g}(\phi X, N) = \overline{g}(\overline{\phi}X, \xi - \frac{1}{2}E) \\ &= -\frac{1}{2}u(X). \end{aligned} \tag{4.3}$$

On the other hand, using $\phi^2(X) = X - \eta(X)e - u(X)U$, we have for any $X \in \Gamma(S(TM))$

$$\phi(\phi_s(X) + u_s(X)E) = X - u(X)U \tag{4.4}$$

which in turn gives

$$\phi_s^2(X) + u_s(\phi_s X)E + u_s(X)\phi_s(E) + u_s(X)u_s(E) = X - u(X)U \tag{4.5}$$

equating the screen and radical parts in (4.5) gives

$$\phi_s^2(X) = X - u(X)U - u_s(X)\phi_s(E) \tag{4.6}$$

$$u_s(\phi_s(X)) = u_s(X)u_s(E) = 0 \tag{4.7}$$

Now using the fact that $\phi E = \overline{\phi}E$ is in S(TM), we deduce that $\phi(E) = \phi_s(E) + u_s(E)E$ implies

$$\phi_s(E) = \phi(E) = \overline{\phi}(E) = -2\overline{\phi}N = -2U \tag{4.8}$$

and that

$$u_s(E) = 0. \tag{4.9}$$

Equation (4.9) justifies why in (4.7), we have $u_s(X) \cdot u_s(E) = 0$. Using (4.8) and (4.3) in (4.6) gives

$$\phi_s^2(X) = X - u(X)(2U), \forall X \in \Gamma(S(TM)). \tag{4.10}$$

Hence the expected structure vector field of M_s is $2U$ poviding that $\phi_s(2U)=0$ and $\eta(2U)=1$ and $\eta_s\circ\phi_s=0$. Indeed we have

$$\eta_s(\xi_s) \;=\; u(2U)=1 \tag{4.11}$$

since $u(U)=\frac{1}{2}$. Now using (4.7) we have $\eta_s\circ\phi_s=u\circ\phi_s=-2u_s\circ\phi_s=0$. Next to show that $\phi_s(\xi_s)=0$, we use

$$\phi(U) \;=\; -\frac{1}{4}E \tag{4.12}$$

from which we deduce that $\phi_s(U)=0$, hence

$$\phi_s(\xi_s) \;=\; \phi_s(2U)=0. \tag{4.13}$$

We now show that the structure $(M_s,\phi_s,\xi_s,\eta_s,g_s)$ is metric. Using

$$g(\phi(X),\phi(Y))=-g(X,Y)+\eta(X)\eta(Y)+u(X)u(Y)$$

and $\eta(X)=0$ for any $X\in\Gamma(S(TM))$, we have

$$g(\phi_s(X)+u_s(X)E,\phi_s(Y)+u_s(Y)E)=-g(X,Y)+u(X)u(Y)$$

which in turn gives

$$g(\phi_s(X),\phi_s(Y))=-g(X,Y)+u(X)u(Y)\forall X,Y\in\Gamma(S(TM))$$

as expected. □

Example 4.2. The Example 3.6 above gives a case of a foliated hypersurface $M=\{(u_1,u_2,u_3,u_4,2u_1-u_3-2u_4),u_i\in\mathbb{R}\}$ for which one finds that $\eta=2du_1-du_3-2du_4$. Hence the paracontact structure reduces to leaves of its screen distribution.

5. Normalized lightlike hypersurfaces in para-Sasakian manifolds.

We now consider the case where $\overline{M}$ is para-Sasakian. We show that the shape operator is expressed in term of the structure tensor field ϕ. We also give necessary and sufficient conditions for M to be totally geodesic and for leaves of an integrable screen distribution to be Para-Sasakian too.

Proposition 5.1. *Let $(\overline{M},\overline{\phi},\overline{\eta},\xi,\overline{g})$ be a para-Sasakian manifold and (M,g) a normalized lightlike hypersurface with ξ as rigging vector field. Then for $X,Y\in\Gamma(TM)$, we have*

$$(\nabla_X\phi)(Y) \;=\; -g(X,Y)e+\eta(Y)X+B(X,Y)U+u(Y)A_N(X) \tag{5.1}$$

$$(\nabla_X u)(Y) \;=\; -g(X,Y)-B(X,\phi Y)-\tau(X)u(Y), \tag{5.2}$$

where $e=\frac{E}{2}$, $U=\overline{\phi}(N)$.

Proof. The proof is straightforward computation using the following equations verified by $\overline{M}$.

$$(\overline{\nabla}_X\overline{\phi})(Y) \;=\; -\overline{g}(X,Y)\xi+\overline{\eta}(Y)X \tag{5.3}$$

$$\overline{\nabla}_X\xi \;=\; -\overline{\phi}(X). \tag{5.4}$$

□

Proposition 5.2. *Under the same hypothesis as in the preceding proposition, we have* $\forall X \in \Gamma(TM)$,

$$\overline{\phi}(X) = A_N(X) + \frac{1}{2}A^*_E(X) + \frac{1}{2}\tau(X)E - \tau(X)N \tag{5.5}$$

$$\overline{\phi}(N) = -\frac{1}{2}A_N(E). \tag{5.6}$$

Proof. For any $X \in \Gamma(TM)$, by (5.4), we have

$$\overline{\nabla}_X N + \frac{1}{2}\overline{\nabla}_X E = -\phi(X) - u(X)N \tag{5.7}$$

then

$$-A_N(X) + \tau(X)N + \frac{1}{2}(\nabla_X E + B(X,E)N) = -\phi(X) - u(X)N,$$

and finally

$$-A_N(X) + \tau(X)N + \frac{1}{2}(-A^*_E(X) - \tau(X)E) = -\phi(X) - u(X)N.$$

Equating the tangent and transversal parts gives

$$u(X) = -\tau(X) \tag{5.8}$$

$$\phi(X) = A_N(X) + \frac{1}{2}A^*_E(X) + \frac{1}{2}\tau(X)E. \tag{5.9}$$

(5.8) and (5.9) gives (5.5).
On the other hand (3.7) gives

$$\overline{\phi}(N) = -\frac{1}{2}\overline{\phi}(E) \in \Gamma(S(TM)).$$

But $\overline{\phi}(E) = \phi(E) = A_N(E)$ by using (5.9). Hence $\overline{\phi}(N) = -\frac{1}{2}A_N(E)$ which is (5.6) as expected. □

Corollary 5.3. *Let* $(\overline{M}, \overline{\phi}, \overline{\eta}, \xi, \overline{g})$ *be a para-Sasakian manifold and* (M, g) *a normalized lightlike hypersurface with* ξ *as rigging vector field.*
(I) The null section E *is geodesic in* (M, ∇) *, i.e.* $\nabla^E_E = 0$.
(II) The screen distribution $ST(M)$ *cannot be conformal.*
(III) The induced connection ∇ *doest not coincide with any of the 'α-connection* ∇^α *being the Riemannian connection of the metric* $g + \alpha\eta \otimes \eta$ *as defined in* [12] *.*

Proof. (I): From (5.9) it follows that $\phi(E) = A_N(E) + \frac{1}{2}\tau(E)E$ which gives $\tau(E) = 0$, since $\phi(E) = \bar{\phi}(E) = A_N(E)$.
(II): If the screen distribution is conformal then there exists $\alpha \in C^\infty(M)$ such that $A_N = \alpha A^\star_E$. Then we have $\bar{\phi}(E) = A_N(E) = \alpha A^\star_E(E) = 0$. Which is a contradiction since we have $\overline{g}(\overline{\phi}E, \overline{\phi}E) = 1$.
(III): Using the similar reasoning as in the proof of Theorem 4.1 in [4] one shows that $\nabla = \nabla^\alpha$ iff $A^\star_E = \alpha A_N$ and $d\alpha + 2\alpha\tau = 0$. This would also lead to the same contradiction as above, namely $\bar{\phi}(E) = 0$. □

The distribution $\overline{\phi}(Rad(TM))$ is nondegenerate since $\overline{g}(\overline{\phi}E, \overline{\phi}E) = 1$, then we can define the unique nondegenerate distribution D_0 by

Definition 5.4.

$$S(TM) \quad = \quad D_0 \perp \overline{\phi}(Rad(TM)). \tag{5.10}$$

This leads to the following decomposition

$$TM \quad = \quad \{\overline{\phi}(Rad(TM) \perp D_0\} \perp Rad(TM). \tag{5.11}$$
$$T\overline{M} \quad = \quad \overline{\phi}(Rad(TM)) \perp D_0 \perp \{Rad(TM) \oplus ltr(TM)\}. \tag{5.12}$$

It is easy to prove the following

Proposition 5.5. *D_0 is $\overline{\phi}$-invariant.*

Now we set

$$D \quad = \quad Rad(TM) \perp D_0 \text{ and } D' = \overline{\phi}(Rad(TM)). \tag{5.13}$$

and it follows that

$$T(M) \quad = \quad D \oplus D' \tag{5.14}$$

Theorem 5.6. *Let $(\overline{M}, \overline{\phi}, \xi, \overline{\eta}, \overline{g})$ be a (2n+1)-para-Sasakian manifold and (M, ϕ, U, η, g) be a normalized lightlike hypersurface of $(\overline{M}, \overline{\phi}, \xi, \overline{\eta}, \overline{g})$.*

Then M is totally geodesic if and only if $\forall X \in \Gamma(TM)$, $\forall Y \in \Gamma(D)$,

$$(\nabla^{\phi}_X)Y \quad = \quad -g(X,Y)e + \eta(Y)X \tag{5.15}$$
$$\frac{1}{2}A_N(X) \quad = \quad g(X,U)e - \frac{1}{2}\nabla^e_X - \phi(\nabla^U_X). \tag{5.16}$$

Proof. Assume that M is totally geodesic. By (5.1), we have for $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$

$$(\nabla^{\phi}_X)Y = -g(X,Y)e + \eta(Y)X,$$

since $u(Y) = \overline{g}(\overline{\phi}(Y), E) = -\overline{g}(Y, \overline{\phi}E) = 0$. On the other hand

$$(\nabla^{\phi}_X)U = -g(X,U)e + \eta(U)X + u(U)A_N(X),$$

then

$$(\nabla^{\phi(U)}_X) - \phi(\nabla^U_X) = -g(X,U)e + \frac{1}{2}A_N(X),$$

since $u(U) = \frac{1}{2}, \eta(U) = 0$. Then finally

$$\frac{1}{2}A_N(X) = g(X,U)e - \frac{1}{2}\nabla^e_X - \phi(\nabla^U_X).$$

Conversely, let $X, Y \in \Gamma(TM)$. By (5.15), $\exists\, \alpha \in C^{\infty}(M)$ such that $Y = Y_D + \alpha U$. We have

$$B(X,Y) \quad = \quad B(X,Y_D) + \alpha B(X,U). \tag{5.17}$$

By (5.1) and for $Y = Y_D$, we have

$$(\nabla^{\phi}_X)Y_D \quad = \quad -g(X,Y_D)e + \eta(Y_D)X + B(X,Y_D)U + u(Y_D)A_N(X). \tag{5.18}$$

By (5.15), we have

$$(\nabla^{\phi}_X)Y_D \quad = \quad -g(X,Y_D)e + \eta(Y_D)X. \tag{5.19}$$

Equating (5.18) and (5.19) gives

$$B(X, Y_D)U + u(Y_D)A_N(X) = 0,$$

which gives $B(X, Y_D) = 0$ since $u(Y_D) = 0$.
On the other hand, by (5.1), we have

$$\begin{aligned}(\nabla^{\phi}_X)U &= -g(X,U)e + \eta(U)X + B(X,U)U + u(U)A_N(X)\\ &= -g(X,U)e + B(X,U)U + \frac{1}{2}A_N(X).\end{aligned}$$

$$\nabla^{\phi U}_X - \phi(\nabla^U_X) = -g(X,U)e + B(X,U)U + \frac{1}{2}A_N(X),$$

and by using (5.16), we have

$$\nabla^{\phi U}_X - \phi(\nabla^U_X) = -g(X,U)e + B(X,U)U + g(X,U)e - \frac{1}{2}\nabla^e_X - \phi(\nabla^U_X),$$

which gives $B(X, U) = 0$. □

The following theorem gives a condition for the leaves of integrable screen distribution to be para-Sasakian.

Theorem 5.7. *Let $(\overline{M}, \overline{\phi}, \xi, \overline{\eta}, \overline{g})$ be a (2n+1)-para-Sasakian manifold and (M, ϕ, U, η, g) be a normalized lightlike hypersurface of $(\overline{M}, \overline{\phi}, \xi, \overline{\eta}, \overline{g})$. Assume that the screen distribution $S(TM)$ is integrable. Then the leaves M_s of $S(TM)$ have a para-Sasakian structure if and only if*

$$A^*_E(PX) = \phi_s(PX) - PX, \ \forall \ X \in \Gamma(TM) \tag{5.20}$$

Proof. Since $\overline{M}$ is para-Sasakian manifold, we have for all $X, Y \in \Gamma(TM)$

$$\begin{aligned}(\overline{\nabla}^{\overline{\phi}}_{PX})PY &= -\overline{g}(PX, PY)\xi\\ &= -\overline{g}(PX, PY)e - \overline{g}(PX, PY)N.\end{aligned} \tag{5.21}$$

On the other hand, we have

$$(\overline{\nabla}^{\overline{\phi}}_{PX})PY = \overline{\nabla}^{\overline{\phi}(PY)}_{PX} - \overline{\phi}(\overline{\nabla}^{PY}_{PX}). \tag{5.22}$$

A straightforward computation gives

$$\begin{aligned}\overline{\nabla}^{\overline{\phi}(PY)}_{PX} &= \overline{\nabla}^{\phi(PY)+u(PY)N}_{PX}\\ &= \nabla^{\phi(PY)}_{PX} + B(PX, \phi(PY))N + PX(u(PY))N + u(PY)\overline{\nabla}^N_{PX}\\ &= \nabla^{*\phi_s(PY)}_{PX} + C(PX, \phi_s(PY))E + PX(u_s(PY))E\\ &- u_s(PY)A^*_E(PX) - u_s(PY)\tau(PX)E\\ &+ B(PX, \phi_s(PY))N + PX(u(PY))N\\ &- u(PY)A_N(PX) + u(PY)\tau(PX)N,\end{aligned} \tag{5.23}$$

$$\begin{aligned}\overline{\phi}(\overline{\nabla}^{PY}_{PX}) &= \overline{\phi}(\nabla^{PY}_{PX} + B(PX, PY)N)\\ &= \phi_s(\nabla^{*PY}_{PX}) + u_s(\nabla^{*PY}_{PX})E + C(PX, PY)\phi E\\ &+ u(\nabla^{PY}_{PX})N + B(PX, PY)\overline{\phi}N.\end{aligned} \tag{5.24}$$

Equations (5.23) and (5.24) in (5.22) give

$$\begin{aligned}(\overline{\nabla}^{\overline{\phi}}_{PX})PY &= (\nabla^{*\phi_s}_{PX})(PY) + \{(\nabla^{*u}_{PX})(PY) + C(PX,PY)\phi E - u_s(PY)\tau(X)\}E \\ &+ \{(\nabla^{u}_{PX})(PY) + B(PX,\phi_s(PY)) + u(PY)\tau(PX)\}N \\ &- B(PX,PY)\overline{\phi}N - C(PX,PY)\phi E \\ &- A_N(PX)u(PY) - u_s(PY)A^*_E(PX). \end{aligned} \tag{5.25}$$

Equating (5.25) and (5.21) give

$$(\nabla^{u}_{PX})(PY) = -g(PX,PY) - B(PX,\phi_s(PY)) - u(PY)\tau(PX) \tag{5.26}$$

$$(\nabla^{*u}_{PX})(PY) = -\frac{1}{2}g(PX,PY) - C(PX,\phi_s(PY)) + u_s(PY)\tau(PX) \tag{5.27}$$

$$\begin{aligned}(\nabla^{*\phi_s}_{PX})(PY) &= B(PX,PY)\overline{\phi}N + C(PX,PY)\phi E \\ &+ A_N(PX)u(PY) + u_s(PY)A^*_E(PX)\end{aligned} \tag{5.28}$$

Equation (5.26) is relation (5.2). By (5.28), we have

$$\begin{aligned}(\overline{\nabla}^{*\phi_s}_{PX})PY &= B(PX,PY)U - 2C(PX,PY)U \\ &+ u(PY)A_N(PX) - \frac{1}{2}u(PX)A^*_E(PX) \\ &= -(-\frac{1}{2}B(PX,PY) + C(PX,PY))(2U) \\ +u \ &(PY)(A_N(PX - \frac{1}{2}A^*_E(PX))) \\ &= -(-\frac{1}{2}g(-A_N(PX),PY) + g(A_N(PX),PY))(2U) \\ &+ u(PY)(A_N(PX - \frac{1}{2}A^*_E(PX))) \\ &= -g(A_N(PX) - \frac{1}{2}A^*_E(PX),PY))(2U) \\ &+ u(PY)(A_N(PX - \frac{1}{2}A^*_E(PX)))\end{aligned} \tag{5.29}$$

If $A^*_E(PX) = \phi_s(PX) - PX$ then $A_N(PX) - \frac{1}{2}A^*_E(PX) = PX$ and by (5.29) we have

$$(\overline{\nabla}^{*\phi_s}_{PX})PY = -g_s(PX,PY)(2U) + u(PY)PX,$$

then $(M_s,\phi_s,\xi_s,\eta_s,g_s)$ is a para-Sasakian manifold.
Conversely if $(M_s,\phi_s,\xi_s,\eta_s,g_s)$ is para-Sasakian i.e.

$$(\overline{\nabla}^{*\phi_s}_{PX})PY = -g_s(PX,PY)(2U) + u(PY)PX \tag{5.30}$$

and then equating (5.29) and (5.30) gives

$$A_N(PX) - \frac{1}{2}A^*_E(PX) = PX.$$

□

Example 5.8. Let $\overline{M} = \mathbb{R}^5$ endowed with the metric $\overline{g}$ given by its following matrix in the cartesian coordinate system $(x_1, x_2, x_3, x_4, x_5)$:

$$[\overline{g}] \;=\; \frac{1}{4}\begin{pmatrix} 1+x_3^2 & -x_3x_4 & 0 & 0 & -x_3 \\ -x_3x_4 & -1+x_4^2 & 0 & 0 & x_4 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -x_3 & x_4 & 0 & 0 & 1 \end{pmatrix} \tag{5.31}$$

Then the Christoffel coefficients of the Levi-Civita connection of $\overline{g}$ are given by:

$$\begin{gathered} \Gamma_{13}^1 = \Gamma_{14}^2 = \Gamma_{12}^4 = -\Gamma_{35}^5 = \tfrac{1}{2}\Gamma_{11}^3 = \tfrac{1}{2}x_3 \\ \Gamma_{23}^1 = \Gamma_{24}^2 = \Gamma_{12}^3 = -\Gamma_{45}^5 = \tfrac{1}{2}\Gamma_{22}^4 = -\tfrac{1}{2}x_4 \\ \Gamma_{35}^1 = \Gamma_{45}^2 = \Gamma_{15}^3 = -\Gamma_{25}^4 = -\tfrac{1}{2} \\ \Gamma_{13}^5 = \tfrac{1}{2}(x_3^2-1), \Gamma_{14}^5 = \Gamma_{23}^5 = -\tfrac{1}{2}x_3x_4, \Gamma_{24}^5 = \tfrac{1}{2}(x_4^2+1), \end{gathered}$$

and $\Gamma_{ij}^k = 0$ for any other triple (i, j, k).
We define an almost paracontact structure $(\overline{\phi}, \xi, \overline{\eta})$ on $\overline{M}$ by:
$\overline{\phi}(\frac{\partial}{\partial x_5}) = 0, \overline{\phi}(\frac{\partial}{\partial x_1}) = -\frac{\partial}{\partial x_3}, \overline{\phi}(\frac{\partial}{\partial x_3}) = -\frac{\partial}{\partial x_1} - x_3\frac{\partial}{\partial x_5}, \overline{\phi}(\frac{\partial}{\partial x_2}) = -\frac{\partial}{\partial x_4}, \overline{\phi}(\frac{\partial}{\partial x_4}) = -\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_5}, \overline{\eta} = \frac{1}{2}(x_3dx_1 - x_4dx_2 - dx_5), \xi = -2\frac{\partial}{\partial x_5}$. Then one shows that $(\overline{M} = \mathbb{R}^5, \overline{\phi}, \xi, \overline{\eta}, \overline{g})$ is a para-Sasakian manifold.

Let us consider the hypersurface $M = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 / x_5 = \frac{1}{2}(\arcsin(x_3) + x_3\sqrt{1+x_3^2} + x_4^2)\}$ of $\overline{M}$.
Around any point $m = (u_1, u_2, u_3, u_4, \frac{1}{2}(\arcsin(u_3) + u_3\sqrt{1+u_3^2} + u_4^2)) \in M$, we consider the local coordinates system (u_1, u_2, u_3, u_4) and set: $V_1 = \frac{\partial}{\partial u_1}$, $V_2 = \frac{\partial}{\partial u_2}$, $V_3 = \frac{\partial}{\partial u_3}$, $V_4 = \frac{\partial}{\partial u_4}$.
The tangent bundle TM at m is spanned by V_1, V_2, V_3, V_4 and one has

$$\Big\{V_1 = \frac{\partial}{\partial x_1}, V_2 = \frac{\partial}{\partial x_2}, V_3 = \frac{\partial}{\partial x_3} + \sqrt{1+x_3^2}\frac{\partial}{\partial x_5}, V_4 = \frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_5}\Big\}. \tag{5.32}$$

It is easy the check that ξ is transversal vector field to M and that the latter is a lightlike hypersurface with the radical distribution $Rad(TM)$ and the lightlike transversal distribution $ltr(TM)$ given by

$$\begin{aligned} Rad(TM) &= span\Big\{E = -2u_3V_1 - 2u_4V_2 - 2\sqrt{1+u_3^2}V_3 + 2u_4V_4\Big\} \\ &= span\Big\{-2x_3\frac{\partial}{\partial x_1} - 2x_4\frac{\partial}{\partial x_2} - 2\sqrt{1+x_3^2}\frac{\partial}{\partial x_3} \\ &\qquad +2x_4\frac{\partial}{\partial x_4} + 2(x_4^2 - x_3^2 - 1)\frac{\partial}{\partial x_5}\Big\} \end{aligned}$$

$$ltr(TM) = span\Big\{N = x_3\frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial x_2} + \sqrt{1+x_3^2}\frac{\partial}{\partial x_3} - x_4\frac{\partial}{\partial x_4} + (x_3^2 - x_4^2 - 1)\frac{\partial}{\partial x_5}\Big\}.$$

The 1-form η induced on M by $\overline{\eta}$ is given by

$$\eta = \frac{1}{2}\Big(u_3du_1 - u_4du_2 - \sqrt{1+u_3^2}du_3 - u_4du_4\Big) \tag{5.33}$$

From which one gets the screen distribution $S(TM)$ which is not integrable and spanned by

$$W_1 = V_1 + \frac{u_3}{\sqrt{1+u_3^2}}V_3, W_2 = V_2 - \frac{u_4}{\sqrt{1+u_3^2}}V_3, W_3 = V_4 - \frac{u_4}{\sqrt{1+u_3^2}}V_3.$$

We also compute

$$\bar{\phi}(E) = 2\alpha W_1 - 2u_4W_2 + 2u_3W_3, \quad \alpha = \sqrt{1+u_3^2}$$

and deduce that the distribution D_0 is spanned by

$$(\alpha u_3u_4 + u_4\alpha^2 + u_4^3 + u_3u_4^2)W_1 - (\alpha - u_3u_4^2 - u_4u_3^2)W_2 \quad and$$

$$(\alpha u_3u_4 + u_4^3 - u_3\alpha^2 + u_3u_4^2)W_1 - (\alpha - u_3u_4^2 - u_4u_3^2)W_3.$$

6. Integrability of some distributions of TM.

In this section we assume that $\overline{M}$ is para-Sasakian and investigate the conditions under which the subdistributions D_0 and D of TM introduced above are integrable distributions.

6.1. **Distribution D_0.** We consider the distribution D_0 defined in (5.10) as orthogonal complementary of $\{\overline{\phi}(Rad(TM))\}$ in $S(TM)$and we set

$$\beta = \overline{\phi}(Rad(TM)) \perp Rad(TM). \tag{6.1}$$

For all $X \in \Gamma(TM), Y \in \Gamma(D_0)$ and $Z \in \Gamma(\beta)$, following the decomposition (5.11) we have

$$\nabla_X^Y = \mathring{\nabla}_X^Y + \mathring{h}(X,Y) \tag{6.2}$$

$$\nabla_X^Z = -\mathring{A}_Z(X) + \nabla_X^\beta Z \tag{6.3}$$

where $\mathring{\nabla}$ is a linear connection on D_0, $\mathring{h} : \Gamma(TM)\times\Gamma(D_0) \longrightarrow \Gamma(\beta)$ is an $C^\infty(M)$-bilinear map, $\mathring{A}_Z$ is $\Gamma(D_0)$-valued $C^\infty(M)$-linear operator on $\Gamma(TM)$ and ∇^β is a linear connection on β.

Lemma 6.1. *Let (M,g) be a normalized lightlike hypersurface of a para-Sasakian manifold $(\overline{M},\overline{\phi},\xi,\overline{\eta},\overline{g})$, then*

$$\overline{g}((\overline{\nabla}_X^{\overline{\phi}})Y,E) = -\overline{g}(X,Y), \quad \forall X,Y \in \Gamma(TM). \tag{6.4}$$

Proof.

$$\begin{aligned}\overline{g}((\overline{\nabla}_X^{\overline{\phi}})Y,E) &= \overline{g}(-\overline{g}(X,Y)\xi + \eta(Y)X,E)\\ &= -\overline{g}(X,Y)\overline{g}(\xi,E) + \eta(Y)\overline{g}(X,E)\\ &= -\overline{g}(X,Y).\end{aligned}$$

□

According to decomposition (6.1), we set

$$\mathring{h}(X,Y) \;=\; \alpha_1(X,Y)\overline{\phi}E + \alpha_2(X,Y)E, \tag{6.5}$$

For all $X \in \Gamma(TM), Y \in \Gamma(D_0$, where we have

$$\begin{aligned} \alpha_1(X,Y) &= \overline{g}(\mathring{h}(X,Y),\overline{\phi}E) \\ \alpha_2(X,Y) &= \overline{g}(\mathring{h}(X,Y),N). \end{aligned} \tag{6.6}$$

By (6.2), one has

$$\nabla_X^Y \;=\; \mathring{\nabla}_X^Y + \alpha_1(X,Y)\overline{\phi}E + \alpha_2(X,Y)E. \tag{6.7}$$

Then

$$\begin{aligned} g(\nabla_X^Y,\overline{\phi}E) &= \overline{g}(\mathring{\nabla}_X^Y + \alpha_1(X,Y)\overline{\phi}E + \alpha_2(X,Y)E,\overline{\phi}E) \\ &= \alpha_1(X,Y), \end{aligned} \tag{6.8}$$

since $g(\phi E,\phi E)=1$. On the other hand we can express α_1 as follows:

$$\begin{aligned} \alpha_1(X,Y) = g(\nabla_X^Y,\overline{\phi}E) &= -\overline{g}(\overline{\phi}\nabla_X^Y,E) = -\overline{g}(-(\overline{\nabla}_X^{\overline{\phi}})Y + \overline{\nabla}_X^{\overline{\phi}Y},E) \\ &= -\overline{g}(X,Y) + \overline{g}(\overline{\phi}Y,\overline{\nabla}_X^E),\ \textit{since}\ (\overline{\nabla}^g = 0) \\ &= -\overline{g}(X,Y) + \overline{g}(\overline{\phi}Y,-A_E^*(X) - \tau(X)E) \\ &= -\overline{g}(X,Y) - B(X,\overline{\phi}Y) \end{aligned}$$

since $g(A_E^*(X),Y) = B(X,Y)$ and D_0 is $\overline{\phi}$-invariant. We also compute $\alpha_2(X,Y) = g(\nabla_X^Y,N) = g(\nabla_X^{*Y} + C(X,Y)E,N) = C(X,Y)$. Then

$$\mathring{h}(X,Y) \;=\; -(g(X,Y) + B(X,\overline{\phi}Y))\overline{\phi}E + C(X,Y)E \tag{6.9}$$

and

$$\nabla_X^Y \;=\; \mathring{\nabla}_X^Y - (g(X,Y) + B(X,\overline{\phi}Y))\overline{\phi}E + C(X,Y)E. \tag{6.10}$$

Theorem 6.2. *Let (M,g) be a normalized lightlike hypersurface of a para-Sasakian manifold $(\overline{M},\overline{\phi},\xi,\overline{\eta},\overline{g})$.*
Then the distribution D_0 is integrable if and only if

$$C(X,Y) = C(Y,X), B(X,\overline{\phi}Y) = B(\overline{\phi}X,Y), \forall X,Y \in \Gamma(D_0). \tag{6.11}$$

Proof. Let $X,Y \in \Gamma(D_0)$. Since ∇ is a torsion free connection, we have

$$\begin{aligned} [X,Y] &= \nabla_X^Y - \nabla_Y^X \\ &= \mathring{\nabla}_X^Y - \mathring{\nabla}_Y^X + B(\overline{\phi}X,Y) - B(X,\overline{\phi}Y))\overline{\phi}E \\ &+ (C(X,Y) - C(Y,X))E. \end{aligned} \tag{6.12}$$

from which the statement in the theorem follows. □

Corollary 6.3. *Let $(\overline{M},\overline{\phi},\xi,\overline{\eta},\overline{g})$ be a para-Sasakian manifold and (M,g) a normalized lightlike hypersurface of $(\overline{M},\overline{\phi},\xi,\overline{\eta},\overline{g})$. Then $\mathring{h}$ is symmetric on D_0 if and only if D_0 is integrable.*

Theorem 6.4. *Let (M,g) be a normalized lightlike hypersurface of a para-Sasakian manifold $(\overline{M},\overline{\phi},\xi,\overline{\eta},\overline{g})$ such that D_0 is integrable and*

$$C(X,Y) \;=\; C(\overline{\phi}X,\overline{\phi}Y), \tag{6.13}$$

for any $X,Y\in\Gamma(D_0)$. Then $trace(\mathring{h}) = -2(n-1)\overline{\phi}E$

Proof. Using decomposition (5.11), we find that rank $D_0 = 2n-2$. Now we consider an orthonormal $\overline{\phi}-$basis $\{e_i,\overline{\phi}e_i\}$ of D_0, $i=1,2\cdots n-1$. We have

$$trace(\mathring{h}) = \sum_{i=1}^{n-1}\mathring{h}(e_i,e_i) - \sum_{i=1}^{n-1}\mathring{h}(\overline{\phi}e_i,\overline{\phi}e_i).$$

Hence, using (6.9) we get

$$\mathring{h}(e_i,e_i) - \mathring{h}(\overline{\phi}e_i,\overline{\phi}e_i) = -2\overline{\phi}E - (C(\overline{\phi}e_i,\overline{\phi}e_i) - C(e_i,e_i))E = -2\overline{\phi}E$$

which leads to the result as stated □

Now using decomposition (5.12) and putting

$$\nu = \overline{\phi}(Rad(TM)) \perp \{Rad(TM)\oplus ltr(TM)\}$$

for any $X\in\Gamma(TM), Y\in D_0$ and $Z\in\Gamma(\nu)$, we have

$$\begin{aligned}\overline{\nabla}_X^Y &= \widehat{\nabla}_X^Y PY + \widehat{h}(X,Y)\\ \overline{\nabla}_X^Z &= -\widehat{A}_Z(X) + \nabla_X^\nu Z\end{aligned}$$

where $\widehat{\nabla}$ is a linear connection on D_0, $\widehat{h}:\Gamma(TM)\times\Gamma(D_0)\longrightarrow\Gamma(\nu)$ is an $C^\infty(M)$-bilinear map, $\widehat{A}_Z$ is an $C^\infty(M)$-linear operator on $\Gamma(TM)$ with values in D_0 and ∇^ν is a linear connection on ν.
We set

$$\widehat{h}(X,Y) = \beta_1(X,Y)\overline{\phi}E + \beta_2(X,Y)E + \beta_3(X,Y)N,$$

for any $X,Y\in\Gamma(D_0)$. Then we easily check that

$$\begin{aligned}\beta_1(X,Y) &= g(\widehat{h}(X,Y),\overline{\phi}E) = -\Big(g(X,Y)+B(X,\overline{\phi}Y)\Big)\overline{\phi}E\\ \beta_2(X,Y) &= g(\widehat{h}(X,Y),N) = C(X,Y)\\ \beta_3(X,Y) &= g(\widehat{h}(X,Y),E) = B(X,Y).\end{aligned}$$

$$\widehat{h}(X,Y) = -\Big(g(X,Y)+B(X,\overline{\phi}Y)\Big)\overline{\phi}E + C(X,Y)E + B(X,Y)N,$$

$$\overline{\nabla}_X^Y = \widehat{\nabla}_X^Y - \Big(g(X,Y)+B(X,\overline{\phi}Y)\Big)\overline{\phi}E + C(X,Y)E + B(X,Y)N.$$

Theorem 6.5. *Let (M,g) be a normalized lightlike hypersurface of a para-Sasakian manifold $(\overline{M},\overline{\phi},\xi,\overline{\eta},\overline{g})$ as above. Assume that the distribution D_0 is integrable and*

$$\begin{aligned}C(X,Y) &= C(\overline{\phi}X,\overline{\phi}Y)\\ B(X,Y) &= B(\overline{\phi}X,\overline{\phi}Y)\end{aligned}$$

for any $X, Y \in \Gamma(D_0)$. Then on D_0 we have

$$trace(\widehat{h}) = -2(n-1)\overline{\phi}E.$$

Proof. We consider an orthonormal $\overline{\phi}-$basis $\{e_i, \overline{\phi}e_i\}$ of D_0, $i = 1, 2 \cdots n-1$. We easily see that

$$trace(\widehat{h}) = trace(\mathring{h}) + \sum_{i=1}^{n-1} \Big(- B(\overline{\phi}e_i, \overline{\phi}e_i) + B(e_i, e_i)\Big) N$$

which in turn gives the result as stated □

6.2. **Distribution D.** We consider the distribution D, defined by

$$D = Rad(TM) \perp D_0. \tag{6.14}$$

and recall

$$D' = \overline{\phi}(Rad(TM)), \quad TM = (D_0 \perp D') \perp Rad(TM) = D \perp D'.$$

For $X \in \Gamma(TM), Y \in \Gamma(D)$ and $Z \in \Gamma(D')$, we have

$$\nabla_X^Y = \widetilde{\nabla}_X^Y + \widetilde{h}(X, Y) \tag{6.15}$$

$$\nabla_X^Z = -\widetilde{A}_Z(X) + \nabla_X^{D'} Z \tag{6.16}$$

where $\widetilde{\nabla}$ is a linear connection on D, $\widetilde{h} : \Gamma(TM) \times \Gamma(D) \longrightarrow \Gamma(D')$ is an $C^\infty(M)$-bilinear map, $\widetilde{A_Z}$ is an $C^\infty(M)$-linear operator with values in D and $\nabla^{D'}$ is a linear connection on D'. We set

$$\widetilde{h}(X, Y) = \sigma_1(X, Y)\overline{\phi}E. \tag{6.17}$$

Then we have

$$\begin{aligned} \sigma_1(X, Y) &= \bar{g}(\nabla_X^Y, \bar{\phi}E) = \bar{g}(\bar{\nabla}_X^Y - B(X, Y)N, \bar{\phi}E) = -\bar{g}(\bar{\phi}(\bar{\nabla}_X^Y), E) \\ &= \bar{g}(-(g(X, Y)\xi + \bar{\eta}(Y)X, E) - \bar{g}(\bar{\nabla}_X^{\bar{\phi}(Y)}, E) \\ &= -(g(X, Y) + B(X, \overline{\phi}Y)) \end{aligned} \tag{6.18}$$

and

$$\nabla_X^Y = \widetilde{\nabla}_X^Y - (g(X, Y) + B(X, \overline{\phi}Y))\overline{\phi}E. \tag{6.19}$$

Hence we have the following result:

Proposition 6.6. *Let (M, g) be a normalized lightlike hypersurface of a para-Sasakian manifold $(\overline{M}, \overline{\phi}, \xi, \overline{\eta}, \overline{g})$ as above. Then the distribution D is integrable if and only if*

$$B(X, \overline{\phi}Y) = B(\overline{\phi}X, Y), \forall X, Y \in \Gamma(D) \tag{6.20}$$

Proof. The result stated follows from Frobenius criterium for integrability and the decomposition $TM = D \perp D'$ since for all $X, Y \in \Gamma(D)$, we have

$$[X, Y] = \widetilde{\nabla}_X^Y - \widetilde{\nabla}_Y^X + (B(\overline{\phi}X, Y) - B(X, \overline{\phi}Y))\overline{\phi}E.$$

□

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[1] Department of Mathematics and Computer Science, University of Douala, PO Box 24157 Douala, Cameroon.

Email address: fngakeu@yahoo.fr

[2] Department of Mathematics, University of Yaoundé I, PO Box 812, Yaoundé, Cameroon

Email address: armeldjomako@yahoo.fr

[3] Department of Mathematics, University of Yaoundé I, PO Box 812, Yaoundé, Cameroon

Email address: bitjongndombol@yahoo.fr