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ON THE NUMERICAL RANGE OF SOME NONLINEAR OPERATORS IN l_p SPACES

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Abstract. The aim of this paper is to study properties of the numerical range $W(T)$ of some nonlinear operators in Banach spaces of sequences l_p

1. Introduction and preliminaries

We recall that the numerical range for a bounded linear operator A in a Hilbert space $\mathcal{H}$ (over a field $\mathbb{K} = \mathbb{R} \vee \mathbb{C}$) with a scalar product $\langle \cdot, \cdot \rangle$ from the very beginning was defined by

$$W(A) = \{\frac{\langle Ax, x \rangle}{\|x\|^2} : x \in \mathcal{H}, x \neq 0\} = \{\langle Ax, x \rangle \ : x \in \mathbb{S}(\mathcal{H})\}, \tag{1.1}$$

where $\mathbb{S}(\mathcal{H}) = \{x \in \mathcal{H} : \|x\| = 1\}$. Essentially this definition and fundamental properties of $W(A)$ goes back to Teopliz, [10].
Given an operator $A : \mathcal{H} \to \mathcal{H}$, we call the real number defined by

$$w(A) = sup\{|\lambda| : \lambda \in W(A)\} \tag{1.2}$$

the numerical radius of operator A and it plays important role in the study of operators not only in Hilbert spaces. The numerical radius w can be a norm; namely we have, see [5],

$$\begin{gathered} w(A) \geq 0, w(A) = 0 \Leftrightarrow A = 0 \\ w(\alpha A) = |\alpha| w(A) \quad (\alpha \in \mathbb{K}) \\ w(A + B) \leq w(A) + w(B). \end{gathered}$$

Even in case of an operator which acts in a finite dimensional Hilbert space calculation of the numerical range and the numerical radius is not always an easy work.
As an illustration we start with a linear operator in two-dimensional Hilbert space giving an elementary approach to this task. Let us take the next

Example 1.1. Given the matrix

$$A = \begin{pmatrix} a & 0 \\ d & a \end{pmatrix} \tag{1.3}$$

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where $a, d \in \mathbb{C}$. The eigenvalue $a = p + iq$ of matrix A, implies $d = \|A - aI\| = \|A - a\|$. A is a linear operator in two-dimensional Hilbert space $\mathbb{C}^2$ with ordinary norm $\|A\| = \max_{i,j} |a_{i,j}|$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Let $f = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ be a vector in $\mathbb{C}^2$, with $|\alpha|^2 + |\beta|^2 = 1$, i.e. $\|f\| = 1$. On the other side we have

$$Af = \begin{pmatrix} a & 0 \\ \|A - a\| & a \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a\alpha \\ \alpha\|A - a\| + a\beta \end{pmatrix}$$

It is easy to calculate $\langle Af, f\rangle = a(|\alpha|^2 + |\beta|^2) + \alpha\overline{\beta}\|A - a\|$ and $|\langle Af, f\rangle - a| = |\alpha\overline{\beta}|\|A - a\|$. More over, $|\langle Af, f\rangle - a|^2 = |\alpha|^2|\beta|^2\|A - a\|^2$. The quadratic equation

$$T^2 - T + \frac{|\langle Af, f\rangle - a|^2}{\|A - a\|^2} = 0$$

must have real solutions $\{T_1, T_2\} = \{|\alpha|^2, |\beta|^2\}$. Therefore, $|\langle Af, f\rangle - a|^2 \leq (\frac{\|A-a\|}{2})^2$ and finally, if we put $x + iy = \langle Af, f\rangle$,

$$W(A) = \{(x, y) : (x - p)^2 + (y - q)^2 \leq \left(\frac{\|A - a\|}{2}\right)^2\} \tag{1.4}$$

From (1.4) we find the numerical radius $w(A) = sup\{|\lambda| : \lambda \in W(A)\} = |a| + \frac{\|A-a\|}{2}$.

The power inequality of the numerical radius of the operator (1.3) is our next goal. If we take A in the form

$$A = a \begin{pmatrix} 1 & 0 \\ \frac{\|A-a\|}{a} & 1 \end{pmatrix}$$

then

$$A^2 = a^2 \begin{pmatrix} 1 & 0 \\ 2\frac{\|A-a\|}{a} & 1 \end{pmatrix}$$

so, for any $n \in \mathbb{N}$, by induction we get

$$A^n = a^n \begin{pmatrix} 1 & 0 \\ n\frac{\|A-a\|}{a} & 1 \end{pmatrix}$$

or

$$A^n = a^{n-1} \begin{pmatrix} a & 0 \\ n\|A - a\| & a \end{pmatrix} = a^{n-1}\widehat{A},$$

where $\widehat{A}$ denotes matrix $\begin{pmatrix} a & 0 \\ n\|A - a\| & a \end{pmatrix}$. Now using relation (1.4) for $\widehat{A}$ we have

$$W(\widehat{A}) = \{(x, y) : (x - p)^2 + (y - q)^2 \leq (\frac{n\|A - a\|}{2})^2\}$$

Since $W(A^n) = a^{n-1}W(\widehat{A})$ then $w(A^n) = |a|^n(1 + \frac{n\|A-a\|}{2|a|})$ and by Bernoulli‘s inequality we have

$$w(A^n) = |a|^n(1 + \frac{n\|A - a\|}{2|a|}) \leq |a|^n\left(1 + \frac{\|A - a\|}{2|a|}\right)^n = [w(A)]^n.$$

Essentially we proved statement (d) of the next

Lemma 1.2. [2] *Let A be bounded and continuous operator in $\mathcal{H}$ and $\alpha, \beta, \lambda \in \mathbb{K}$ and I is identity map. The numerical range (1.1) has the following properties:*
(a) $W(\alpha I + \beta A) = \{\alpha + \beta\lambda : \lambda \in W(A)\}$*;*
(b) $W(A)$ *is convex;*
(c) *In case* $\mathcal{H} = \mathbb{C}^n, W(A)$ *is compact;*
(d) *In case* $\mathcal{H} = \mathbb{C}^2, W(A)$ *is the convex hull of an ellipse whose foci are two complex eigenvalues of A; if there is only one eigenvalue λ, the ellipse is a circle with center λ and radius* $\frac{\|A-\lambda I\|}{2}$.

The first definition of a numerical range for bounded and continuous operator F in Hilbert spaces was given by Zarantonello [11],

$$W_Z(F) = \left\{ \frac{\langle Fx - Fy, x - y\rangle}{\|x - y\|^2} : x, y \in \mathcal{H}, x \neq y \right\}.$$

This definition coincides with (1.1) in the linear case.
It is not so clear who was the first to give definition of a numerical range for nonlinear operators in Banach spaces. Defining the numerical range of a bounded linear operator in a Banach space requires to recall additional facts; see [4]. Given a Banach space $\mathcal{B}$ with dual $\mathcal{B}^*$, we use $\langle x, \ell\rangle$ to denote the value of linear functional $\ell \in \mathcal{B}^*$ at point $x \in \mathcal{B}$. Recall that duality map $\mathcal{D} : \mathcal{B} \to \mathcal{B}^*$ is defined by

$$\mathcal{D}(x) = \{\ell_x \in \mathcal{B}^* : \langle x, \ell_x\rangle = \| x \|^2, \| \ell_x \| = \| x \|\} \quad (x \in \mathcal{B}).$$

Since

$$\mathcal{D}(tx) = \{\ell_{tx} \in \mathcal{B}^* : \langle tx, \ell_{tx}\rangle = \| tx \|^2, \| \ell_{tx} \| = \| tx \|\} = t\mathcal{D}(x),$$

this map is homogeneous. In general case this is multi-valued map. However, if $\mathcal{D}$ is single-valued, the space $\mathcal{B}$ is called smooth space.

Definition 1.3. [8] Given a linear space $\mathcal{X}$ over $\mathbb{K}$, the mapping $[\cdot, \cdot] : \mathcal{X} \times \mathcal{X} \to \mathbb{K}$ is called semi-inner product if it has following properties (for all $x, y, z \in \mathcal{X}, \alpha, \beta \in \mathbb{K}$):

(1) $[x, x] > 0$ if and only if $x \neq 0$;
(2) $[x + y, z] = [x, z] + [y, z]$;
(3) $[\alpha x, y] = \alpha[x, y]$;
(4) $[x, \beta y] = \overline{\beta}[x, y]$;
(5) $|[x, y]|^2 \leq [x, x] \cdot [y, y]$.

Now, if $[x, y]$ is a semi-inner product on $\mathcal{X}$, one may define a norm on this space putting

$$\|x\| = \sqrt{[x, x]}$$

Conversely, on any normed linear space $\mathcal{X}$ we can define a semi-inner product by means of the duality map $\mathcal{D}$. For any selection $\ell_x \in \mathcal{D}(x)$ for all $x \in \mathcal{X}$ we put

$$[x, y] = \langle y, \ell_x\rangle \quad (y \in \mathcal{X}). \tag{1.5}$$

For the nonlinear superposition operator $F : l_p \to l_p$ generated by a function $f(s, u) : \mathbb{N} \times \mathbb{C} \to \mathbb{C}$ putting $Fx(s) = f(s, x(s))$, also known as Nemytskij's operator, we recall several properties, namely

Theorem 1.4. [3]*Let* $1 \leq p, q < \infty$. *Then the following properties are equivalent:*
(i) the superposition operator F *acts from* l_p *to* l_q ,
(ii) there exist functions $a(s) \in l_q$ *and constants* $\delta > 0, b \geq 0, n \in \mathbb{N}$, *for which*

$$|f(s,u)| \leq a(s) + b|u|^{\frac{p}{q}} \quad (s \geq n, |u| < \delta);$$

(iii) for any $\varepsilon > 0$ *there exist a function* $a_\varepsilon(s) \in l_q$ *and constants* $\delta_\varepsilon > 0, b_\varepsilon \geq 0, n_\varepsilon \in \mathbb{N}$, *for which* $\|a_\varepsilon(s)\|_q < \varepsilon$ *and*

$$|f(s,u)| \leq a_\varepsilon(s) + b_\varepsilon|u|^{\frac{p}{q}} \quad (s \geq n_\varepsilon, |u| < \delta_\varepsilon). \tag{1.6}$$

2. Main results

In this section, we assume that $\mathcal{B} = l_p$, where $1 < p < \infty$ is a complex Banach space of complex sequences $x(s), s \in \mathbb{N}$, with norm

$$\|x(s)\|_p = (\sum_{s\in\mathbb{N}} |x(s)|^p)^{\frac{1}{p}}$$

and with $\mathcal{D}(x) = \{\ell_x\}$ given by product

$$\langle y, \ell_x\rangle = \frac{1}{\|x\|^{p-2}} \sum_{s\in\mathbb{N}} |x(s)|^{p-2}\overline{x(s)}y(s) \quad (x \in l_p \setminus \{0\}, y \in l_p). \tag{2.1}$$

Lemma 2.1. *The* $l_p; p > 1$ *is a smooth space with* $\mathcal{D}(x) = \{\ell_x\}$ *where* ℓ_x *is defined by* (2.1).

Proof. It is well known that the linear functional ℓ_x in Hilbert space l_2 has unique representation by the scalar product

$$(y, \ell_x) = \sum_{s=1}^{\infty} \overline{x(s)}y(s), \tag{2.2}$$

where $y \in l_2$ is unique. Moreover, we first consider the problem of multipliers from l_p to l_q.We recall that a function $c(s)$ is called a multiplier from l_p to l_q, if $c(s)h(s) \in l_q$ for some $h(s) \in l_p$. The set l_p/l_q of all multipliers from l_p to l_q with the norm $\|c\|_{p,q} = \sup_{\|h\|\leq 1} \|ch\|_q$ is a Banach space; see [1]. It is known (see [3]) that relation

$$lp/lq = \begin{cases} l_{pq(p-q)^{-1}} & \text{if } \ p > q \\ l_\infty & \text{if } \ p \leq q \end{cases}$$

holds.
Put $\tau(s) = \left(\frac{|x(s)|}{\|x\|}\right)^{p-2}$ and $q = \frac{p}{p-1}$, then we have

$$\sum_{s=1}^{\infty} [\tau(s)]^{\frac{p}{p-2}} = \sum_{s=1}^{\infty} [(\frac{|x(s)|}{\|x\|})^{p-2}]^{\frac{p}{p-2}} = \sum_{s=1}^{\infty} \frac{|x(s)|^p}{\|x\|_p^p} = 1.$$

Therefore, multiplier $\tau \in l_{\frac{p}{p-2}}, p > 2$. On the other side, if we take $x \in l_p\setminus\{0\}$ and $y \in l_p$ then $\tau(s)y(s)$ belongs $l_{\frac{p}{p-1}}$, and from (2.2) we get (2.1). In case $1 < p < 2$, $\tau(s) \in l_\infty$, because $\sup_{s\in\mathbb{N}} |\tau(s)| = \sup_{s\in\mathbb{N}}(\frac{|x(s)|}{\|x\|})^{p-2} = 1$ and for $x \in l_p\setminus\{0\}, y \in l_p$ we have (2.1). So it is proved that $\mathcal{D} = \{\ell_x\}$ is a single-valued map. □

Here and in what follows the semi-inner product, see (1.5), we denote by

$$[x,y] = \langle y, \ell_x\rangle, (x, y \in l_p). \tag{2.3}$$

Here we should consider when a nonlinear superposition operator $F : l_p \to l_p$ generated by a function $f(s,u) : \mathbb{N}\times\mathbb{C} \to \mathbb{C}$ by $Fx(s) = f(s, x(s))$ is continuous. We denote by P_D the operator of multiplication by the characteristic function χ_D of the set $D \subset \mathbb{N}$ acting in each space l_p. For brevity we will put $P_n = P_{\{n+1,n+2,\dots\}}$. Now we have the next

Theorem 2.2. *Let $1 \le p, q < \infty$ and let the superposition operator F , generated by the function $f(s,u)$, acts from l_p to l_q. This operator is continuous if and only if each of the functions $f(s,u); s \in \mathbb{N}$, are continuous.*

Proof. Let the functions $f(s,u)(s \in \mathbb{N})$ be continuous and let $x_0 \in l_p$. Moreover, let ε be an arbitrary positive number and let δ_ε and n_ε be numbers for which, for the corresponding $a_\varepsilon, (\|a_\varepsilon\|_q \le \varepsilon)$ and $b_\varepsilon \ge 0$, be as in Theorem 1.3, i.e. relation (1.6) holds. Put $\eta = \frac{\delta_\varepsilon}{2}$, and let $\widetilde{n}$ be a natural number such that $\widetilde{n} \ge n_\varepsilon$ and $\|P_{\widetilde{n}} x_0\| \le \eta \le (b_\varepsilon^{-1}\varepsilon)^{\frac{q}{p}}$. Then for $\|x - x_0\|_p \le \eta$ and $s > \widetilde{n}$ we have the inequalities $|x(s)| \le \delta_\varepsilon$, and therefore in view of (1.6),

$$|f(s, x_0(s))| \le a_\varepsilon(s) + b_\varepsilon|x_0(s)|^{\frac{p}{q}}$$

$$|f(s, x(s))| \le a_\varepsilon(s) + b_\varepsilon|x(s)|^{\frac{p}{q}}.$$

Hence using inequality

$$\|Fx - Fx_0\|_q \le \left(\sum_{s=1}^{s=\widetilde{n}} |f(s,x(s)) - f(s, x_0(s))|^q\right)^{\frac{1}{q}} + \|FP_{\widetilde{n}}x\|_q + \|FP_{\widetilde{n}}x_0\|_q$$

it follows that

$$\|Fx - Fx_0\|_q \le \left(\sum_{s=1}^{s=\widetilde{n}} |f(s,x(s)) - f(s, x_0(s))|^q\right)^{\frac{1}{q}} + \\ +2\|a_\varepsilon\|_q + b_\varepsilon(\|P_{\widetilde{n}}x(s)\|_p^{\frac{p}{q}} + \|P_{\widetilde{n}}x_0(s)\|_p^{\frac{p}{q}}. \tag{2.4}$$

On the other side we have

$$\|P_{\widetilde{n}}x(s)\|_p = (\textstyle\sum_{s=\widetilde{n}+1}^{\infty} |x(s)|^p)^{\frac{1}{p}} = \\ (\textstyle\sum_{s=\widetilde{n}+1}^{\infty} |x(s) - x_0(s) + x_0(s)|^p)^{\frac{1}{p}} \le \|x - x_0\|_p + \|P_{\widetilde{n}}x_0\|_p$$

Now from (2.4) we get

$$\|Fx - Fx_0\|_q \le \left(\sum_{s=1}^{s=\widetilde{n}} |f(s,x(s)) - f(s, x_0(s))|^q\right)^{\frac{1}{q}} + \\ +2\|a_\varepsilon\|_q + b_\varepsilon(\|P_{\widetilde{n}}x(s)\|_p^{\frac{p}{q}} + \|P_{\widetilde{n}}x_0(s)\|_p^{\frac{p}{q}}) \le \\ \le \left(\sum_{s=1}^{s=\widetilde{n}} |f(s,x(s)) - f(s, x_0(s))|^q\right)^{\frac{1}{q}} + \\ +3\varepsilon + b_\varepsilon((b_\varepsilon^{-1}\varepsilon)^{\frac{q}{p}} + \|x - x_0\|_p)^{\frac{p}{q}}. \tag{2.5}$$

Since the functions $f(s,u)$, $s \in \mathbb{N}$, are continuous, we can take $0 < \theta < \eta$, such that for $\|x - x_0\|_p < \theta$ the first term on the right-hand side of the relation (2.5) is not greater then ε; moreover, the last term in this part of inequality is not greater then 2ε. So we conclude that for $\|x - x_0\|_p < \theta$ we have inequality $\|Fx - Fx_0\|_q < 6\varepsilon$, and therefor the operator F is continuous at the point x_0.
As for necessity of the conditions of the theorem, it is sufficient to notice that for each $s \in \mathbb{N}$ the complex function $f(s,u)$ is a superposition of the next three functions:

the imbedding $\sigma_s(u) : u \mapsto u\chi_{\{s\}}$ of the complex plane $\mathbb{C}$ into l_p;
the operator F;
the surjection $\varepsilon_s(y) : y \mapsto y(s)$ of the space l_q into complex plane $\mathbb{C}$.

Since the first function $u \mapsto u\chi_{\{s\}}$ i. e. $u \mapsto (0, 0, ..., s, ...)$ is continuous, because (for any $u, v \in \mathbb{C}$) we have $\|\sigma_s(u) - \sigma_s(v)\|_p = \|u\chi_{\{s\}} - v\chi_{\{s\}}\|_p = |u - v|$, and the third function $y \mapsto y(s)$ is continuous because

$$|\varepsilon_s(y_1) - \varepsilon_s(y_2)| = |y_1(s) - y_2(s)| \leq \|y_1 - y_2\|_q$$

the superposition $f(s,u) = \varepsilon_s \circ F \circ \sigma_s(u)$ is continuous if and only if F is continuous. □

Now we pass to the numerical ranges of nonlinear operators in Banach spaces $l_p; p > 1$. This can be made either by means of the duality map or using semi-inner product, [2]. Since l_p is a smooth space we prefer use of semi-inner product, and a Rhodius definition of the numerical range and numerical radius ([9]), as following

$$W_R(F) = \left\{ \frac{[Fx - Fy, x - y]}{\|x - y\|^2} : x - y \in l_p \backslash \{0\} \right\},$$

where the semi-inner product $[\cdot, \cdot]$ is defined by (2.3) and $w_R(F) = \{|\lambda| : \lambda \in W_R(F)\}$. Note that this definition is given in [2] .
The function $f(s,u) = a(s) + b(s)u$, where $a(s) \in l_p$ is a complex function of the natural argument s and $b(s) \in l_\infty$, generates a continuous superposition operator $Fx(s) = f(s, x(s))$ in l_p (Theorem 1.4 and Theorem 2.2).

Theorem 2.3. *Let F be the superposition operator generated by the complex function $f(s,u) = a(s) + b(s)u$, where $a \in l_p, p > 1$ and $b \in l_\infty$. Then the numerical range $W_R(F)$ is a convex set.*

Proof. Note that $|f(s,u)| \leq |a(s)| + |b||u|$ and in view of Theorem 1.4, the operator F acts in l_p space. Moreover operator F has form $Fx = a + Lx$ where $a \in l_p$ is fixed and L is a linear operator. Since $W_R(\alpha L + \beta) = \alpha W_R(L) + \beta$ (see Lema 1.2) we can suppose $\frac{[Lx,x]}{\|x\|^2} = \xi$, and $\frac{[Ly,y]}{\|y\|^2} = \eta$, $\xi \neq \eta$ as arbitrary elements in $W_R(L)$. Since there exist $\alpha, \beta \in \mathbb{C}$ $(\alpha = (\xi - \eta)^{-1}; \beta = \eta(\eta - \xi)^{-1})$ such that: $\alpha\xi + \beta = 1$ and $\alpha\eta + \beta = 0$ we can take that $\frac{[Lx,x]}{\|x\|^2} = 1$, and $\frac{[Ly,y]}{\|y\|^2} = 0$ are given arbitrary elements. Now it should be sufficient to prove that the segment [0,1] belongs to the numerical range $W_R(\alpha L + \beta) = \alpha W_R(L) + \beta$ (see Lema 1.2). Indeed, if

$\alpha\frac{[Lh,h]}{\|h\|^2}+\beta = t$, we have $\alpha\frac{[Lh,h]}{\|h\|^2}+\beta = t(\alpha\xi+\beta)+(1-t)(\alpha\eta+\beta) = \alpha[t\xi+(1-t)\eta]+\beta$. So we take $\xi = 1$ and $\eta = 0$. On the other side, if

$$h(t) = tx + (1-t)y, (0 \leq t \leq 1; x, y \in l_p \setminus \{0\}),$$

then it is clear that $h(t) \neq 0$, for any $t \in [0,1]$. Namely, if we suppose that for some $t_0 \in (0,1)$, $h(t_0) = 0$, then $y = \frac{t_0}{t_0-1}x$, and $\frac{[Lx,x]}{\|x\|^2} = b = \frac{[Ly,y]}{\|y\|^2}$; contradiction. Note that $\phi(t) = \frac{[Fh(t),h(t)]}{\|h(t)\|^2} = \frac{t(1-b)\|x\|^2-(1-t)b\|y\|^2}{\|h\|^2} + b$ is a real function. Moreover, $\phi(0) = 0$ and $\phi(1) = 1$ and since $\phi(t)$ is continuous on segment $[0,1]$ one can conclude $[0,1] \subseteq R_\phi$, where R_ϕ denotes range of function ϕ. □

Let

$$\mathcal{C} = \{T : l_p \to l_p\}$$

denotes the set of all continuous operators $T : l_p \to l_p$. Note that operators $Fx = a(s) + b(s)x(s)$ belongs $\mathcal{C}$, where $b(s) \in l_p/l_p$ can be a constant bounded complex sequence $b(s) = b, s \in \mathbb{N}$ too.
In $\mathcal{C}$ we introduce the class $\mathcal{S}$ of singular operators acting in l_p spaces. We say ([6]) that, for a given complex number λ, an operator $S \in \mathcal{S}$ if at least one of the next statements is not true:

(1) $(\lambda I - S)$ is injection;
(2) $(\lambda I - S)$ is surjection;
(3) $(\lambda I - S)^{-1} \in \mathcal{C}$.

Recall that an operator is regular if it is not singular. Class of regular operators in $\mathcal{C}$ we denote by $\mathcal{T}$.
For the class of all continuous operators F on Banach space l_p we define the characteristics

$$[T]_{Lip} = sup_{x\neq y}\frac{\|Tx - Ty\|}{\|x - y\|}, \tag{2.6}$$

and

$$[T]_{lip} = inf_{x\neq y}\frac{\|Tx - Ty\|}{\|x - y\|}$$

and we say T is Lipshviz continuous (shortly Lip-continuous) if $[T]_{Lip} < \infty$. We can also define the norms in $\mathcal{C}$, if $[T]_B < +\infty$, by

$$[T]_B = sup_{x\neq 0}\frac{\|Tx\|}{\|x\|} \tag{2.7}$$

and, if $T(0) = 0$, by (2.6).
For the class of all continuous operators F on Banach space l_p over field $\mathbb{C}$ the Rhodius resolvent set ([9]) is given by:

$$\rho_R(F) = \{\lambda \in \mathbb{C} : F \quad \textit{is regular} \}$$

and the Rhodius spectrum is the set $\sigma_R(F) = \mathbb{C} \setminus \rho_R(F)$.
Here and in the sequel we call a bijection $F : l_p \to l_p$ a lipeomorphism if both

F and F^{-1} is Lip-continuous. For the class of all Lip-continuous operators F on Banach space l_p over field $\mathbb{C}$ the Kachurovskij resolvent set ([9]) is given by

$$\rho_K(F) = \{\lambda \in \mathbb{C} : (\lambda I - F) \text{ is lipeomorphism}\}$$

and the Kachurovskij spectrum is the set $\sigma_K(F) = \mathbb{C} \setminus \rho_K(F)$.
For our farther study we need to introduce, [7]:

$$\sigma_{lip}(F) = \{\lambda \in \mathbb{C} : [\lambda I - F]_{lip} = 0\}.$$

Using (1.2) and relations (2.6) and (2.7) one can prove that for any $T \in \mathcal{C}$ we have

$$c[T]_B \leq w_R(T) \leq [T]_B; \quad c[T]_{Lip} \leq w_K(T) \leq [T]_{Lip}. \tag{2.8}$$

where $w_K(T) = \sup\{|\lambda| : \lambda \in W_R(T)\}$ and $0 < c < 1$ is a constant.
Since there is no matter which norm we would take, here and forward notation is $\|T\|$. So, for any $\lambda \in W_R(T)$, $|\lambda| = \frac{|[Tx,x]|}{\|x\|^2} \leq \frac{\|Tx\|\|x\|}{\|x\|^2} = \frac{\|Tx\|_p}{\|x\|} \leq \|T\|$ therefore $w_R(T) \leq \|T\|$. As for the first inequality of (2.8), if we suppose that for any $0 < c < 1$ holds $c\|T\| > w_R(T)$, then we get result $\|T\| > \frac{1}{c}w_R(T)$ with $\frac{1}{c}$ arbitrary large number, e.g. $\|T\| = \infty$; contradiction.
It is well known [see [2]] the chain of inequalities

$$[T]_{lip} \leq w_R(T) \leq w_K(T) \leq [T]_B \leq [T]_{Lip}.$$

Now in $\mathcal{C}$ we have two pair equivalent norms and those indicate the same topology in space $\mathcal{C}$. If we take $d(F, G) = \|F - G\|$, as usually, $\mathcal{C}$ becomes a metric space.

Lemma 2.4. *([2]) The characteristics $[F]_{lip}$ and $[F]_{Lip}$ have the following properties ($F, G \in \mathcal{C}$):*

(1) $[F]_{lip} - [G]_{Lip} \leq [F + G]_{lip} \leq [F]_{lip} + [G]_{Lip}$
(2) $|[F]_{lip} - [G]_{lip}| \leq [F - G]_{Lip}$
(3) $[F]_{lip} > 0$ *implies F is injection and closed.*

The set $\sigma_{lip}(F) = \{\lambda \in \mathbb{C} : [\lambda I - F]_{lip} = 0\}$ is closed. Firstly, we see that

$$\sigma_{lip}(F) \subseteq \{\lambda \in \mathbb{C} : [F]_{lip} \leq |\lambda| \leq [F]_{Lip}\}. \tag{2.9}$$

Indeed, if $\lambda \in \sigma_{lip}(F)$ is fixed, then $[\lambda I - F]_{lip} = 0$ (Lemma 2.4 (1)) implies

$$[F]_{lip} - [\lambda I]_{Lip} \leq 0 \quad \text{and} \quad [\lambda I]_{lip} - [F]_{Lip} \leq 0.$$

Consequently, $[F]_{lip} \leq [\lambda I]_{Lip} = |\lambda| = [\lambda I]_{lip} \leq [F]_{Lip}$, what we claimed by (2.9). On the other side, using relations (2) from Lemma 2.4, we have

$$|[\lambda I - F]_{lip} - [\mu I - F]_{lip}| \leq |\lambda - \mu| \quad (\lambda, \mu \in \mathbb{C})$$

therefore $\sigma_{lip}(F)$ is closed set.

Theorem 2.5. *Let F be the Lip-continuous operator in $l_p; p > 1$ and $0 \in \sigma_{lip}(F)$. Then $W_R(F) = \sigma_{lip}(F)$. So $W_R(F)$ is compact.*

Proof. We claim that $W_R(F) = \sigma_{lip}(F)$. Suppose that $\lambda \in \mathbb{C}$ does not belong $\overline{W_R(F)} \supseteq W_R(F)$ then $d_\lambda = dis(\lambda, W_R(F)) = \inf_{x \neq y} |\lambda - \frac{[Fx-Fy, x-y]}{\|x-y\|^2}| > 0$ but then we, for any $x \neq y$, have

$$d_\lambda < |\lambda - \frac{[Fx - Fy, x - y]}{\|x - y\|^2}| = \frac{|[\lambda(x - y), x - y] - [Fx - Fy, x - y]|}{\|x - y\|^2} =$$

$$= \frac{|[\lambda(x-y)-(Fx-Fy),x-y]|}{\|x-y\|^2} \leq \frac{\|\lambda(x-y)-(Fx-Fy)\|\|x-y\|}{\|x-y\|^2} =$$
$$= \frac{\|\lambda x - Fx - (\lambda y - Fy)\|}{\|x-y\|}.$$

From the last relations we conclude that $[\lambda I - F]_{lip} \geq d_\lambda > 0$ and λ is not in $\sigma_{lip}(F)$.

On the other side if $\lambda \neq 0$ and $\lambda \notin \sigma_{lip}(F)$ then $[I - F/\lambda]_{lip} > 0$; $\lambda I - F = \lambda(I - F/\lambda)$ is at list an closed injection (see relations (3) from Lemma 2.4). Moreover, the operator $\lambda I - F$ is a lipeomorphism ([2], Theorem 3.6), therefore $\lambda \notin \sigma_K(F)$. So $|\lambda| > w_K(F)$ and since $w_K(F) \leq [F]_{Lip}$ (see (2.8)) we have that $|\lambda| \geq [F]_{Lip}$. Now if one suppose for this λ that $\lambda \in W_R(F)$ we obtain

$$[F]_{Lip} \leq |\lambda| = \frac{|[Fx-Fy,x-y]|}{\|x-y\|^2} \leq \frac{\|Fx-Fy\|\|x-y\|}{\|x-y\|^2} \leq [F]_{Lip}$$

hence

$$|[Fx-Fy,x-y]| = \|Fx-Fy\|\|x-y\|.$$

From the last equation we conclude that there exists $\alpha \in \mathbb{C}$ such that $Fx - Fy = \alpha(x-y)$. Now we have that $x \neq y$ implies

$$|\alpha| = |\alpha|\frac{\|x-y\|^2}{\|x-y\|^2} = \frac{|[\alpha(x-y),x-y]|}{\|x-y\|^2} = \frac{|[Fx-Fy,x-y]|}{\|x-y\|^2} = |\lambda|$$

therefore $x \neq y$ implies $\lambda x - Fx = \lambda y - Fy$ hence $\lambda \in \sigma_K(F)$; contradiction. Finally we have $\lambda \notin W_R(F)$ and $W_R(F) \subseteq \sigma_{lip}(F)$.

It is clear that $W_R(F)$ is bounded. Moreover, closedness follows from $W_R(F) = \sigma_{lip}(F)$. □

Corollary 2.6. *Let Σ be a given compact set in complex plane. Then there exist a nonlinear Lip-continuous operator F which acts in l_p such that $\Sigma = W_R(F)$.*

Proof. First note that if $B = \{b(s) : s \in \mathbb{N}\}$ is a dense set in Σ then we can take complex function $f(s,u) = a(s) + b(s)u$ where $a(s)$ is a fixed sequence in l_p. And it is easy note that the function $f(s,u)$ generates superposition operator $Fx = a(s) + b(s)x$ in l_p spaces since $(b(s))_{s\in\mathbb{N}}$ is bounded, see Lemma 2.1.On the other hand we have

$$[F]_{Lip} = \sup_{x\neq y}\frac{\|Fx-Fy\|}{\|x-y\|} = \sup_{x\neq y}\frac{\|b(s)(x-y)\|}{\|x-y\|} = \sup_{\|h\|=1}\|b(s)h\|_p \leq \|b(s)\|_\infty < \infty$$

hence operator F is Lip-continuous. Moreover, if $\lambda \in W_R(F)$ then

$$|\lambda| = \frac{|[Fx-Fy,x-y]|}{\|x-y\|^2} = \frac{|b(s)|[(x-y),x-y]}{\|x-y\|^2} = |b(s)|$$

therefore $\lambda \in B \subseteq \Sigma$. On the other side, if $b(s_i)$ is an arbitrary element in B then one can take $x_0 = (0,0,...,x(s_i),0,0,...)$ and $y_0 = (0,0,...,y(s_i),0,0,...)$ and calculation

$$\frac{|[Fx_0-Fy_0,x_0-y_0]|}{\|x_0-y_0\|^2} = \frac{|b(s_i)|[(x_0-y_0),x_0-y_0]}{\|x_0-y_0\|^2} = |b(s_i)|$$

show that $b(s_i)$ belongs to $W_R(F)$. Since Σ is closed we have $\Sigma \subseteq W_R(F)$. □

Note that supposition $0 \in \sigma_{lip}(F)$ in Theorem 2.5 is essential. That fact one can illustrate by the following

Example 2.7. Let $F : l_p \to l_p; p > 1$ be defined by

$$F(x(1), x(2), x(3), ...) = (\|x\|, x(1), x(2), x(3), ...).$$

It is known (see [2], Example 5.3) that $\sigma_{lip}(F) = \{\lambda \in \mathbb{C} : 1 \leq |\lambda| \leq 2^{\frac{1}{p}}\}$, so that $0 \notin \sigma_{lip}(F)$.Let $\ell_x, x \neq 0$ be functional given by (2.1). For $\alpha \in \overline{\mathbb{D}} = \{\xi \in \mathbb{C} : |\xi| \leq 1\}$ and $x = (\overline{\alpha}, 0, (1-|\alpha|^p)^{\frac{1}{p}}, 0, 0, ...) \in l_p$ we get $\|x\| = 1$ and calculating ℓ_x at element $Fx = (1, \overline{\alpha}, 0, (1-|\alpha|^p)^{\frac{1}{p}}, 0, ...)$ we obtain

$$\langle Fx, \ell_x \rangle = \frac{1}{\|x\|^{p-2}} \sum_{s \in \mathbb{N}} |x(s)|^{p-2} \overline{x(s)} Fx(s) = |\alpha|^{p-2}\alpha.$$

So, we have

$$\overline{\mathbb{D}} = \{|\alpha|^{p-2}\alpha : |\alpha| \leq 1\} \subseteq W_R(F).$$

In other words, $W_R(F) \neq \sigma_{lip}(F)$.

Example 2.8. Let $e \in \mathbb{S}(l_p)$ be fixed, and operator F defined by $Fx = \|x\|e$ then $W_R(F)$ is compact and convex set. It is not hard to calculate $[F]_{lip}$ and $[F]_{Lip}$, namely

$$[F]_{lip} = \inf_{x \neq y} \frac{\|Fx - Fy\|}{\|x - y\|} = \inf_{x \neq y} \frac{|\|x\| - \|y\||}{\|x - y\|} = 0$$

because there exist $x \neq y$ with $\|x\| = \|y\|$, e.g. $x = e_i; y = e_j (i \neq j); e_i, e_j \in \mathbb{S}(l_p)$. And

$$[F]_{Lip} = \sup_{x \neq y} \frac{\|Fx - Fy\|}{\|x - y\|} = \sup_{x \neq y} \frac{|\|x\| - \|y\||}{\|x - y\|} \leq 1.$$

Since $F(0) = 0$ we can take $x \neq 0$ arbitrary and $y = 0$ then obtain $[F]_{Lip} = 1$. So we have $\sigma_{lip}(F) \subseteq \overline{\mathbb{D}}$. We have to show that $\sigma_{lip}(F) \supseteq \overline{\mathbb{D}}$ is truth. Indeed, $0 \in \sigma_{lip}(F)$ and for all $0 < |\lambda| \leq 1$ choosing

$$\widehat{x} = \frac{|\lambda| + 1}{2|\lambda|} \overline{\lambda} e, \quad \widehat{y} = \frac{|\lambda| - 1}{2|\lambda|} \overline{\lambda} e$$

we easy obtain $F\widehat{x} = \frac{1}{2}(1 + |\lambda|)e$ and $F\widehat{y} = \frac{1}{2}(1 - |\lambda|)e$. Consequently,

$$[\lambda I - F]_{lip} = \inf_{x \neq y} \frac{\|\lambda x - Fx - (\lambda y - Fy)\|}{\|x - y\|} \leq \frac{\|\lambda \widehat{x} - F\widehat{x} - (\lambda \widehat{y} - F\widehat{y})\|}{\|\widehat{x} - \widehat{y}\|} = |\lambda| - |\lambda| = 0$$

which implies the relation $\overline{\mathbb{D}} \subseteq \sigma_{lip}(F)$. The statement follows from Theorem 2.6.

Now we introduce and observe in $\mathcal{C}$ the following mappings:

$$\varphi : F \to W_R(F), \quad \psi : F \to w_R(F), \quad \theta : F \to \sigma_R(F). \tag{2.10}$$

Whether those functions will be continuous in some sense it depends on the topology in $\mathcal{C}$ and $\mathbb{C}$. We are also interested whether mapping $F \to \sigma_R(F)$ is continuous with respect to any topology. Here we note that for linear operators in a Hilbert space answer on the last question is negative; see Problem 103 in [5], see also [6] . In other words there exist the operators with large spectra in any neighbourhood of operators with small spectra.

Theorem 2.9. *The function φ, given in (2.10), is continuous with respect to the uniform (norm) topology.*

Proof. In what sense the numerical range $W_R(F)$ is a continuous function of its argument F? The best way to ask the question is in terms of the Hausdorff metric for compact subsets of the complex plane. To define that metric, write $M + (\varepsilon) = \{z + \alpha : z \in M, |\alpha| < \varepsilon\}$ for each set M of complex numbers and each positive number ε. In this notation, if M and N are compact sets, the Hausdorff distance $d(M, N)$ between them is the infimum of all positive numbers ε such that both $M \subseteq N + (\varepsilon)$ and $N \subseteq M + (\varepsilon)$. Since the Hausdorff metric is defined for compact sets, the appropriate function to discuss, in general, is $\overline{W(F)}$, not $W(F)$. As for the continuity question, it still has as many interpretations as there are topologies for operators, but we stay here on uniform continuity. Let $\varepsilon > 0$ be given and $\|F - G\| < \varepsilon$. Since

$$\frac{|[(F-G)x, x]|}{\|x\|^2} \leq \frac{\|(F-G)x\|}{\|x\|} \leq \|F - G\| < \varepsilon,$$

we have for $\lambda \in W_R(F)$, i.e.

$$\lambda = \frac{[Fx, x]}{\|x\|^2} = \frac{[Gx, x]}{\|x\|^2} + \frac{[(F-G)x, x]}{\|x\|^2}$$

From the last relation we conclude $\lambda \in W_R(G) + (\varepsilon)$ hence $W_R(F) \subseteq W_R(G) + (\varepsilon)$ and changing the roles of F and G, we get $W_R(G) \subseteq W_R(F) + (\varepsilon)$. Moreover, the Hausdorff distance $d(W_R(F), W_R(G)) = 0$.

As a consequence it is immediately clear: if φ is continuous with respect to any topology, then so is $\psi(F) = w_R(F)$, and consequently, if ψ is discontinuous, then so is φ.

Indeed, let us take $\|F - G\| < \varepsilon$ for any non-negative ε and suppose that the function φ is continuous. Since

$$\sup\{|\lambda| : \lambda \in W_R(F)\} \leq \sup\{|\lambda| : \lambda \in W_R(G) + (\varepsilon)\},$$

we have $w_R(F) \leq w_R(G) + \varepsilon$ i.e., $w_R(F) - w_R(G) \leq \varepsilon$. Analogously one can get $w_R(G) - w_R(F) \leq \varepsilon$. Therefore, we have: $\|F - G\| < \varepsilon \Rightarrow |\psi(F) - \psi(G)| \leq \varepsilon$. □

Example 2.10. Let now $a(s), s \in \mathbb{Z}$ be a sequence from $l_p(\mathbb{Z})$ and

$$x = (\cdots, x(-3), x(-2), x(-1), x(0), x(1), x(2), x(3), \cdots) \in l_p(\mathbb{Z})$$

then, for any $n \in \mathbb{N}$, $F = a(s) + L_0$, $F_n = a(s) + L_n$ are continuous operators where linear operator

$$L_n x = (\cdots, h(-3), h(-2), h(-1), h(0), h(1), h(2), h(3), \cdots)$$

with $h(s)$ defined by

$$h(s) = \begin{cases} x(s-1), s \neq 0 \\ \frac{1}{n} x(-1), s = 0. \end{cases}$$

The operator L_0 is defined by

$$L_0 x = (\cdots, h(-3), h(-2), h(-1), h(0), h(1), h(2), h(3), \cdots)$$

and

$$h(s) = \begin{cases} x(s-1), s \neq 0 \\ 0, \quad s = 0. \end{cases}$$

As for spectra of the operator L_0 we have that $(\lambda I - L_0)x = (\lambda I - L_0)y$ for $\lambda = 0$ implies $L_0 x = L_0 y$ hence

$$(\cdots, x(-3), x(-2), 0, x(0), x(1), x(2), \cdots) = (\cdots, y(-3), y(-2), 0, y(0), y(1), y(2), \cdots)$$

so that $0I - L_0$ is injection. Since equation $L_0 x = 0$ has non-trivial solution $\widehat{x} = (\cdots, 0, 0, x(-1), 0, 0, 0, \cdots)$ we conclude $0 \in \sigma_R(L_0)$.
Let now λ be a complex number. Then $(\lambda I - L_0)x = (\lambda I - L_0)y$ implies $x = y$ so that $\lambda I - L_0$ is injective operator. However, equation $(I - L_0)x = 0$ has non-trivial solution $\widehat{x} = (\cdots, x(-2), 0, x(0), x(1), x(2), \cdots)$, so that $1 \in \sigma_R(L_0)$. Let be $|\lambda| > 1$. Whether the operator $(\lambda I - L_0)^{-1}$ is continuous for this value of λ? First if $y \in l_p(\mathbb{Z})$ then there exist $x \in l_p(\mathbb{Z})$ so $(\lambda I - L_0)x = y$. Indeed, if we take $x = (\cdots, \frac{1}{\lambda-1}y(-2), \frac{1}{\lambda}y(-1), \frac{1}{\lambda-1}y(0), \frac{1}{\lambda-1}y(1), \cdots)$ then

$$\begin{aligned} \|x\| = \|(\cdots, \tfrac{1}{\lambda-1}y(-2), \tfrac{1}{\lambda}y(-1), \tfrac{1}{\lambda-1}y(0), \tfrac{1}{\lambda-1}y(1), \cdots)\| = \\ = |\tfrac{1}{\lambda-1}|(\textstyle\sum_{s\neq -1} |y(s)|^p)^{\frac{1}{p}} + |\tfrac{1}{\lambda} - \tfrac{1}{\lambda-1}||y(-1)| \leq \tfrac{2}{|\lambda|-1}\|y\| \end{aligned}$$

and $(\lambda I - L_0)x = y$. The operator $(\lambda I - L_0)^{-1}$ is a bijection. On the other side, for all λ with $|\lambda| > 1$, we have

$$\begin{gathered} \|(\lambda I - L_0)^{-1}x - (\lambda I - L_0)^{-1}y\| = \\ = \|(\cdots, \tfrac{1}{\lambda-1}x(-2), \tfrac{1}{\lambda}x(-1), \tfrac{1}{\lambda-1}x(0), \tfrac{1}{\lambda-1}x(1), \cdots) - \\ -(\cdots, \tfrac{1}{\lambda-1}y(-2), \tfrac{1}{\lambda}y(-1), \tfrac{1}{\lambda-1}y(0), \tfrac{1}{\lambda-1}y(1), \cdots)\|_p = \\ = \|(\cdots, \tfrac{1}{\lambda-1}(x(-2) - y(-2)), \tfrac{1}{\lambda}(x(-1) - y(-1)), \tfrac{1}{\lambda-1}(x(0) - y(0)), \\ \tfrac{1}{\lambda-1}(x(1) - y(1)), \cdots)\| \leq \tfrac{2}{|\lambda|-1}\|x - y\| \end{gathered}$$

From the last inequalities we conclude that $(\lambda I - L_0)^{-1}$ is a continuous operator for any $\lambda, |\lambda| > 1$, and finally $\sigma_R(F) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. In the same way one can prove that $\sigma_R(F_n) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Now we have

$$\|F_n x - Fx\| = \|L_n x - L_0 x\| = \left(\sum_{s=-\infty}^{s=\infty} |L_n x(s) - L_0 x(s)|^p \right)^{\frac{1}{p}} \leq \frac{1}{n}\|x\|$$

we conclude that $\|F_n - F\| = sup_{x\neq 0} \frac{\|L_n x - L_0 x\|}{\|x\|} \leq \frac{1}{n} \to 0, n \to \infty$, however $\theta(F_n) = \sigma_R(F_n) = \sigma_R(L_n) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and $\theta(F) = \sigma_R(F) = \sigma_R(L_0) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.
On the other side if $z \in \sigma_R(L_n)$ then $z = e^{i\omega}$ and if $w \in \sigma_R(L_0)$ then $w = \rho e^{i\omega}$; $\rho \leq 1$. Now it is clear that $\sigma_R(L_n) \subseteq \sigma_R(L_0) + (\varepsilon)$, for any $\varepsilon \geq 1$. If $w \in \sigma_R(L_0)$ then $w + \alpha \in \sigma_R(L_n) + (\varepsilon)$ if and only if $|w + \alpha| = 1 \leq |w| + |\alpha|$ and $|\alpha| \geq 1 - \rho$, therefor the Hausdorff distance of the spectra is $d(\sigma_R(F_n), \sigma_R(F)) = 1$, and the function θ is not continuous.

Upper semi-continuity for a set-valued function such as $\sigma_R(F)$ has the metric definition, [6]: to each open set Λ that includes $\sigma_R(F)$ there corresponds a positive number ε such that if $\|F - G\| < \varepsilon$ then $\sigma_R(G) \subseteq \Lambda$.

Theorem 2.11. *The function θ given by (2.10), is upper semi-continuous in the norm topology.*

Proof. It is well known that set of regular operators is open in metric space $\mathcal{C}$(see [6]) Let $\mathcal{S}$ be the set of all singular operators (on a fixed l_p space). Let F be fixed operator and $\varrho(\lambda)$ be the distance (in the metric space of operators) from $\lambda I - F$ to set $\mathcal{S}$. The function $\varrho(\lambda) = d(\lambda I - F, \mathcal{S}) = \inf_{T\in\mathcal{S}} \|\lambda I - F - T\|$ is continuous. Indeed,

$$|\varrho(\lambda_1) - \varrho(\lambda_2)| = |d(\lambda_1 I - F, \mathcal{S}) - d(\lambda_2 I - F, \mathcal{S})| \leq$$
$$\leq |d(\lambda_1 I - F, T) - d(\lambda_2 I - F, T)| \leq d(\lambda_1 I - F, \lambda_2 I - F) =$$
$$= \|\lambda_1 I - \lambda_2 I\| = |\lambda_1 - \lambda_2|.$$

Let Λ be an open set which includes $\sigma_R(F)$, and $\Delta = \{\lambda : |\lambda| \leq 1 + \|F\|\}$. If $\lambda \in \Delta\backslash\Lambda$, then $\varrho(\lambda) > 0$. Indeed, since $\mathcal{S}$ is closed, from $\varrho(\lambda) = 0$, i.e. $d(\lambda I - F, \mathcal{S}) = 0$ it follows $\lambda I - F \in \mathcal{S}$, and we have $\lambda \in \sigma_R(F)$; contradiction. Since $\Delta\backslash\Lambda$ is compact, there exists a positive number ε such that $\varrho(\lambda) \geq \varepsilon$ for all $\lambda \in \Delta\backslash\Lambda$. Without loss of generality we may assume that $\varepsilon < 1$. Suppose now that $\|F - G\| < \varepsilon$ hence it follows that $\|\lambda I - F - (\lambda I - G)\| = \|F - G\| < \varepsilon \leq \varrho(\lambda)$ for all $\lambda \in \Delta\backslash\Lambda$. This implies that $\lambda I - G$ is not in $\mathcal{S}$, and hence that λ does not belong $\sigma_R(G)$. In conclusion we have $\sigma_R(G) \cap (\Delta\backslash\Lambda) = \emptyset$. At the same time, if $\lambda \in \sigma_R(G)$, then

$$|\lambda| \leq \|G\| = \|F + (G - F)\| \leq \|F\| + \|F - G\| < \|F\| + \varepsilon < 1 + \|F\|$$

so that $\sigma_R(G) \subset \Delta$. From these two properties of $\sigma_R(G)$ we get $\sigma_R(G) \subset \Lambda$. □

It is not hard to see that the results from this paper one can carry over on the nonlinear operators in many other functional spaces.

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References

1. J. Appell, P.P. Zabreiko, *Nonlinear superposition operators*, Cambridge University Press, Cambridge 1990.
2. J. Appell,E. De Pascale, A. Vignoli, *Non Linear Spectral Theory*, Walter de Gruyter - Berlin - New York, 2004, 408 p.
3. F. Dedagich, P.P. Zabreiko, *On the superposition operators in l_p spaces*, Sibirskij matematičeskij zhurnal XXVIII, N^o1. Engl.transl., Plenum Publishing Corporation, 1987.
4. J. Diestel, *Geometry of Banach spaces - Selected Topics*, Lecture Notes in Math. 485, Springe - Verlag, Berlin 1975.
5. P.R. Halmos, *A Hilbert space problem book*,D.Van Nostrand Company,Inc., Princeton, New Jersey, 1967, 350 p.
6. E.Hille, R.S. Phillips, *Functional Analysis and Semigroups*, Amer. Math. Soc. Collquium Publ. 31, Providence, 1957.
7. I.J. Maddox, A.W. Wickstead,*The spectrum of uniformly Lipschitz mappings*, Proc.Roy. Irish Acad. Sect. A 89 (1989), 101-114.
8. R.H. Martin, *Nonlinear operator and differential equqtions in Banach spaces*, J. Wiley, New York, 1976.
9. A. Rhodius, *Der numerische wertebereish fur nicht notwendig lineare Abbildungen in nicht notwendig lokalkonvexen Raumen*, Math. Nachr. 72 (1976), 169-180.

10. O. Teoplitz, *Das algebraische Analogon zu einem Satz von Féjèr*, Math.Z.2 (1918), 187-197.
11. E.H. Zerantonello, *The closure of the numerical range conteins the spectrum*, Bull. Amer. Math.Soc. 70, (1964),781-787.

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