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STRONGLY QUASILINEAR ELLIPTIC SYSTEMS IN SOBOLEV SPACES WITH PERTURBATIONS THROUGH YONG MEASURES

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Abstract. We use the Galerkin method to obtain weak solutions for strongly quasilinear elliptic systems in the Sobolev space $W_0^{1,p}(\Omega;\mathbb{R}^m)$ of the form

$$G\varpi(x) = f(x, \varpi, D\varpi) \text{ in } \Omega, \text{ with } \varpi(x) = 0 \text{ on } \partial\Omega,$$

where the term $G\varpi(x)$ includes perturbations. Our approach involves the use of Young measures to obtain the desired results.

1. Introduction and preliminaries

Our aim is to study a variational approach to obtain weak solutions $\varpi \in \mathbb{R}^m$ ($m \in \mathbb{N}^*$) for the following class of quasilinear elliptic system

$$\begin{aligned} G\varpi(x) &= f(x, \varpi, D\varpi), && \text{in } \Omega \\ \varpi(x) &= 0, && \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

where $G\varpi(x) = -\text{div}\big(\sigma(x, \varpi, D\varpi) + \Theta(\varpi)\big) + g\big(x, \varpi - \Upsilon(\varpi)\big) + \alpha(x)|\varpi|^{p-2}\varpi$. Here, $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) is a bounded open subset, $\partial\Omega$ denotes its boundary, and the term f is an element of $W^{-1,p'}(\Omega;\mathbb{R}^m)$, which is the dual space of $W_0^{1,p}(\Omega;\mathbb{R}^m)$, where $1 < p < \infty$. $\mathbb{M}^{m\times n}$ refers to the real vector space of $m \times n$ matrices endowed with the inner product $A_{ij}B_{ij}$ and conventional summation. For the functions σ, g, Θ, Υ, α, and f, specific conditions need to be met, which we will elaborate on later.

As a field of application of nonlinear PDEs, we first find according to a chronological graduation the study of wave propagation that has attributed an important advance to its development, then we would look at the equations related to biological and chemical phenomena, and then we would continue to work with equations related to nonlinear optics, acoustics, solid mechanics, fluid dynamics, quantum field theory, plasma physics, image processing and engineering. The study of each problem in this field requires an appropriate approach. Consequently, the methods used are diverse and depend on the working spaces and the nature of the problems to be addressed, as well as the data and conditions that

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must be satisfied. For example, in [3], the authors used truncation techniques and the generalized monotonicity method in functional spaces to prove the existence of renormalized solutions for $L^\infty-$data. In [16], the authors used the topological degree theory of Berkovits and Mustonen to study the existence of weak solutions for a class of nonlinear parabolic problems. Lastly, in [1], the authors aim to prove the existence of an entropy solution for a unilateral problem with measure data in Musielak-Orlicz spaces.

Our paper has the advantage that we address a class of problems where for functions σ, monotonicity in the variables $(\varpi, \xi) \in \mathbb{R}^m \times \mathbb{M}^{m\times n}$ or strict monotonicity with respect to the 3rd variable that satisfy only (C0)− (C2) do not generally work for the standard methods applied to monotone operators cited in [25, 24, 13, 9, 23], developed by Višik, Minty, Browder, Brezis and Lions and those cited in [19, 10, 26, 15], developed by Hess, (Brézis and Browder), Webb and El Moumni. It is unnecessary to apply the standard pointwise monotonicity requirement for σ, therefore we'll use a more integrated, weaker form of monotonicity known as quasimonotonicity. However, there is no requirement to employ for σ the standard pointwise monotonicity condition, and instead, we make use of a weaker integrated form of monotonicity known as quasimonotonicity (i.e "monotonicity in integrated form″), for more details we refer to [30, 14].

In the context of Sobolev spaces, Norbert Hungerbühler treated in [21] the following system

$$\begin{cases} -\operatorname{div}\sigma(x, u(x), Du(x)) & = f \quad \text{in } \Omega, \\ u & = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

with $f \in W^{-1,p'}(\Omega)$ depends only on the first variable. To arrive at his result, the author employed Young measures and weak monotonicity over σ.
Augsburger and Hungerbuhler have treated a system somewhat generalizing that of [4], which corresponds to a diffusion problem whose resolution is done under conditions for σ similar to those quoted in the problem (1.2), and proved that it has a weak solution. Their system is as follows

$$\begin{cases} -\operatorname{div}\sigma(x, u(x), Du(x)) = & v(x) + f(x, u) + \operatorname{div} g(x, u) \quad \text{in } \Omega, \\ u = & 0 \quad \text{on } \partial\Omega. \end{cases}$$

Bensoussan, Boccardo and Murat have shown in [8] that the nonlinear elliptic equation $A(u) + g(x, u, Du) = h(x)$ has a solution, with A being a Leray-Lions operator defined on $W_0^{1,p}(\Omega)$ and with values in $W^{-1,p'}(\Omega)$, g is a nonlinear term verifying the natural growth with respect to Du, but not necessarily with respect to u, as well as the sign condition $g(x, u, \xi)u \geq 0$, and the right-hand side $h \in W^{-1,p'}(\Omega)$.
Some works have dealt with systems using Young measures, in which the right-hand side contains data f dependent on u and Du. One can mention [6], where the authors studied the problem

$$\begin{cases} -\operatorname{div}\sigma(x, u, Du) = & f(x, u, Du) \quad \text{in } \Omega, \\ u = & 0 \quad \text{on } \partial\Omega, \end{cases}$$

where the solution is proven within the framework of Orlicz-Sobolev spaces, while limiting to weak monotonicity conditions.

Inspired by previous research, in particular [8, 21], we are interested in proving the existence of solution for the problem (1.1) in the context of the weak topology and in Sobolev spaces $W_0^{1,p}(\Omega, \mathbb{R}^m)$ while trying to generalize them. To achieve our main result, we use Young measures theory to find solutions under refined conditions, for weak monotonicity assumptions, and for both standard and nonstandard growth conditions. This notion of Young measures combined with Galerkin's method allows to show the required compactness of approximate solutions.

This paper is composed of six sections. In the second section, we focus on the spaces used, as well as their main properties and we review some results and properties concerning the Young measures. In the third section, we give the conditions and the main results concerning the weak solutions of our problem. In the fourth section, we construct a sequence (ϖ_j) of approximate solutions using Galerkin's method and some preliminary estimates. In the fifth section, we give the necessary properties concerning the sequence constructed in section 4, and we show some useful lemmas for the proof of the main theorem. While the sixth section is reserved to prove the existence of weak solutions.

2. Preliminaries

First, we recall some useful properties of the spaces $L^p(\Omega;\mathbb{R}^m)$ and $W^{1,p}(\Omega;\mathbb{R}^m)$. Then, we briefly review established results on Young measures, highlighting their properties that will help us in the sequel. For more details, the reader can consult references [2] and [12].

2.1. Functional spaces $L^p(\Omega;\mathbb{R}^m)$ and $W^{1,p}(\Omega;\mathbb{R}^m)$.

Let $(n \geq 2)$, Ω be an open subset of $\mathbb{R}^n$, $1 \leq p < +\infty$ and $m \geq 1$. For $a \in \mathbb{R}^m$, $|a|$ denotes its euclidean norm, where the inner product in $\mathbb{R}^m$ is denoted by the dote ' . ', and for $A \in \mathbb{M}^{m\times n}$, $|A|$ denotes its norm when regarded as a vector in $\mathbb{R}^{mn}$ ($\mathbb{M}^{m\times n}$ is endowed with reduced $\mathbb{R}^{mn}$ topology).

For vector-valued functions $\varpi = (\varpi^1, ..., \varpi^m)^t : \Omega \to \mathbb{R}^m$, $D\varpi$ denotes the $m \times n$ matrix of its distribution partial derivatives and is defined as follows

$$D\varpi = \begin{pmatrix} \frac{\partial \varpi^1}{\partial x^1} & \frac{\partial \varpi^1}{\partial x^2} & . & . & . & \frac{\partial \varpi^1}{\partial x^n} \\ \frac{\partial \varpi^2}{\partial x^1} & \frac{\partial \varpi^2}{\partial x^2} & . & . & . & \frac{\partial \varpi^2}{\partial x^n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \frac{\partial \varpi^m}{\partial x^1} & \frac{\partial \varpi^m}{\partial x^2} & . & . & . & \frac{\partial \varpi^m}{\partial x^n} \end{pmatrix},$$

with $x = (x^1, ..., x^n)^t$.

Lebesgue space $L^p(\Omega)$ is defined by

$$L^p(\Omega) = \Big\{\text{measurable functions } \varpi : \Omega \to \mathbb{R} \text{ such that } \int_\Omega |\varpi(x)|^p dx < +\infty\Big\},$$

endowed with the norm

$$\|\varpi\|_{L^p(\Omega)} := \|\varpi\|_p = \left(\int_\Omega |\varpi(x)|^p dx\right)^{\frac{1}{p}}.$$

For $m \geq 1$, $L^p(\Omega;\mathbb{R}^m)$ is defined by

$$L^p(\Omega;\mathbb{R}^m) = \Big\{ \varpi = (\varpi^1, ..., \varpi^m) \in \mathbb{R}^m \text{ such that } \varpi^i \in L^p(\Omega) \text{ for } 1 \leq i \leq m \Big\}.$$

When $m = 1$, $L^p(\Omega;\mathbb{R})$ is noted simply by $L^p(\Omega)$. One can identify the dual of the space $L^p(\Omega;\mathbb{R}^m)$ with $L^{p'}(\Omega;\mathbb{R}^m)$ (where $p' = \frac{p}{p-1}$ denote the exponent conjugate to p).

One defines the Sobolev space $W^{1,p}(\Omega;\mathbb{R}^m)$ by

$$W^{1,p}(\Omega;\mathbb{R}^m) = \Big\{\varpi \in L^p(\Omega;\mathbb{R}^m) \text{ and } \; D\varpi \in L^p(\Omega;\mathbb{M}^{m\times n})\Big\}.$$

It is endowed with the norm

$$\|\varpi\|_{W^{1,p}(\Omega;\mathbb{R}^m)} = \|\varpi\|_{1,p} = \|\varpi\|_p + \|D\varpi\|_p.$$

We denote by $C_0^j(\Omega;\mathbb{R}^m)$ the space of C^j functions with compact support in Ω ($j = 0, 1, \dots, \infty$). Consequently, and according to [12], the space $W_0^{1,p}(\Omega;\mathbb{R}^m)$ being the closure of $C_0^1(\Omega;\mathbb{R}^m)$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ with respect to the norm $\|\varpi\|_{1,p}$. Note that the density of $C_0^\infty(\Omega;\mathbb{R}^m)$ in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ allows us to replace $C_0^1(\Omega;\mathbb{R}^m)$ with $C_0^\infty(\Omega;\mathbb{R}^m)$ in the definition of $W_0^{1,p}(\Omega;\mathbb{R}^m)$.

We can identify the dual of the space $W_0^{1,p}(\Omega;\mathbb{R}^m)$ with $W^{-1,p'}(\Omega;\mathbb{R}^m)$.

Proposition 2.1. [2, 12]
Assume $1 \leq p < +\infty$*:*

(1) $L^p(\Omega;\mathbb{R}^m)$*,* $W^{1,p}(\Omega;\mathbb{R}^m)$*, and* $W_0^{1,p}(\Omega;\mathbb{R}^m)$ *are Banach spaces and separable.*
(2) $L^p(\Omega;\mathbb{R}^m)$*,* $W^{1,p}(\Omega;\mathbb{R}^m)$*, and* $W_0^{1,p}(\Omega;\mathbb{R}^m)$ *are reflexive as soon as* $1 < p < \infty$*.*

Remark 2.2. The separability of these spaces means that they contain a dense countable part, and consequently, one can approximate the problem by problems of dimension finite. Reflexivity allows us to extract from any bounded sequence of one of these spaces a weakly convergent subsequence in it.

The work in Sobolev spaces requires very often the use of some so-called Sobolev injections. We recall here one of these injections that we apply in the next and given by the Rellich Kondrachov theorem.

Proposition 2.3. *(Rellich-Kondrachov)* [12, Theorem 9.16]
Ω *being bounded and of class* C^1 *and* $p < n$*. Then the following compact injections follow:*

$$W^{1,p}(\Omega;\mathbb{R}^m) \hookrightarrow L^q(\Omega;\mathbb{R}^m) \quad \forall q \in [1, p^*) \quad \textit{where} \quad p^* = \frac{np}{n-p}.$$

In particular, since $p < p^*$ *then*

$$W^{1,p}(\Omega;\mathbb{R}^m) \hookrightarrow L^p(\Omega;\mathbb{R}^m) \quad \textit{with compact injection for all } p \ (\textit{and all } n).$$

In order to establish the essential estimates that are used to demonstrate the desired results, we need in what follows the Holder and Poincaré inequalities.

Proposition 2.4. *(Hölder's inequality)* [12, Theorem 4.6]
If $1 \leq p \leq \infty$, $\varpi \in L^p(\Omega;\mathbb{R}^m)$ *and* $\varpi^* \in L^{p'}(\Omega;\mathbb{R}^m)$, *then* $\varpi\varpi^* \in L^1(\Omega;\mathbb{R}^m)$ *and*

$$\int_\Omega |\varpi\varpi^*|dx \leq \|\varpi\|_p\|\varpi^*\|_{p'}.$$

Corollary 2.5. *(Poincaré's inequality)*[12]
Assume that Ω *is a bounded open set of* $\mathbb{R}^n$ *and* $1 \leq p < \infty$. *Then there exists a constant* C *(depending on* Ω *and* p*) so that for all* $\varpi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$, *we have*

$$\|\varpi\|_p \leq C\|D\varpi\|_p.$$

In particular, the two norms $\|D\varpi\|_p$ and $\|\varpi\|_{1,p}$ are equivalent on $W_0^{1,p}(\Omega;\mathbb{R}^m)$.

2.2. Brief reminder on Young measures.

In this subsection, we shortly review some established Young measures outcomes and we provide some of their properties. For further detail, we refer to [7, 17, 20]. $C_0(\mathbb{R}^m)$ represents the space of real valued continuous functions ψ with compact support in $\mathbb{R}^m$ (i.e. $\lim_{|\xi|\to\infty}\psi(\xi) = 0$) which is a Banach space endowed with the L^∞-norm. We identify the dual of $C_0(\mathbb{R}^m)$ to $\mathcal{M}(\mathbb{R}^m)$ which is in fact the space of signed Radon measures possessing the property of finite mass.
For all $\psi \in C_0(\mathbb{R}^m)$ and $\nu : \Omega \to \mathcal{M}(\mathbb{R}^m)$

$$\langle \nu, \psi\rangle = \int_{\mathbb{R}^m} \psi(\xi)d\nu(\xi), \text{ denotes the duality pairing between these spaces.}$$

The fundamental theorem of Young measures is given in the following result:

Theorem 2.6 (**Young, Tartar, Ball** [14])**.**
Let $\Omega \subset \mathbb{R}^n$ *be Lebesgue measurable (not necessarily bounded) and* $\varpi_j : \Omega \to \mathbb{R}^m$, $j = 1, 2, ...$, *be a sequence of Lebesgue measurable functions. Then there exists a subsequence* (ϖ_k) *and a family* $\{\nu_x\}_{x\in\Omega}$ *of non-negative Radon measures on* $\mathbb{R}^m$, *such that*

(i) $\|\nu_x\| = \int d\nu_x(\xi) \leq 1$ *for almost every* $x \in \Omega$
(ii) $\psi(\varpi_k) \rightharpoonup^* \overline{\psi}$ *weakly* in* $L^\infty(\Omega)$ *for all* $\psi \in C_0^0(\mathbb{R}^n)$, *where* $\overline{\psi}(x) = \langle \nu_x, \psi\rangle$
(iii) *If for all* $R > 0$

$$\lim_{L\to\infty}\sup_{k\in\mathbb{N}}\left|\left\{x \in \Omega \cap B(0, R) : \ |\varpi_k(x)| \geq L\right\}\right| = 0 \tag{2.1}$$

then

$$\|\nu_x\| = 1 \quad \textit{for almost every } x \in \Omega, \tag{2.2}$$

and for all measurable $A \subset \Omega$ *there holds*

$$\left\{\begin{array}{l} \psi(\varpi_k) \rightharpoonup \overline{\psi} = \langle \nu_x, \psi\rangle \textit{ weakly in } L^1(A) \textit{ for a continuous function} \\ \psi : \mathbb{R}^m \to \mathbb{R} \textit{ provided the sequence } \psi(\varpi_k) \textit{ is weakly precompact in } L^1(A). \end{array}\right. \tag{2.3}$$

Here, $C_0^0(\mathbb{R}^m) = C_0(\mathbb{R}^m)$ *and* $\Big|.\Big|$ *denotes the Lebesgue measure restricted to* Ω.

Remark 2.7. It should be noted that (2.1), (2.2), and (2.3) are equivalent (one can see the proof in [22, Theorem 1.2]).

Lemma 2.8 ([17]).
Assume that the sequence (ϖ_j) is bounded in $L^\infty(\Omega;\mathbb{R}^m)$. Then, there exists a subsequence (ϖ_k) of (ϖ_j) and for a.e.$x \in \Omega$ a Borel probability measure ν_x on $\mathbb{R}^m$ such that for each $\psi \in C(\mathbb{R}^m)$ we have

$$\psi(\varpi_k) \overset{*}{\rightharpoonup} \overline{\psi} \quad \text{weakly in } L^\infty(\Omega), \quad \text{where} \quad \overline{\psi}(x) = \int_{\mathbb{R}^m} \psi(\xi) d\nu_x(\xi) \quad (a.e.\ x \in \Omega).$$

Note that, if we fix the subsequence (ϖ_k) of (ϖ_j), then (ν_x) is obtained in a unique way and, according to (i) of Theorem 2.6, it is a sub-probability family on $\mathbb{R}^m$ (means that, for a.e. $x \in \Omega$: $\|\nu_x\| \le 1$).

Definition 2.9 ([17]).
We call $(\nu_x)_{x\in\Omega}$ the family of Young measures associated with the subsequence (ϖ_k).
We also say that $(\nu_x)_{x\in\Omega}$ generated by the subsequence (ϖ_k).

It should be noted that if $\Omega' \subset \Omega$ is any measurable subset and $\psi : \Omega' \times \mathbb{R}^m \to \mathbb{R}$ is a Carathéodory function such that $\{\psi(., \varpi_k)\}$ is sequentially weakly relatively compact in $L^1(\Omega')$, then, under hypothesis (2.1) and according to [7], we have

$$\lim_{k\to\infty} \int_\Omega \psi(x, \varpi_k(x))dx = \int_\Omega \langle \nu_x, \psi(x, .)\rangle dx.$$

This property is also equivalent to (2.1), (2.2), and (2.3).
It is also due to Ball that for $\psi \in L^1(\Omega; C_0(\mathbb{R}^m))$,

$$\lim_{k\to\infty} \int_\Omega \psi(x, \varpi_k(x))dx = \int_\Omega \langle \nu_x, \psi(x, .)\rangle dx,$$

as soon as (ϖ_k) generates the Young measure ν_x.

Theorem 2.6 has very important applications which serve in the sequel as the fundamental tools. Among the applications that help us in the rest of this article, we cite the following results.

Proposition 2.10 ([20]).
Assume $|\Omega| < \infty$, then

(1) *If ν_x is the Young measure associated with the (whole) sequence (ϖ_j), then the following equivalence holds:*

$$\varpi_j \to \varpi \quad \text{in measure} \quad \text{if and only if} \quad \nu_x = \delta_{\varpi(x)} \text{ for all each } x \in \Omega.$$

(2) *If the sequences $\varpi_j : \Omega \to \mathbb{R}^m$ and $\varpi_j^* : \Omega \to \mathbb{R}^d$ generate the Young measure $\delta_{\varpi(x)}$ and ν_x respectively, then (ϖ_j, ϖ_j^*) generates the Young measure $\delta_{\varpi(x)} \otimes \nu_x$.*

Lemma 2.11 (Fatou-Lemma [14]).
Let $F : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ be a Carathéodory function and $\varpi_j : \Omega \to \mathbb{R}^m$

a sequence of measurable functions such that $\varpi_j \to \varpi$ *in measure and such that* $(D\varpi_j)$ *generates the Young measure* ν_x. *Then*

$$\liminf_{j\to\infty} \int_\Omega F(x, \varpi_j(x), D\varpi_j(x))dx \geq \int_\Omega \int_{\mathbb{M}^{m\times n}} F(x, \varpi, \xi) d\nu_x(\xi) dx$$

provided that the negative part $F^-(x, \varpi_j(x), D\varpi_j(x))$ *is equiintegrable.*

3. The main results

It is assumed that the function Θ, Υ, and α fulfil the following conditions:
(A0) $\Theta : \mathbb{R}^m \to \mathbb{M}^{m\times n}$ is assumed to be linear, continuous and bounded by a constant $c_0 > 0$, such that

$$|\Theta(\varpi)| \leq c_0 \quad \text{for all} \quad \varpi \in \mathbb{R}^m.$$

(A1) $\Upsilon : \mathbb{R}^m \to \mathbb{R}^m$ is assumed to be continuous, and there exists a constant $C_\Upsilon \geq 0$ depending on the diameter of Ω (denoted as diam(Ω)) and the exponent p, such that

$$C_\Upsilon < \frac{1}{diam(\Omega)} \Big(\frac{1}{2}\Big)^{\frac{1}{p}},$$

satisfying, for any $\varpi, \varpi^* \in \mathbb{R}^m$,

$$|\Upsilon(\varpi) - \Upsilon(\varpi^*)| \leq C_\Upsilon |\varpi - \varpi^*| \quad \text{and} \quad \Upsilon(0) = 0.$$

(A2) $\alpha : \Omega \to \mathbb{R}$ is assumed to be measurable and bounded, i.e., there exists $\alpha_0 > 0$, $\alpha_1 > 0$, such that $\alpha \in L^\infty(\Omega)$, $0 < \alpha_0 \leq \alpha \leq \alpha_1 < \infty$.

It is assumed that the function σ satisfies the following conditions:
(C0) (Continuity)
$\sigma : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function, i.e. $\sigma(., \varpi, \xi)$ is measurable for every $(\varpi, \xi) \in \mathbb{R}^m \times \mathbb{M}^{m\times n}$ and $\sigma(x, ., .)$ is continuous for almost every $x \in \Omega$.

(C1) (Growth and coercivity)
There exists $a_1 \in L^{p'}(\Omega)$, $b \in L^1(\Omega)$, $c_1, c_1' \geq 0$, $c_2 > 0$, and $0 < q < n(p-1)/(n-p)$, such that the two inequalities hold

$$|\sigma(x, \varpi, \xi)| \leq a_1(x) + c_1|\varpi|^q + c_1'|\xi|^{p-1},$$
$$\sigma(x, \varpi, \xi) : \xi \geq c_2|\xi|^p - b(x),$$

where ' : ' means the inner product in $\mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$.
(C2) (Monotonicity)
One of the following conditions is satisfied by σ:

(1) $\sigma(x, \varpi, .)$ is a C^1 function and is monotone for any $(x, \varpi) \in \Omega \times \mathbb{R}^m$, here monotonicity means that for any $(x, \varpi) \in \Omega \times \mathbb{R}^m$

$$\big(\sigma(x, \varpi, \xi) - \sigma(x, \varpi, \xi^*)\big) : (\xi - \xi^*) \geq 0 \text{ for any } \xi, \xi^* \in \mathbb{M}^{m\times n}.$$

(2) There exists a function $W : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ satisfying, for any $(x, \varpi) \in \Omega \times \mathbb{R}^m$, the following two properties:
  (i) $\sigma(x, \varpi, \xi) = \frac{\partial W}{\partial \xi}(x, \varpi, \xi) = D_\xi W(x, \varpi, \xi)$, for all $\xi \in \mathbb{M}^{m\times n}$,

(ii) $W(x,\varpi,.)$ is $\mathcal{C}^1$ and convex.

(3) σ is strictly monotone, that is:
(i) σ is monotone,
(ii) for all $(x,\varpi)\in\Omega\times\mathbb{R}^m$
$\big(\sigma(x,\varpi,\xi)-\sigma(x,\varpi,\xi^*)\big):(\xi-\xi^*)=0$ if and only if $\xi=\xi^*$,
for all $\xi,\xi^*\in\mathbb{M}^{m\times n}$.

(4) σ is strictly p-quasimonotone with respect to the 3rd variable, i.e., for all $(x,\varpi)\in\Omega\times\mathbb{R}^m$

$$\int_{\mathbb{M}^{m\times n}}\big(\sigma(x,\varpi,\xi)-\sigma(x,\varpi,\overline{\xi})\big):(\xi-\overline{\xi})d\nu(\xi)>0.$$

Here $\nu=\{\nu_x\}_{x\in\Omega}$ represents any family of Young measures that does not restrict to a Dirac measure and is generated by a sequence in $L^p(\Omega)$ for a.e. $x\in\Omega$ and $\overline{\xi}=\langle\nu_x,id\rangle$.

(5) σ is strictly quasimonotone with respect to the 3rd variable, that is, there is $\tilde{\alpha}>0$ verifying for any $x\in\Omega$ and $\varpi,\varpi^*\in W_0^{1,p}(\Omega;\mathbb{R}^m)$,

$$\int_\Omega\big(\sigma(x,\varpi,D\varpi)-\sigma(x,\varpi,D\varpi^*)\big):(D\varpi-D\varpi^*)dx\geq\tilde{\alpha}\int_\Omega|D\varpi-D\varpi^*|^p dx.$$

It is assumed that the function g satisfies the following conditions:

(C3)

(i) $g:\Omega\times\mathbb{R}^m\to\mathbb{R}^m$ is a Carathéodory function, as defined in **(C0)**.

(ii) For all $\varpi,\varpi^*\in\mathbb{R}^m$,
$|g(x,\varpi-\Upsilon(\varpi^*))|\leq a_2(x)+c_3|\varpi-\Upsilon(\varpi^*)|^\beta$, where $c_3\geq0$, $0<\beta<p-1$, and $a_2\in L^{p'}(\Omega)$.

(iii) For all $\varpi,\varpi^*\in\mathbb{R}^m$,
$g(x,\varpi-\Upsilon(\varpi^*)).\varpi\geq0$.

It is assumed that the function f satisfies the following conditions:

(C4)

(i) $f:\Omega\times\mathbb{R}^m\times\mathbb{M}^{m\times n}\to\mathbb{R}^m$ is a Carathéodory function, as defined in **(C0)**,

(ii) For all $(\varpi,\xi)\in\mathbb{R}^m\times\mathbb{M}^{m\times n}$,
$|f(x,\varpi,\xi)|\leq a_3(x)+c_4|\varpi|^\gamma+c_4'|\xi|^{\gamma'}$, where $c_4,c_4'\geq0$, $0<\gamma,\gamma'<p-1$, and $a_3\in L^{p'}(\Omega)$.

Remark 3.1. The Carathéodory conditions **(C0)**, **(C3)(i)** and **(C4)(i)** ensure the measurability on Ω of $\sigma\big(.,\varpi(.),\tilde{\varpi}(.)\big)$, $g\big(.,\varpi(.)\big)$ and $f\big(.,\varpi(.),\tilde{\varpi}(.)\big)$ for measurable functions ϖ and $\tilde{\varpi}$ defined on Ω and having their values in $\mathbb{R}^m$ and $\mathbb{M}^{m\times n}$, respectively, (see, [29, Appendix, page 1013]).
Conditions **(C1)**, **(C3)(ii)** and **(C4)(ii)** which denote growth and coercivity, are standard. We use them to ensure the boundedness and the equiintegrability of the functions σ, g and f, and consequently, to construct approximate solutions using the Galerkin method and to proceed to the limit.
Particularly, the functions $f\big(.,\varpi(.),D\varpi(.)\big).\varpi(.)$ and $g\big(.,\varpi(.)\big).\varpi(.)$ are elements of $L^1(\Omega)$ when $\varpi\in W_0^{1,p}(\Omega,\mathbb{R}^m)$.
By using conventional techniques, the strict monotonicity requirement **(C2)(c)** guarantees the presence of weak solutions (see e-g [23]). The crucial distinction

is that, contrary to what is typically assumed in earlier works, it is not necessary to use the monotonicity in the variables (ϖ, ξ) in **(a)**, **(b)**, **(d)** and **(e)** of **(C2)** or the strict monotonicity which was considered by the operators of Leray-Lions (see e-g [13, 23]). More precisely, condition **(b)** of **(C2)** behaves with the potential $W(x, \varpi, \xi)$ by taking it uniquely convex instead of strictly convex in ξ, and replace $\sigma(x, \varpi, \xi)$ in problem (1.2) by $\frac{\partial W}{\partial \xi}(x, \varpi, \xi)$. In this case, we must point out that W is locally affine at the points where it is not strictly convex. Thus, by the bias of the Young measure associated with the subsequence of gradients of approximating solutions, we have the possibility of passing locally to this subsequence's limit (for more details see [4]). Notably, σ is strictly monotone if W is strictly convex, this allows us to apply the standard methods. While condition **(d)** of **(C2)** defines the strict p-quasimonotonicity in terms of gradient Young measures. At the end, condition **(e)** of **(C2)** designates monotonicity in integrated form.

Definition 3.2. We say that a measurable function $\varpi \in W_0^{1,p}(\Omega; \mathbb{R}^m)$ is a weak solution for the problem (1.1), if for all $\psi \in W_0^{1,p}(\Omega; \mathbb{R}^m)$, the following egality holds

$$\int_\Omega \big((\sigma(x, \varpi, D\varpi) + \Theta(\varpi)\big) : D\psi dx + \int_\Omega g\big(x, \varpi - \Upsilon(\varpi)\big).\psi dx + \int_\Omega \alpha(x)|\varpi|^{p-2}\varpi.\psi dx = \int_\Omega f(x, \varpi, D\varpi).\psi dx.$$

The theorem below is the main result of the paper.

Theorem 3.3.
*If σ satisfies the conditions **(C0)**-**(C2)**, then the Dirichlet problem (1.1) has a weak solution $\varpi \in W_0^{1,p}(\Omega; \mathbb{R}^m)$ for any Θ, Υ, and α satisfying **(A0)**-**(A2)**, any g satisfying **(C3)**, and any $f \in W^{-1,p'}(\Omega; \mathbb{R}^m)$ satisfying **(C4)**.*

Remark 3.4. A simple model for our case can be the following system:

$$\begin{cases} -\text{div}\left(|D\varpi|^{p-2}D\varpi\right) + |\varpi|^{p-2}\varpi & = f(x, \varpi, D\varpi) \quad \text{in } \Omega, \\ \varpi & = 0 \quad \text{on } \partial\Omega, \end{cases}$$

and as a potential, we can take $W(x, \varpi, \xi) = \frac{1}{p}|\xi|^p$.

Before proving the theorem, we need several ingredients. We start by introducing the following operator

$$\begin{aligned} &\mathscr{T} : W_0^{1,p}(\Omega; \mathbb{R}^m) \to W^{-1,p'}(\Omega; \mathbb{R}^m) \\ &\quad \varpi \mapsto \Big(\psi \mapsto \int_\Omega \big((\sigma(x, \varpi, D\varpi) + \Theta(\varpi)\big) : D\psi dx + \int_\Omega g(x, \varpi - \Upsilon(\varpi)).\psi dx \\ &\qquad + \int_\Omega \alpha(x)|\varpi|^{p-2}\varpi.\psi dx - \int_\Omega f(x, \varpi, D\varpi).\psi dx\Big). \end{aligned}$$

Note that in everything that follows, we use C as a generic constant whose value can change from one line to the next.

Lemma 3.5.
*If **(A0)**-**(A2)** and **(C0)**-**(C4)** hold, then the following properties are verified for arbitrary* $\varpi \in W_0^{1,p}(\Omega, \mathbb{R}^m)$ *and* $f \in W^{-1,p'}(\Omega; \mathbb{R}^m)$

(1) *The functional* $\mathscr{T}(\varpi)$ *is well defined, linear and bounded.*
(2) *The restriction of* $\mathscr{T}$ *to any finite linear subspace* V *of* $W_0^{1,p}(\Omega, \mathbb{R}^m)$ *is continuous.*

Proof. (1) Let ϖ be an arbitrary element in $W_0^{1,p}(\Omega, \mathbb{R}^m)$. We can easily verify by the linearity of the integral that $\mathscr{T}(\varpi)$ is linear. Without losing generality, we can assume that $q = \gamma = \gamma' = \beta' = p-1$. So by using the growth condition in **(C1)**, **(C3)(ii)** and **(C4)(ii)**, and using the continuous embedding $W_0^{1,p}(\Omega; \mathbb{R}^m) \hookrightarrow L^p(\Omega)$, and the fact that $|\kappa + \tilde{\kappa}|^p \leq 2^{p-1}(|\kappa|^p + |\tilde{\kappa}|^p)$ (for $\kappa, \tilde{\kappa} \in \mathbb{R}$ and $p > 1$), we will have

$$\int_\Omega |\sigma(x, \varpi, D\varpi)|^{p'} dx \leq C \int_\Omega \left(|a_1(x)|^{p'} + |\varpi|^p + |D\varpi|^p\right) \mathrm{d}x < \infty, \tag{3.1}$$

$$\int_\Omega |g(x, \varpi - \Upsilon(\varpi))|^{p'} dx \leq C \int_\Omega \left(|a_2(x)|^{p'} + |\varpi|^p + |D\varpi|^p\right) \mathrm{d}x < \infty, \tag{3.2}$$

and

$$\int_\Omega |f(x, \varpi, D\varpi)|^{p'} dx \leq C \int_\Omega \left(|a_3(x)|^{p'} + |\varpi|^p + |D\varpi|^p\right) \mathrm{d}x < \infty, \tag{3.3}$$

where C depends on c_1, c_1' and p' in (3.1), depends on c_3, c_3' and p' in (3.2), and depends on c_4, c_4' and p' in (3.3). Hence, it follows from the Hölder inequality, from condition **(A0)**, and with the continuous embedding $W_0^{1,p}(\Omega; \mathbb{R}^m) \hookrightarrow L^p(\Omega)$, that for any $\psi \in W_0^{1,p}(\Omega; \mathbb{R}^m)$,there exists a constant $C > 0$ such that

$$\begin{aligned}
|\langle \mathscr{T}(\varpi), \psi\rangle| &= \Big| \int_\Omega \big(\sigma(x, \varpi, D\varpi) + \Theta(\varpi)\big) : D\psi dx + \int_\Omega g(x, \varpi - \Upsilon(\varpi)).\psi dx \\
&\qquad + \int_\Omega \alpha(x)|\varpi|^{p-2}\varpi.\psi dx - \int_\Omega f(x, \varpi, D\varpi).\psi dx \Big| \\
&\leq \int_\Omega \big(|\sigma(x, \varpi, D\varpi)| + |\Theta(\varpi)|\big)|D\psi| dx + \int_\Omega |g(x, \varpi - \Upsilon(\varpi))||\psi| dx \\
&\qquad + \int_\Omega |\alpha(x)||\varpi|^{p-1}|\psi| dx + \int_\Omega |f(x, \varpi, D\varpi)||\psi| dx \\
&\leq \big\|\sigma(x, \varpi, D\varpi)\big\|_{p'} \|D\psi\|_p + c_0\|D\psi\|_1 + \big\|g(x, \varpi - \Upsilon(\varpi))\big\|_{p'} \|\psi\|_p \\
&\qquad + \|\alpha\|_\infty \|\varpi\|_p^{p-1} \|\psi\|_p + \|f\|_{p'} \|\psi\|_p \\
&\leq C\|\psi\|_{1,p}.
\end{aligned}$$

Thus, $\mathscr{T}(\varpi)$ is well defined and bounded.

(2) Let s be the dimension of V which is a finite subspace of $W_0^{1,p}(\Omega; \mathbb{R}^m)$ having the basis $(e_i)_{i=1,..,s}$. For $\varpi_j = a_j^i e_i$ and $\varpi = a^i e_i$ (with conventional summation), we suppose that $\varpi_j \to \varpi$ in V. Hence, the sequence whose general term is $a_j = (a_j^i)_{1\leq i\leq s}$ converges to $a = (a^i)_{1\leq i\leq s}$ in $\mathbb{R}^s$. This implies that the two sequences (ϖ_j) and $(D\varpi_j)$ are convergent almost everywhere to ϖ and $D\varpi$

respectively. Moreover, we can bound $\|\varpi_j\|_p$ and $\|D\varpi_j\|_p$ by a constant noted by C. Indeed, we have

$$\int_\Omega |\varpi_j - \varpi|^p dx \to 0 \quad \text{and} \quad \int_\Omega |D\varpi_j - D\varpi|^p dx \to 0 \quad \text{as} \quad j \to \infty,$$

then by [11, Théorème IV.9.], there is a subsequence of (ϖ_j) (again designated by (ϖ_j)) and $g_1, g_2 \in L^1(\Omega)$ such that $|\varpi_j - \varpi|^p \le g_1$ and $|D\varpi_j - D\varpi|^p \le g_2$. Consequently, we have

$$|\varpi_j|^p = |\varpi_j - \varpi + \varpi|^p \le 2^{p-1}\big(|\varpi_j - \varpi|^p + |\varpi|^p\big) \le 2^{p-1}\big(g_1 + |\varpi|^p\big).$$

In the same way we have

$$|D\varpi_j|^p \le 2^{p-1}\big(g_2 + |D\varpi|^p\big).$$

Therefore, based on the continuity conditions **(A0)**, **(A1)**, **(C0)**, **(C3)(i)**, and **(C4)(i)**, we infer that sequences $\big(\sigma(x, \varpi_j(x), D\varpi_j(x)) : D\psi\big)$, $\big(\Theta(\varpi_j(x)) : D\psi\big)$, $\big(g(x, \varpi_j(x), D\varpi_j(x)).\psi\big)$ and $\big(f(x, \varpi_j(x), D\varpi_j(x)).\psi\big)$ converge almost everywhere to $\sigma(x, \varpi(x), D\varpi(x)) : D\psi$, $\Theta(\varpi(x)) : D\psi$, $g(x, \varpi(x), D\varpi(x)).\psi$ and $f(x, \varpi(x), D\varpi(x)).\psi$ respectively. On the other hand, these four sequences are equiintegrable due to conditions **(A0)**, **(A1)**, **(C1)**, **(C3)(ii)**, and **(C4)(ii)**. Moreover, since $\alpha(x)|\varpi_j|^{p-2}\varpi_j \to \alpha(x)|\varpi|^{p-2}\varpi$ in $L^{p'}(\Omega; \mathbb{R}^m)$, it follows, by Vitali's Theorem, for all $\psi \in W_0^{1,p}(\Omega; \mathbb{R}^m)$

$$\begin{aligned}
\|\mathscr{T}(\varpi_j) - \mathscr{T}(\varpi)\|_{-1,p'} &= \sup_{\|\psi\|_{1,p}=1} \big|\langle \mathscr{T}(u_j) - \mathscr{T}(\varpi), \psi\rangle\big| \\
&\le C\Big(\|\sigma(x, \varpi_j, D\varpi_j) - \sigma(x, \varpi, D\varpi)\|_{p'} + \|\Theta(\varpi_j) - \Theta(\varpi)\|_{p'} \\
&+ \|g(x, \varpi_j - \Upsilon(\varpi_j)) - g(x, \varpi - \Upsilon(\varpi))\|_{p'} + \|\alpha(x)|\varpi_j|^{p-2}\varpi_j - \alpha(x)|\varpi|^{p-2}\varpi\|_{p'} \\
&\qquad + \|f(x, \varpi_j, D\varpi_j) - f(x, \varpi, D\varpi)\|_{-1,p'}\Big) \\
&\le C.
\end{aligned}$$

Thus the restriction of $\mathscr{T}$ to V is continuous. □

4. Galerkin approximation

We establish approximating solutions by Galerkin's method and some *a priori* estimates. Indeed, the separability of $W_0^{1,p}(\Omega, \mathbb{R}^m)$ ensures the existence of an increasing sequence of finite dimensional subspaces (V_j) in the sense of inclusion, such that $\cup_{j\in\mathbb{N}} V_j$ is dense in $W_0^{1,p}(\Omega, \mathbb{R}^m)$. Now, we fix j in $\mathbb{N}$ and we suppose V_j of finite dimension equal to s. We define the following map

$$\mathscr{S} : \mathbb{R}^s \to \mathbb{R}^s, \qquad \begin{pmatrix} a^1 \\ a^2 \\ . \\ . \\ a^s \end{pmatrix} \mapsto \begin{pmatrix} \langle \mathscr{T}(a^i e_i), e_1\rangle \\ \langle \mathscr{T}(a^i e_i), e_2\rangle \\ . \\ . \\ \langle \mathscr{T}(a^i e_i), e_s\rangle \end{pmatrix}.$$

Lemma 4.1.
$\mathscr{S}$ verifies the following properties:

(1) $\mathscr{S}$ *is continuous.*
(2) $\mathscr{S}(a) \cdot a \to \infty$ *as* $\|a\|_{\mathbb{R}^s} \to \infty$.

Proof.
(1) The restriction from $\mathscr{T}$ to V_j remains continuous, this allows us to deduce that of $\mathscr{S}$.
(2) Let $a \in \mathbb{R}^s$ and set $\varpi = a^i e_i \in V_j$ (with conventional summation). First of all, we have $\|a\|_{\mathbb{R}^s} \to \infty$ as soon as $\|\varpi\|_{1,p} \to \infty$. Indeed,

$$\|\varpi\|_{1,p}^p \leq \left(\sum_{i=1}^{s} |a^i|.\|e_i\|_{1,p} \right)^p \leq \max_{1 \leq i \leq s} \|e_i\|_{1,p}^p \left(\sum_{i=1}^{s} |a^i| \right)^p \leq C \|a\|_{\mathbb{R}^s}^p. \tag{4.1}$$

Where the constant C is independent of ϖ.
However, since $p > 1$, there exists $\overline{c} = \frac{c_2}{2c_0} > 0$ such that $\int_\Omega |D\varpi| dx \leq \overline{c} \int_\Omega |D\varpi|^p dx$. According to Hölder's inequality, the coercivity condition in **(C1)**, the condition in **(C3)(iii)**, and the definition of $\mathscr{T}$, we get

$$\begin{aligned}
\langle \mathscr{T}(\varpi), \varpi \rangle &= \int_\Omega \big(\sigma(x, \varpi, D\varpi) + \Theta(\varpi) \big) : D\varpi dx + \int_\Omega g(x, \varpi - \Upsilon(\varpi)).\varpi dx \\
&\quad + \int_\Omega |\alpha(x)||\varpi|^{p-2} \varpi.\varpi dx - \int_\Omega f(x, \varpi, D\varpi).\varpi dx \\
&\geq \int_\Omega (c_2 |D\varpi|^p - b(x)) dx - c_0 \int_\Omega |D\varpi| dx - \int_\Omega |f(x, \varpi, D\varpi)||\varpi| dx \\
&\geq \frac{c_2}{2} \|D\varpi\|_p^p - \|b\|_{L^1} - \|f\|_{-1,p'} \|\varpi\|_{1,p}.
\end{aligned}$$

Moreover, Poincaré's inequality ensures the existence of a constant $\tilde{c} > 0$ verifying for any $\varpi \in W^{1,p}(\Omega; \mathbb{R}^m)$, $\|\varpi\|_p \leq \tilde{c} \|D\varpi\|_p$, or even $\|\varpi\|_{1,p} \leq (1 + \tilde{c}) \|D\varpi\|_p$.
It follows that

$$\langle \mathscr{T}(\varpi), \varpi \rangle \geq \frac{c_2}{2(1+\tilde{c})^p} \|\varpi\|_{1,p}^p - \|b\|_{L^1} - \|f\|_{-1,p'} \|\varpi\|_{1,p} \longrightarrow \infty \text{ as } \|\varpi\|_{1,p} \to \infty.$$

Hence, $\mathscr{S}(a).a \longrightarrow \infty$ as $\|a\|_{\mathbb{R}^s} \to \infty$ since $\mathscr{S}(a).a = \langle \mathscr{T}(a^i e_i), a^i e_i \rangle = \langle \mathscr{T}(\varpi), \varpi \rangle$. □

Lemma 4.2.

(1) *For any* $j \in \mathbb{N}$, *there is* $\varpi_j \in V_j$ *verifying for all* $\psi \in V_j$

$$\langle \mathscr{T}(\varpi_j), \psi \rangle = 0. \tag{4.2}$$

(2) *The sequence already constructed in 1) is uniformly bounded in* $W_0^{1,p}(\Omega; \mathbb{R}^m)$, *i.e., satisfying*

$$\|\varpi_j\|_{1,p} \leq R \quad \textit{for any} \quad j \in \mathbb{N}, \tag{4.3}$$

for some constant $R > 0$ *independent of* j.

Proof.

(1) By Lemma 4.1, $\mathscr{S}(a).a \to \infty$ as $\|a\|_{\mathbb{R}^s} \to \infty$. Hence, there is $R > 0$ such that $\mathscr{S}(a).a > 0$ as soon as $a \in \partial B_R(0) \subset \mathbb{R}^s$. Consequently, thanks to topological arguments, $\mathscr{S}(x) = 0$ admits a solution that belongs to $B_R(0)$ (see [28], Proposition 2.8). Therefore, there is $\varpi_j \in V_j$ for any $j \in \mathbb{N}$ so that $\langle \mathscr{T}(\varpi_j), \psi \rangle = 0$ for any $\psi \in V_j$.

(2) In the process of proving lemma 4.1, we have $\langle \mathscr{T}(\varpi), \varpi \rangle \longrightarrow \infty$ as $\|\varpi\|_{1,p} \to \infty$. Consequently, there is $R > 0$ so that whenever $\|\varpi\|_{1,p} > R$, one has $\langle \mathscr{T}(\varpi), \varpi \rangle > 1$. However, the sequence obtained in 1) verifies $\langle \mathscr{T}(\varpi_j), \varpi_j \rangle = 0$ for all $\varpi_j \in V_j$. Whence the uniform boundness of the sequence (ϖ_j).

□

5. The Young measure generated by the Galerkin approximation

In this section, we use Young measure as a powerful device to deal with the difficulties encountered when weak convergence does not work properly for nonlinear functions and operators. first we give the most important properties concerning the sequences of the Galerkin approximations constructed previously. Then, we show a few helpful lemmas required for the main theorem's proof. The features of gradient sequences in connection to the weak limit and Young's measure are gathered in the subsequent lemma. Throughout this section, we denote by ϖ_j the Galerkin approximation sequence.

Lemma 5.1.
For the sequence (ϖ_j) already defined in (4.2), there exists a Young measure ν_x associated with the sequence (or at least a subsequence) of gradients $(D\varpi_j)$, which has the following properties:

(1) *The sequence $(D\varpi_j)$ is bounded in $L^p(\Omega; \mathbb{M}^{m\times n})$.*
(2) *$\|\nu_x\| = 1$ for almost every $x \in \Omega$.*
(3) *$D\varpi_j \rightharpoonup \langle \nu_x, id \rangle$ weakly in $L^1(\Omega; \mathbb{M}^{m\times n})$.*
(4) *$\langle \nu_x, id \rangle = D\varpi(x)$ for almost every $x \in \Omega$.*

Proof.

(1) According to 2) of lemma 4.1, there is a constant $R > 0$ independent of j, verifying for sequence $\varpi_j \in V_j$ defined in (4.2), $\|\varpi_j\|_{1,p} \leq R$ for any $j \in \mathbb{N}$. Or again, $\|D\varpi_j\|_p \leq R$ for any $j \in \mathbb{N}$. Just take $C = R^p > 0$ to get

$$\int_\Omega |D\varpi_j|^p dx \leq C \text{ for any } j \in \mathbb{N}.$$

(2) We have,

$$L^p \left|\left\{x \in \Omega \cap B_R(0) : \ |D\varpi_j| \geq L\right\}\right| \leq \int_{\{x\in\Omega\cap B_R(0) :\, |D\varpi_j|\geq L\}} |D\varpi_j|^p dx \leq \int_\Omega |D\varpi_j|^p dx.$$

Therefore, under condition (1) and for any $R > 0$,

$$\sup_{j\in\mathbb{N}} \big|\big\{x \in \Omega \cap B_R(0) : \ |D\varpi_j| \geq L\big\}\big| \leq \frac{C}{L^p} \to 0 \quad \text{as} \quad L \to \infty.$$

By virtue of Theorem 2.6, there exists a Young measure ν_x generated by the subsequence of $(D\varpi_j)$ (again denoted by $(D\varpi_j)$). We conclude by (iii) of the same lemma that $\|\nu_x\| = 1$.

(3) By reflexivity of $L^p(\Omega; \mathbb{M}^{m\times n})$ $(1 < p)$ and the fact that $\mathbb{M}^{m\times n}$ can be identified with $\mathbb{R}^{mn}$, and taking into account (1) and according to Eberlein-Šmulian theorem [11], we can extract a subsequence (also noted by $D\varpi_j$) which converges weakly in $L^p(\Omega; \mathbb{M}^{m\times n})$. By Theorem 2.6(iii), taking $\psi = id$, one has

$$D\varpi_j \rightharpoonup \langle \nu_x, id\rangle \quad \text{weakly in } L^1(\Omega; \mathbb{M}^{m\times n}).$$

(4) According to (4.3), (ϖ_j) is uniformly bounded sequence in $W_0^{1,p}(\Omega; \mathbb{R}^m)$. Hence, we may extract a subsequence (again represented by (ϖ_j)) that weakly converges to an element denoted by ϖ in $W_0^{1,p}(\Omega; \mathbb{R}^m)$. Moreover, for a subsequence of (ϖ_j), we have by Rellich-Kondrachov theorem [2] that $\varpi_j \to \varpi$ in $L^p(\Omega; \mathbb{R}^m)$ and $D\varpi_j \rightharpoonup D\varpi$ in $L^p(\Omega; \mathbb{M}^{m\times n})$. Consequently, by the embedding $L^p(\Omega; \mathbb{M}^{m\times n}) \subset L^1(\Omega; \mathbb{M}^{m\times n})$, we have $D\varpi_j \rightharpoonup D\varpi$ in $L^1(\Omega; \mathbb{M}^{m\times n})$. The uniqueness of limit implies by (3) that, $\langle \nu_x, id\rangle = D\varpi(x)$ for a.e. $x \in \Omega$.

□

We state below the so-called div-curl inequality, which is an important tool to prove existence results for certain classes of PDEs, including certain types of nonlinear problems. It will serve us to prove that the weak limit of a suitably chosen sub-sequence of Galerkin approximations ϖ_j is in fact a solution of (1.1).

Lemma 5.2.
*Assume that **(C0)-(C4)** hold, the Young measure ν_x associated with the gradient $D\varpi_j$ and $\varpi_j \rightharpoonup \varpi$ in $W^{1,p}(\Omega; \mathbb{R}^m)$, then the following inequality holds*

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x, \varpi, \xi) : (\xi - D\varpi) d\nu_x(\xi) dx \leq \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x, \varpi, D\varpi) : (\xi - D\varpi) d\nu_x(\xi) dx. \tag{5.1}$$

Proof.
Consider the sequences $I_{j,1}$, $I_{j,2}$ and I_j defined by

$$I_{j,1} = \sigma(x, \varpi_j, D\varpi_j) : (D\varpi_j - D\varpi) \quad , \quad I_{j,2} = -\sigma(x, \varpi, D\varpi) : (D\varpi_j - D\varpi),$$

and

$$I_j = I_{j,1} + I_{j,2}.$$

Since $\varpi \in L^p(\Omega; \mathbb{R}^m)$, $D\varpi \in L^p(\Omega; \mathbb{M}^{m\times n})$, we get from **(C1)** that $\sigma(x, \varpi, D\varpi) \in L^{p'}(\Omega; \mathbb{M}^{m\times n})$. By (3) of Lemma 5.1 which expresses the weak convergence of

$(D\varpi_j)$ and (4) of the same Lemma, we have

$$\liminf_{j\to\infty}\int_\Omega I_{j,2}dx = -\int_\Omega \sigma(x,\varpi,D\varpi):\Big(\int_{\mathbb{M}^{m\times n}}\xi d\nu_x(\xi)-D\varpi\Big)dx = 0.$$

Therefore

$$\liminf_{j\to\infty}\int_\Omega I_j dx = \liminf_{j\to\infty}\int_\Omega I_{j,1}dx.$$

Both $\big(\sigma(x,\varpi_j,D\varpi_j):D\varpi_j\big)^-$ and $\big(\sigma(x,\varpi_j,D\varpi_j):D\varpi\big)^-$ are equiintegrable. In fact, for a measurable subset Ω' of Ω and under the condition of coercivity in **(C1)**,

$$\sigma(x,\varpi_j,D\varpi_j):D\varpi_j \geq c_2|D\varpi_j|^p - b(x) \geq -b(x),$$

which gives

$$\int_{\Omega'}\big|\min\big(\sigma(x,\varpi_j,D\varpi_j):D\varpi_j,0\big)\big|dx \leq \int_{\Omega'}|b(x)|dx < \infty.$$

Whence the equiintegrability of $\big(\sigma(x,\varpi_j,D\varpi_j):D\varpi_j\big)^-$.
Then, by Hölder's inequality, we write

$$\int_{\Omega'}\big|\big(\sigma(x,\varpi_j,D\varpi_j):D\varpi\big)\big|dx \leq \left(\int_{\Omega'}|\sigma(x,\varpi_j,D\varpi_j|^{p'}dx\right)^{\frac{1}{p'}}\left(\int_{\Omega'}|D\varpi|^p dx\right)^{\frac{1}{p}}.$$

According to **(C1)** and (4.3), the first integral of the last inequality is uniformly bounded independently of j. And if we choose Ω' of sufficiently small measure, the second integral will be arbitrarily small. Consequently, $I_{j,1}^-$ is equiintegrable. Since (ϖ_j) is uniformly bounded in $W_0^{1,p}(\Omega;\mathbb{R}^m)$, we have by Lemma 3.1 of [4], $\varpi_j \to \varpi$ in measure for a subsequence again named (ϖ_j). From where, by Lemma 2.11 we get

$$\liminf_{j\to\infty}\int_\Omega I_j dx \geq \int_\Omega\int_{\mathbb{M}^{m\times n}}\big(\sigma(x,\varpi,\xi)-\sigma(x,\varpi,D\varpi)\big):(\xi-D\varpi)d\nu_x(\xi)dx.$$

To complete the proof, we will show that $\liminf_{j\to\infty}\int_\Omega I_j dx \leq 0$. Indeed, according to Mazur's theorem, there exists a sequence $(\varpi_j^*)\subset W_0^{1,p}(\Omega;\mathbb{R}^m)$ so that each ϖ_j^* is in convex linear combination of $\{\varpi_1,..,\varpi_j\}$ satisfying $\varpi_j^*\to\varpi$ in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ (see, e.g., [27, Theorem 2]). Which means that ϖ_j^* belongs to the same space V_j as ϖ_j.
However,

$$\liminf_{j\to\infty}\int_\Omega I_j dx = \liminf_{j\to\infty}\int_\Omega I_{j,1}dx = \liminf_{j\to\infty}\Big(\int_\Omega \sigma(x,\varpi_j,D\varpi_j):(D\varpi_j - D\varpi_j^*)dx + \int_\Omega \sigma(x,\varpi_j,D\varpi_j):(D\varpi_j^*-D\varpi)dx\Big).$$

Choosing $\varpi_j - \varpi_j^*$ as a test function in (4.2) and using the Hölder inequality, we get

$$\Big| \int_\Omega \sigma(x, \varpi_j, D\varpi_j) : (D\varpi_j - D\varpi_j^*)dx \Big|$$
$$= \Big| \int_\Omega f(x, \varpi_j, D\varpi_j).(\varpi_j - \varpi_j^*)dx - \int_\Omega \Theta(\varpi_j) : (D\varpi_j - D\varpi_j^*)dx$$
$$- \int_\Omega g(x, \varpi_j - \Upsilon(\varpi_j)).(\varpi_j - \varpi_j^*)dx - \int_\Omega \alpha(x)|\varpi_j|^{p-2}\varpi.(\varpi_j - \varpi_j^*)dx \Big|$$
$$\leq \big(\big\| f(x, \varpi_j, D\varpi_j) \big\|_{p'} + \big\| g(x, \varpi_j - \Upsilon(\varpi_j)) \big\|_{p'} + \alpha_1 \|\varpi_j\|_p^{p-1} \big) \|\varpi_j - \varpi_j^*\|_p$$
$$+ c_0 \|D\varpi_j - D\varpi_j^*\|_1,$$

because $|f(x, \varpi_j, D\varpi_j)|^{p'}$ and $|g(x, \varpi_j - \Upsilon(\varpi_j))|^{p'}$ are bounded by an integrable function according to the growth condition of f and g in (**C3**)(ii), (**C4**)(ii) and by 4.3 , and Θ is continuous and bounded by c_0.
On the other hand, by the construction of ϖ_j^* we have

$$\|\varpi_j - \varpi_j^*\|_p \leq \|\varpi_j - \varpi\|_p + \|\varpi_j^* - \varpi\|_p \to 0 \text{ as } j \to \infty,$$

and

$$\|D\varpi_j - D\varpi_j^*\|_p \leq \|D\varpi_j - D\varpi\|_p + \|D\varpi_j^* - D\varpi\|_p \to 0 \text{ as } j \to \infty.$$

According to embedding $L^p(\Omega; \mathbb{R}^m) \subset L^1(\Omega; \mathbb{R}^m)$

$$\|D\varpi_j - D\varpi_j^*\|_1 \to 0 \text{ as } j \to \infty.$$

Whence,

$$\int_\Omega \sigma(x, \varpi_j, D\varpi_j) : (D\varpi_j - D\varpi_j^*)dx \to 0 \text{ as } j \to \infty.$$

Similar to this, by (4.3) and condition **(C1)**, we have $|\sigma(x, \varpi_j, D\varpi_j)|^{p'}$ is uniformly bounded in j. And by definition of ϖ_j^*,

$$\|D\varpi_j^* - D\varpi\|_p \longrightarrow 0 \quad \text{as } j \to \infty.$$

Thus, by Hölder's inequality

$$\int_\Omega \sigma(x, \varpi_j, D\varpi_j) : (D\varpi_j^* - D\varpi)dx \longrightarrow 0 \quad \text{as } j \to \infty.$$

Accordingly,

$$\liminf_{j\to\infty} \int_\Omega I_j dx \leq 0.$$

Hence the desired result. □

The following lemma expresses the localisation of the support for ν_x.

Lemma 5.3.
Suppose (5.1) holds, then for almost every $x \in \Omega$

$$\big(\sigma(x, \varpi, \xi) - \sigma(x, \varpi, D\varpi)\big) : (\xi - D\varpi) = 0 \quad \textit{on supp } \nu_x. \tag{5.2}$$

Proof. From (5.1), we deduce that

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \big(\sigma(x,\varpi,\xi) - \sigma(x,\varpi,D\varpi)\big) : (\xi - D\varpi) d\nu_x(\xi) dx \le 0.$$

The integrand in the previous inequality also cannot be negative owing to the monotonicity of σ, therefore it must be null compared to the product measure $d\nu_x(\xi) \otimes dx$. From where

$$\big(\sigma(x,\varpi,\xi) - \sigma(x,\varpi,D\varpi)\big) : (\xi - D\varpi) = 0 \quad \text{on supp}\, \nu_x.$$

□

6. Proof of Theorem 3.3

We begin by presenting some lemmas for σ fulfilling **(C0)**-**(C2)**.

Lemma 6.1.
*If σ fulfills the conditions **(C0)**-**(C2)** and $\varpi_j \rightharpoonup \varpi$ in $W_0^{1,p}(\Omega;\mathbb{R}^m)$, then for all $\psi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$, we have*

$$\int_\Omega \sigma(x,\varpi_j,D\varpi_j) : D\psi dx \to \int_\Omega \sigma(x,\varpi,D\varpi) : D\psi dx \text{ as } j\to\infty. \tag{6.1}$$

Proof.
To prove this lemma for a subsequence of ϖ_j, we treat five cases that correspond to the cases of **(C2)**.
Case (a): At first, we affirm that on supp ν_x

$$\sigma(x,\varpi,\xi) : \zeta = \sigma(x,\varpi,D\varpi) : \zeta + \big(\nabla_\xi \sigma(x,\varpi,D\varpi)\zeta\big) : (D\varpi - \xi), \tag{6.2}$$

for almost $(x,\zeta) \in \Omega \times \mathbb{M}^{m\times n}$ and where ∇_ξ denotes the derivative relative to the third variable of σ.
Indeed, we develop the following expression for any $\tau \in \mathbb{R}$,

$$\big(\sigma(x,\varpi,\xi) - \sigma(x,\varpi,D\varpi+\tau\zeta)\big) : (\xi - D\varpi - \tau\zeta),$$

then, using (5.2), we obtain

$$\begin{aligned}&\big(\sigma(x,\varpi,\xi) - \sigma(x,\varpi,D\varpi+\tau\zeta)\big) : (\xi - D\varpi - \tau\zeta)\\ &= \sigma(x,\varpi,D\varpi) : (\xi - D\varpi) - \sigma(x,\varpi,\xi) : \tau\zeta - \sigma(x,\varpi,D\varpi+\tau\zeta) : (\xi - D\varpi - \tau\zeta).\end{aligned}$$

By monotonicity of σ, we get

$$-\sigma(x,\varpi,\xi) : \tau\zeta \ge -\sigma(x,\varpi,D\varpi) : (\xi - D\varpi) + \sigma(x,\varpi,D\varpi+\tau\zeta) : (\xi - D\varpi - \tau\zeta).$$

And since

$$\sigma(x,\varpi,D\varpi+\tau\zeta) = \sigma(x,\varpi,D\varpi) + \nabla_\xi\sigma(x,\varpi,D\varpi)\tau\zeta + o(\tau),$$

it follows that

$$\begin{aligned}&-\sigma(x,\varpi,D\varpi) : (\xi - D\varpi) + \sigma(x,\varpi,D\varpi+\tau\zeta) : (\xi - D\varpi - \tau\zeta)\\ &\qquad= -\sigma(x,\varpi,D\varpi) : (\xi - D\varpi) + \sigma(x,\varpi,D\varpi+\tau\zeta) : (\xi - D\varpi)\\ &\qquad\qquad - \sigma(x,\varpi,D\varpi+\tau\zeta) : \tau\zeta\\ &\qquad= \tau\big[\nabla_\xi\sigma(x,\varpi,D\varpi)\zeta : (\xi - D\varpi) - \sigma(x,\varpi,D\varpi) : \zeta\big] + o(\tau).\end{aligned}$$

From which

$$-\sigma(x,\varpi,\xi):\tau\zeta \geq \tau\big[\nabla_\xi\sigma(x,\varpi,D\varpi)\zeta:(\xi-D\varpi)-\sigma(x,\varpi,D\varpi):\zeta\big]+o(\tau). \quad (6.3)$$

Given that the sign of τ is arbitrary, equation (6.2) naturally follows from inequality (6.3).
$\sigma(x,\varpi_j,D\varpi_j)$ is equiintegrable since it is bounded, and its weak L^1-limit is determined by Ball's theorem as

$$\overline{\sigma}=\int_{\mathbb{M}^{m\times n}}\sigma(x,\varpi,\xi)d\nu_x(\xi)=\int_{\operatorname{supp}\nu_x}\sigma(x,\varpi,\xi)d\nu_x(\xi).$$

Using our affirmation (6.2) to get

$$\begin{aligned}\overline{\sigma}&=\int_{\operatorname{supp}\nu_x}\sigma(x,\varpi,D\varpi)d\nu_x(\xi)+\big(\nabla_\xi\sigma(x,\varpi,D\varpi)\big)^t\int_{\operatorname{supp}\nu_x}(D\varpi-\xi)d\nu_x(\xi)\\ &=\int_{\operatorname{supp}\nu_x}\sigma(x,\varpi,D\varpi)d\nu_x(\xi)\\ &\qquad+\big(\nabla_\xi\sigma(x,\varpi,D\varpi)\big)^t\underbrace{\Big(D\varpi\int_{\operatorname{supp}\nu_x}d\nu_x(\xi)-\int_{\operatorname{supp}\nu_x}\xi d\nu_x(\xi)\Big)}_{=:0}\\ &=\sigma(x,\varpi,D\varpi).\end{aligned}$$

According to the Eberlein-Šmulian theorem [11], the sequence $\sigma(x,\varpi_j,D\varpi_j)$ converges weakly in $L^{P'}(\Omega,\mathbb{M}^{m\times n})$, since $\sigma(x,\varpi_j,D\varpi_j)$ is bounded and $L^{P'}(\Omega,\mathbb{M}^{m\times n})$ is reflexive ($p'>1$). Hence by uniqueness of limit, $\sigma(x,\varpi,D\varpi)$ is its weak $L^{P'}$-limit (for a subsequence). Hence (5.2) is verified for each $\psi\in W_0^{1,p}(\Omega;\mathbb{R}^m)$.
Case (b): Let us put

$$E_x=:\Big\{\xi\in\mathbb{M}^{m\times n}:W(x,\varpi,\xi)=W(x,\varpi,D\varpi)+\sigma(x,\varpi,D\varpi):(\xi-D\varpi)\Big\},$$

with $x\in\Omega$ and $\varpi\in\mathbb{R}^m$.
First, we will show that $\operatorname{supp}\nu_x\subset E_x$, for almost every $x\in\Omega$.
The monotonicity of σ implies, for $\xi\in\operatorname{supp}\nu_x$ and $\tau\in[0,1]$, that

$$(1-\tau)\big(\sigma(x,\varpi,D\varpi+\tau(\xi-D\varpi))-\sigma(x,\varpi,\xi)\big):(D\varpi-\xi)\geq 0, \quad (6.4)$$

and according to (5.2),

$$(1-\tau)\big(\sigma(x,\varpi,\xi)-\sigma(x,\varpi,D\varpi)\big):(\xi-D\varpi)=0 \text{ for any } \tau\in[0,1]. \quad (6.5)$$

When we subtract (6.5) from (6.4), we obtain

$$(1-\tau)\big(\sigma(x,\varpi,D\varpi+\tau(\xi-D\varpi))-\sigma(x,\varpi,D\varpi)\big):(D\varpi-\xi)\geq 0 \text{ for any } \tau\in[0,1]. \quad (6.6)$$

We apply the monotonicity of σ again to obtain

$$\big(\sigma(x,\varpi,D\varpi+\tau(\xi-D\varpi))-\sigma(x,\varpi,D\varpi)\big):\tau(\xi-D\varpi)\geq 0 \text{ for any } \tau\in[0,1]. \quad (6.7)$$

Using (6.6) and (6.7), we conclude that

$$\big(\sigma(x,\varpi,D\varpi+\tau(\xi-D\varpi))-\sigma(x,\varpi,D\varpi)\big):(\xi-D\varpi)=0 \text{ for any } \tau\in[0,1].$$

Hence

$$\sigma(x,\varpi,D\varpi):(\xi-D\varpi)=\sigma(x,\varpi,D\varpi+\tau(\xi-D\varpi)):(\xi-D\varpi) \text{ for any } \tau\in[0,1]. \tag{6.8}$$

Since,

$$W(x,\varpi,\xi)=W(x,\varpi,D\varpi)+\big(W(x,\varpi,\xi)-W(x,\varpi,D\varpi)\big),$$

and in accordance with **(C2)(b)** we have

$$W(x,\varpi,\xi)=W(x,\varpi,D\varpi)+\int_0^1\sigma(x,\varpi,D\varpi+\tau(\xi-D\varpi)):(\xi-D\varpi)\,\mathrm{d}\tau.$$

Combining the last equality with (6.8), we get

$$W(x,\varpi,\xi)=W(x,\varpi,D\varpi)+\sigma(x,\varpi,D\varpi):(\xi-D\varpi).$$

So, it follows that $\xi\in E_x$ and consequently $\operatorname{supp}\nu_x\subset E_x$.

Thanks to the convexity of W relative to the third variable, we get for any $\xi\in\mathbb{M}^{m\times n}$ the following inequality

$$W(x,\varpi,\xi)\geq W(x,\varpi,D\varpi)+\sigma(x,\varpi,D\varpi):(\xi-D\varpi).$$

Let $A(\xi)=W(x,\varpi,\xi)$ and $B(\xi)=A(\xi)+\sigma(x,\varpi,D\varpi):(\xi-D\varpi)$.

For $\zeta\in\mathbb{M}^{m\times n}$, $\tau\in\mathbb{R}$ and $\xi\in E_x$, and taking into account that $W(x,\varpi,.)$ is $\mathcal{C}^1$, we get

$$\frac{A(\xi+\tau\zeta)-A(\xi)}{\tau}\geq\frac{B(\xi+\tau\zeta)-B(\xi)}{\tau}\quad\text{if } \tau>0,$$
$$\frac{A(\xi+\tau\zeta)-A(\xi)}{\tau}\leq\frac{B(\xi+\tau\zeta)-B(\xi)}{\tau}\quad\text{if } \tau<0.$$

So, $D_\xi A=D_\xi B$, and therefore, for all $\xi\in E_x\supset\operatorname{supp}\nu_x$, we have

$$\sigma(x,\varpi,\xi)=\sigma(x,\varpi,D\varpi). \tag{6.9}$$

Consequently

$$\overline{\sigma}(x):=\int_{\mathbb{M}^{m\times n}}\sigma(x,\varpi,\xi)d\nu_x(\xi)=\int_{\operatorname{supp}\nu_x}\sigma(x,\varpi,D\varpi)d\nu_x(\xi)=\sigma(x,\varpi,D\varpi). \tag{6.10}$$

To complete the proof we introduce the Carathéodory function $\Phi(x,\varphi,\xi)=|\sigma(x,\varphi,\xi)-\overline{\sigma}(x)|$. Moreover, consider the sequence $\Phi_j(x)=\Phi(x,\varpi_j,D\varpi_j)$, which is equiintegrable since $\sigma(x,\varpi_j,D\varpi_j)$ is equiintegrable. As a result,

$$\Phi_j\rightharpoonup\overline{\Phi}\quad\text{weakly in } L^1(\Omega),$$

with

$$\begin{aligned}\overline{\Phi}(x)&=\int_{\mathbb{R}^m\times\mathbb{M}^{m\times n}}\Phi(x,\varphi,\xi)d\delta_{\varpi(x)}(\varphi)\otimes d\nu_x(\xi)\\&\overset{(6.10)}{=}\int_{\mathbb{M}^{m\times n}}|\sigma(x,\varpi,\xi)-\sigma(x,\varpi,D\varpi)|d\nu_x(\xi)\\&=\int_{\operatorname{supp}\nu_x}|\sigma(x,\varpi,\xi)-\sigma(x,\varpi,D\varpi)|d\nu_x(\xi)\\&\overset{(6.9)}{=}0.\end{aligned}$$

Consequently, given $\Phi_j \geq 0$, we have $\Phi_j \to 0$ strongly in $L^1(\Omega)$.
Thus, (5.2) is obtained for any $\psi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ due to Vitali's theorem.

For the remaining three cases, we first show that the sequence $D\varpi_j \to D\varpi$ in measure (or again for a not relabeled subsequence).

Case (c): Using Lemma 5.3 and the strict monotonicity of σ, we conclude that $\operatorname{supp}\nu_x = \{D\varpi(x)\}$. Consequently, $\nu_x = \delta_{D\varpi(x)}$ for almost every $x \in \Omega$. Therefore, Proposition 2.10(i) implies $D\varpi_j \to D\varpi$ in measure as $j \to \infty$.

Case (d): Let $\Omega' \subset \Omega$ be a subset of positive Lebesgue measure and assume that ν_x is not a Dirac measure on it. Since $\overline{\xi} = \langle \nu_x, id\rangle = D\varpi$, it follows

$$\int_{\mathbb{M}^{m\times n}} \sigma(x,\varpi,\overline{\xi}) : \xi d\nu_x(\xi) = \sigma(x,\varpi,D\varpi) : \int_{\mathbb{M}^{m\times n}} \xi d\nu_x(\xi) \tag{6.11}$$

$$= \int_{\mathbb{M}^{m\times n}} \sigma(x,\varpi,\overline{\xi}) : \overline{\xi} d\nu_x(\xi). \tag{6.12}$$

Then, by (6.11) and the strict p-quasimonotonicity of σ relative to the third variable cited in **(C2)(d)**, one gets for a.e. $x \in \Omega'$

$$\int_{\mathbb{M}^{m\times n}} \sigma(x,\varpi,\xi) : \xi d\nu_x(\xi) > \int_{\mathbb{M}^{m\times n}} \sigma(x,\varpi,\xi) : \overline{\xi} d\nu_x(\xi).$$

We integrate this last inequality on Ω to obtain

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,\varpi,\xi) : \xi d\nu_x(\xi) dx > \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,\varpi,\xi) : \overline{\xi} d\nu_x(\xi) dx. \tag{6.13}$$

Using lemma 5.2 and taking into account (6.11), we get

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,\varpi,\xi) : \overline{\xi} d\nu_x(\xi) dx \geq \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,\varpi,\xi) : \xi d\nu_x(\xi) dx \tag{6.14}$$

The inequalities (6.13) and (6.14) lead to a contraction. Therefore, there exists a function h such that $\nu_x = \delta_{h(x)}$ for almost every $x \in \Omega$. Then

$$h(x) = \int_{\mathbb{M}^{m\times n}} \xi d\delta_{h(x)}(\xi) = \int_{\mathbb{M}^{m\times n}} \xi d\nu_{(x)}(\xi) = D\varpi(x).$$

This implies that $\nu_x = \delta_{D\varpi(x)}$ for almost every $x \in \Omega$. Therefore, we can deduce from Proposition 2.10(i) that $D\varpi_j \to D\varpi$ in measure on Ω as $j \to \infty$.

Case (e): By the condition of strict quasimonotonicity of σ relative to the third variable cited in **(C2)(e)** , we have

$$\begin{aligned}\int_\Omega |D\varpi_j - D\varpi|^p dx \leq \frac{1}{\alpha_0} \int_\Omega \big(\sigma(x,\varpi_j,D\varpi_j) - \sigma(x,\varpi,D\varpi)\big) : (D\varpi_j - D\varpi) dx \\ + \frac{1}{\alpha_0} \int_\Omega \big(\sigma(x,\varpi,D\varpi)) - \sigma(x,\varpi_j,D\varpi))\big) : (D\varpi_j - D\varpi) dx.\end{aligned}$$

Through the proof of Lemma 5.2, we find the following intermediate result

$$\liminf_{j\to\infty} \int_\Omega \big(\sigma(x,\varpi_j,D\varpi_j) - \sigma(x,\varpi,D\varpi)\big) : (D\varpi_j - D\varpi) dx = \liminf_{j\to\infty} \int_\Omega I_j dx \leq 0.$$

And since $\varpi_j \rightharpoonup \varpi$ in $W_0^{1,p}(\Omega;\mathbb{R}^m)$, it follows that ϖ_j is bounded in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ $\big($note that $W^{1,p}(\Omega;\mathbb{R}^m) \subset L^p(\Omega;\mathbb{R}^m)$ with compact embeding$\big)$. So $\varpi_j \to \varpi$ in $L^p(\Omega;\mathbb{R}^m)$. Therefore, Hölder inequality and continuity condition **(C0)** of σ allow to write

$$\liminf_{j\to\infty} \int_\Omega \big(\sigma(x,\varpi,D\varpi) - \sigma(x,\varpi_j,D\varpi)\big) : (D\varpi_j - D\varpi)dx = 0.$$

Consequently

$$\liminf_{j\to\infty} \int_\Omega |D\varpi_j - D\varpi|^p dx = 0.$$

Thus, $D\varpi_j \to D\varpi$ in measure as $j \to \infty$ for a subsequence.

To get the desired result in the last three cases, we proceed as follows: the fact that $\varpi_j \rightharpoonup \varpi$ in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ and that $W^{1,p}(\Omega;\mathbb{R}^m) \subset L^p(\Omega;\mathbb{R}^m)$ (with compact embeding) allow to write that $\varpi_j \to \varpi$ in $L^p(\Omega;\mathbb{R}^m)$ and $\varpi_j \to \varpi$ in measure. By extracting a subsequence from (ϖ_j) suitably if necessary [18, Theorem 2.30], we can deduce that $\varpi_j \to \varpi$ and $D\varpi_j \to D\varpi$ for almost every $x \in \Omega$. Hence, it follows from **(C0)**, for an arbitrary $\psi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$, that

$$\sigma(x,\varpi_j,D\varpi_j) : D\psi \to \sigma(x,\varpi,D\varpi) : D\psi \quad \text{almost everywhere.}$$

The measure of Ω being finite, so

$$\sigma(x,\varpi_j,D\varpi_j) : D\psi \to \sigma(x,\varpi,D\varpi) : D\psi \quad \text{in measure.}$$

Moreover, it follows from the growth condition in **(C1)**, the uniform bound (4.3) and the Hölder inequality that $(\sigma(x,\varpi_j,D\varpi_j) : D\psi)_j$ is equiintegrable (see the proof of Lemma 5.2). Therefore, the Vitali Theorem implies

$$\int_\Omega \sigma(x,\varpi_j,D\varpi_j) : D\psi dx \to \int_\Omega \sigma(x,\varpi,D\varpi) : D\psi dx \text{ as } j \to \infty \text{ for any } \psi \in \cup_{j\in\mathbb{N}} V_j.$$

And by the density of $\cup_{j\in\mathbb{N}} V_j$ in $W_0^{1,p}(\Omega;\mathbb{R}^m)$, we get the desired result for any $\psi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$. □

So far, we have the components needed to show Theorem 3.3. For this, it is enough to prove that there is $\varpi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ verifying $\langle \mathscr{T}(\varpi), \psi\rangle = 0$ for any $\psi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$. Indeed, by the uniform boundedness of the sequence ϖ_j in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ of Lemma 4.1, and by Eberlein-Šmulian theorem [11], there is a subsequence (also denoted by ϖ_j) and $\varpi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ verifying $\varpi_j \rightharpoonup \varpi$ in $W_0^{1,p}(\Omega;\mathbb{R}^m)$. So, $\varpi_j \to \varpi$ in $L^p(\Omega;\mathbb{R}^m)$ since $W^{1,p}(\Omega;\mathbb{R}^m) \subset L^p(\Omega;\mathbb{R}^m)$ with compact embeding, and therefore, $\varpi_j \to \varpi$ in measure. By extracting an appropriate subsequence if necessary, we may deduce that (ϖ_j) and $(D\varpi_j)$ converge almost everywhere respectively to ϖ and $D\varpi$. Then it follows from the continuity conditions **(A0)**, **(C3)(i)** and **(C4)(i)** that $\Theta(\varpi_j) : D\psi(x) \to \Theta(\varpi) : D\psi(x)$, $g(x,\varpi_j - \Upsilon(\varpi_j)).\psi(x) \to g(x,\varpi - \Upsilon(\varpi)).\psi(x)$ and $f(x,\varpi_j,D\varpi_j).\psi(x) \to f(x,\varpi,D\varpi).\psi(x)$ almost everywhere. Subsequently, $\Theta(\varpi_j) : D\psi(x)$, $g(x,\varpi_j - \Upsilon(\varpi_j)).\psi(x)$ and $f(x,\varpi_j,D\varpi_j).\psi(x)$ are equiintegrable according to boundedness condition in **(A0)**, to growth conditions **(C3)(ii)** and **(C4)(ii)**, the uniform bound (4.3) and Hölder inequality (see the proof of Lemma 5.2).

Hence by vitali's convergence theorem $\Theta(\varpi_j) : D\psi(x)$, $g(x,\varpi_j - \Upsilon(\varpi_j)).\psi(x)$ and

$f(x, \varpi_j, D\varpi_j).\psi(x)$ converge in $L^1(\Omega)$ to $\Theta(\varpi) : D\psi(x)$, $g(x, \varpi - \Upsilon(\varpi)).\psi(x)$ and $f(x, \varpi_j, D\varpi_j).\psi(x)$ respectively.
Therefore

$$\int_\Omega \Theta(\varpi_j) : D\psi dx \to \int_\Omega \Theta(\varpi) : D\psi dx \text{ as } j \to \infty \text{ for any } \psi \in \underset{k\in\mathbb{N}}{\cup} V_k,$$

$$\int_\Omega g(x, \varpi_j - \Upsilon(\varpi_j)).\psi dx \to \int_\Omega g(x, \varpi - \Upsilon(\varpi)).\psi dx \text{ as } j \to \infty \text{ for any } \psi \in \underset{k\in\mathbb{N}}{\cup} V_k,$$

and

$$\int_\Omega f(x, \varpi_j, D\varpi_j).\psi dx \to \int_\Omega f(x, \varpi, D\varpi).\psi dx \text{ as } j \to \infty \text{ for any } \psi \in \underset{k\in\mathbb{N}}{\cup} V_k.$$

Furthermore, by simple calculation, it is easy to verify that

$$\alpha(x)|\varpi_j|^{p-2}\varpi_j \to \alpha(x)|\varpi|^{p-2}\varpi \text{ in } L^{p'}(\Omega; \mathbb{R}^m).$$

It follows that

$$\int_\Omega \alpha(x)|\varpi_j|^{p-2}\varpi_j.\psi dx \to \int_\Omega \alpha(x)|\varpi|^{p-2}\varpi.\psi dx \text{ as } j \to \infty \text{ for any } \psi \in \underset{k\in\mathbb{N}}{\cup} V_k.$$

And by the density of $\cup_{k\in\mathbb{N}} V_k$ in $W_0^{1,p}(\Omega; \mathbb{R}^m)$, we get for any $\psi \in W_0^{1,p}(\Omega; \mathbb{R}^m)$,

$$\int_\Omega \Theta(\varpi_j) : D\psi dx \to \int_\Omega \Theta(\varpi) : D\psi dx \text{ as } j \to \infty,$$

$$\int_\Omega g(x, \varpi_j - \Upsilon(\varpi_j)).\psi dx \to \int_\Omega g(x, \varpi - \Upsilon(\varpi)).\psi dx \text{ as } j \to \infty,$$

$$\int_\Omega \alpha(x)|\varpi_j|^{p-2}\varpi_j.\psi dx \to \int_\Omega \alpha(x)|\varpi|^{p-2}\varpi.\psi dx \text{ as } j \to \infty,$$

and

$$\int_\Omega f(x, \varpi_j, D\varpi_j).\psi dx \to \int_\Omega f(x, \varpi, D\varpi).\psi dx \text{ as } j \to \infty.$$

On the other hand, it follows from (6.1) that for any $\psi \in W_0^{1,p}(\Omega; \mathbb{R}^m)$, we have

$$\int_\Omega \sigma(x, \varpi_j, D\varpi_j) : D\psi dx \to \int_\Omega \sigma(x, \varpi, D\varpi) : D\psi dx \text{ as } j \to \infty.$$

Therefore

$$\langle \mathscr{T}(\varpi_j), \psi\rangle \to \langle \mathscr{T}(\varpi), \psi\rangle \text{ as } j \to \infty; \text{for any } \psi \in W_0^{1,p}(\Omega; \mathbb{R}^m).$$

According to Lemma 4.2 1), the density of $\cup_{j\in\mathbb{N}} V_j$ in $W_0^{1,p}(\Omega; \mathbb{R}^m)$ and Mazur's theorem (Theorem 2 of [27]), we have for all $\psi \in W_0^{1,p}(\Omega; \mathbb{R}^m)$, $\langle \mathscr{T}(\varpi_j), \psi\rangle = 0$. This implies that for all $\psi \in W_0^{1,p}(\Omega; \mathbb{R}^m)$, $\langle \mathscr{T}(\varpi), \psi\rangle = 0$.
In conclusion, ϖ is effectively a weak solution for the system (1.1), and Theorem 3.3 is successfully proved.

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