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DERIVATIONS AND DIMENSIONALLY NILPOTENT DERIVATIONS IN LIE TRIPLE ALGEBRAS

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Abstract. In this paper, we first study derivations in non nilpotent Lie triple algebras. We determine the structure of derivation algebra according to whether it admits an idempotent or a pseudo-idempotent. We study the multiplicative structure of non nil dimensionally nilpotent Lie triple algebras. We show that when $n = 2p + 1$ the adapted basis coincides with the canonical basis of the gametic algebra $G(2p + 2, 2)$ or this one obviously associated to a pseudo-idempotent and if $n = 2p$ then the algebra is either one of the precedent case or a conservative Bernstein algebra.

1. Introduction

A $n + 1$ finite dimensional algebra A is dimensionally nilpotent if there is a derivation $d : A \longrightarrow A$ such that $d^{n+1} = 0$ and $d^n \neq 0$. This notion has been studied by G.F. Leger and P.L. Manley[5] for Lie algebras, J.M. Osborn [9] for Jordan algebras, Micali and Ouattara[6] for genetic algebras. Recently, V. Eberlin [3] has deepened the work of the authors of [5] in his thesis. Regarding Jordan algebras, Osborn shows that every dimensionally nilpotent Jordan K-algebra is either nilpotent or satisfies $A/Rad(A) \simeq K$.

We study the case of non nilpotent dimensionally Lie triple algebras. In an adapted basis we characterize the multiplicative structure of these algebras with respect to the parity of n. More precisely we show that when $n = 2p + 1$, the adapted basis coincides with canonical basis of the gametic algebra $G(2p + 2, 2)$ [11] or this one obviously associated to a pseudo-idempotent. If $n = 2p$ then this algebra is either one of the precedent case or a train algebra of rank 3 which is a Jordan algebra [10]. Since Jordan algebras are also Lie triple ones the final corollary describes non nilpotent dimensionally nilpotent Jordan algebras.

2. Preliminaries

A *Lie triple algebra* is a commutative algebra satisfying

$$2x(x(xy)) + yx^3 = 3x(yx^2) \tag{2.1}$$

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while a *Jordan algebra* is a commutative algebra satisfying

$$x^2(yx) = (x^2y)x.$$

Every Jordan algebra satisfies identity (2.1).

Theorem 2.1 ([4])**.** *Let A be a Lie triple algebra and L the ideal generated by the associators (x^2, x, x). Then $L^2 = 0$ and A/L is a Jordan algebra.*

Definition 2.2. A *pseudo-idempotent* of A is a non-zero element e such that there is $t \neq 0$ in L satisfying $e^2 = e + t$ and $et = \frac{1}{2}t$.

Theorem 2.3 ([2])**.** *Every Lie triple non nil-algebra contains either a non-zero idempotent, or a pseudo-idempotent.*

Definition 2.4. An ideal I of an algebra A is said to be *characteristic* if $d(I) \subseteq I$ for every derivation d of A. An ideal I of an algebra A is said to be *d-invariant* if $d(I) \subseteq I$ for a given derivation d of A.

3. Characterization of derivations

In this paragraph we study the derivations in Lie triple non nil-algebras. We give a characterization, distinguishing two cases: with an idempotent or with a pseudo-idempotent.

3.1. **Lie triple algebras with idempotent.** With respect to a non-zero idempotent e, the algebra A admits the following Peirce decomposition $A = A_e(1) \oplus A_e(1/2) \oplus A_e(0)$. Relations between Peirce components and the products of their elements are ruled by the following lemma:

Lemma 3.1 ([2, Lemme 2.2])**.** *Let $A = A_e(1) \oplus A_e(1/2) \oplus A_e(0)$ be the Peirce decomposition of A relative to a non-zero idempotent. Then*

(*i*) $A_e(1/2)A_e(1/2) \subseteq A_e(1) + A_e(0)$, $A_e(\lambda)A_e(\lambda) \subseteq A_e(\lambda)$,
$A_e(\lambda)A_e(1/2) \subseteq A_e(1/2)$, $A_e(\lambda)A_e(1-\lambda) = 0$, $(\lambda = 0, 1)$;

(*ii*) $(x_1y_1)a_{1/2} = x_1(y_1a_{1/2}) + y_1(x_1a_{1/2})$,
$(x_0y_0)a_{1/2} = x_0(y_0a_{1/2}) + y_0(x_0a_{1/2})$;

(*iii*) $[x_1(x_{1/2}a_{1/2})]_1 = [(x_1x_{1/2})a_{1/2} + (x_1a_{1/2})x_{1/2}]_1$,
$[x_0(x_{1/2}a_{1/2})]_0 = [(x_0x_{1/2})a_{1/2} + (x_0a_{1/2})x_{1/2}]_0$;

(*iv*) $[(x_1x_{1/2})y_{1/2}]_0 = [(x_1y_{1/2})x_{1/2}]_0$,
$[(x_0x_{1/2})y_{1/2}]_1 = [(x_0y_{1/2})x_{1/2}]_1$;

(*v*) $x_0(y_1a_{1/2}) = y_1(x_0a_{1/2})$;

(*vi*) $x_{1/2}(x_{1/2}^2)_1 = x_{1/2}(x_{1/2}^2)_0 = \frac{1}{2}x_{1/2}^3$;

(*vii*) $(x_{1/2}y_{1/2})_0z_{1/2} + (y_{1/2}z_{1/2})_0x_{1/2} + (z_{1/2}x_{1/2})_0y_{1/2}$
$= (x_{1/2}y_{1/2})_1z_{1/2} + (y_{1/2}z_{1/2})_1x_{1/2} + (z_{1/2}x_{1/2})_1y_{1/2}$.

Since A is e-stable, i.e. $A_e(\lambda)A_e(1/2) \subseteq A_e(1/2)$ and $[(x_\lambda x_{1/2})y_{1/2}]_{1-\lambda} = [(x_\lambda y_{1/2})x_{1/2}]_{1-\lambda}$ with $\lambda = 0, 1$, calculations on derivations give results similar to [1, Corollary 2], precisely.

Theorem 3.2. *Every derivation d of A is determined and only defined by a quadruplet $(d(e), f_d, g_d, h_d)$ with $f_d \in End_K(A_e(1/2))$, $g_d \in Der_K(A_e(0))$ and $h_d \in Der_K(A_e(1))$ satisfying the following conditions:*

(i) $d(e) \in A_e(1/2)$;
(ii) $d(x_1) = h_d(x_1) + 2d(e)x_1$;
(iii) $d(x_{1/2}) = f_d(x_{1/2}) + 2(d(e)x_{1/2})_0 - 2(d(e)x_{1/2})_1$;
(iv) $d(x_0) = g_d(x_0) - 2d(e)x_0$;
(v) $h_d(x_1y_1) = h_d(x_1)y_1 + x_1h_d(y_1)$;
(vi) $g_d(x_0y_0) = g_d(x_0)y_0 + x_0g_d(y_0)$;
(vii) $h_d((x_{1/2}y_{1/2})_1) = [f_d(x_{1/2})y_{1/2} + x_{1/2}f_d(y_{1/2})]_1$;
(viii) $g_d((x_{1/2}y_{1/2})_0) = [f_d(x_{1/2})y_{1/2} + x_{1/2}f_d(y_{1/2})]_0$;
(ix) $f_d(x_1x_{1/2}) = h_d(x_1)x_{1/2} + x_1f_d(x_{1/2})$;
(x) $f_d(x_0x_{1/2}) = g_d(x_0)x_{1/2} + x_0f_d(x_{1/2})$.

Proposition 3.3. *Let A be a Lie triple algebra and $A = A_e(1) \oplus A_e(1/2) \oplus A_e(0)$ the Peirce decomposition of A with respect to an idempotent $e \neq 0$. Subspaces $J_\lambda = \{x_\lambda \in A_e(\lambda) \mid x_\lambda A_e(1/2) = 0\}$ $(\lambda = 0, 1)$ and $J = J_0 \oplus J_1$ are characteristic ideals of A and the quotient algebra A/J is a Jordan algebra.*

Proof. Considering $J_\lambda = \ker(S_\lambda)$, with $S_\lambda : A_e(\lambda) \to End_K(A_e(1/2))$, $x_\lambda \mapsto S_\lambda(x_\lambda)$ and $S_\lambda(x_\lambda) : a_{1/2} \mapsto x_\lambda a_{1/2}$. We know by ([7]) that J_λ is an ideal of $A_e(\lambda)$ $(\lambda = 0, 1)$ and since $A_e(\lambda)A_e(1/2) \subseteq A_e(1/2)$, then $J = J_1 + J_0$ is an ideal of A such that A/J is a Jordan algebra ([8], Proposition 6.7).

Let's consider $d \in Der_K(A)$, $x_\lambda \in J_\lambda(e)$ and $a_{1/2} \in A_e(1/2)$. We have $0 = d(x_\lambda a_{1/2}) = x_\lambda d(a_{1/2}) + d(x_\lambda)a_{1/2}$.
But $d(a_{1/2}) = f_d(a_{1/2}) + 2(d(e)a_{1/2})_0 - 2(d(e)a_{1/2})_1$, therefore we have $x_\lambda d(a_{1/2}) = 0$ because $x_\lambda(d(e)a_{1/2})_\lambda = [(x_\lambda d(e))a_{1/2} + (x_\lambda a_{1/2})d(e)]_\lambda$. Hence $d(x_\lambda)a_{1/2} = 0$ with $\lambda = 0, 1$. But, on the one hand we have $d(x_1) = h_d(x_1) - 2d(e)x_1$, and $0 = d(x_1)a_{1/2} = h_d(x_1)a_{1/2}$ and then $h_d(x_1) \in J_1$, on the other hand we have $d(x_0) = g_d(x_0) - 2d(e)x_0$, with $0 = d(x_0)a_{1/2} = g_d(x_0)a_{1/2}$ and then $g_d(x_0) \in J_0$. Hence $d(J_\lambda) \subseteq J_\lambda$ and we conclude that $d(J) \subseteq J$. □

3.2. **Lie triple algebras with pseudo-idempotent.**

Lemma 3.4 ([2], Proposition 4.3)**.** *Let $L = L_e(1) \oplus L_e(1/2) \oplus L_e(0)$ and $A = A_e(1) \oplus A_e(1/2) \oplus A_e(0)$ be the respective Peirce decomposition of L and A, with respect to the pseudo-idempotent e, satisfying $e^2 = e + t$ with $t \in L_e(1/2)$ fixed. Then*

(i) $A_e(0)L_e(1/2) \subseteq L_e(1/2)$, $A_e(1)L_e(1/2) \subseteq L_e(1/2)$, $A_e(1)L_e(1) \subseteq L_e(1)$, $A_e(0)L_e(0) \subseteq L_e(0)$, $A_e(0)L_e(1) = A_e(1)L_e(0) = 0$, $A_e(1/2)L_e(0) = A_e(1/2)L_e(1) = A_e(1/2)L_e(1/2) = 0$;
(ii) $A_e(1)A_e(0) \subseteq L_e(1/2)$, $A_e(0)A_e(1/2) \subseteq A_e(1/2)$, $A_e(1)A_e(1/2) \subseteq A_e(1/2)$, $A_e(0)A_e(0) \subseteq A_e(0) + L_e(1/2)$, $A_e(1)A_e(1) \subseteq A_e(1) + L_e(1/2)$, $A_e(1/2)A_e(1/2) \subseteq A_e(1) + A_e(0)$;
(iii) $(x_0y_0)_{1/2} = 4(x_0t)y_0 = 4(y_0t)x_0$;
$(x_1y_1)_{1/2} = 4(x_1t)y_1 = 4(y_1t)x_1$;
$(x_0y_1)_{1/2} = 4(x_0t)y_1 = 4(y_1t)x_0$;
(iv) $(x_1y_1)a_{1/2} = x_1(y_1a_{1/2}) + y_1(x_1a_{1/2})$;
(v) $(x_0y_0)a_{1/2} = x_0(y_0a_{1/2}) + y_0(x_0a_{1/2})$;
(vi) $x_0(y_1a_{1/2}) = y_1(x_0a_{1/2})$;
(vii) $[x_0(x_{1/2}a_{1/2})]_0 = [(x_0x_{1/2})a_{1/2} + (x_0a_{1/2})x_{1/2}]_0$;

$(viii)$ $[x_1(x_{1/2}a_{1/2})]_1 = [(x_1x_{1/2})a_{1/2} + (x_1a_{1/2})x_{1/2}]_1$;
(ix) $[(x_0x_{1/2})y_{1/2}]_1 = [(x_0y_{1/2})x_{1/2}]_1$;
$[(x_1x_{1/2})y_{1/2}]_0 = [(x_1y_{1/2})x_{1/2}]_0$;
(x) $(x_{1/2}y_{1/2})_0z_{1/2} + (y_{1/2}z_{1/2})_0x_{1/2} + (z_{1/2}x_{1/2})_0y_{1/2}$
$= (x_{1/2}y_{1/2})_1z_{1/2} + (y_{1/2}z_{1/2})_1x_{1/2} + (z_{1/2}x_{1/2})_1y_{1/2}$.

Lemma 3.5. *Let A be a Lie triple algebra and e a pseudo-idempotent of A: $e^2 = e + t$, $et = \frac{1}{2}t$, $t^2 = 0$ with $t \in L$. For every derivation d of A, we have*

$$d(t) = 0 \text{ and } d(e) \in A_e(1/2).$$

Proof. Let's consider $d \in Der_K(A)$. Since $e^2 = e+t$, we have $2ed(e) = d(e)+d(t)$. Setting $d(e) = [d(e)]_1+[d(e)]_{1/2}+[d(e)]_0$, we have $d(t) = [d(e)]_1-[d(e)]_0$. Because of $2et = t$, we deduce $2ed(t) + 2d(e)t = d(t)$. We have $2d(e)t = -[d(e)]_1 - [d(e)]_0$. We know that $t \in L_e(1/2)$ and $L_e(1/2)$ is an ideal of A. It follows that $[d(e)]_1 = [d(e)]_0 = 0$. □

Theorem 3.6. *Every derivation d of A is determined and only defined by a quadruplet $(d(e), f_d, g_d, h_d)$ with $f_d \in End_K(A_e(1/2))$, $g_d \in End_K(A_e(0))$ and $h_d \in End_K(A_e(1))$ satisfying the following conditions:*
(i) $d(e) \in A_e(1/2)$;
(ii) $d(x_1) = h_d(x_1) + 2d(e)x_1$;
(iii) $d(x_{1/2}) = f_d(x_{1/2}) + 2(d(e)x_{1/2})_0 - 2(d(e)x_{1/2})_1$;
(iv) $d(x_0) = g_d(x_0) - 2d(e)x_0$;
(v) $h_d((x_1y_1)_1) = [h_d(x_1)y_1 + x_1h_d(y_1)]_1$;
$f_d((x_1y_1)_{1/2}) = [h_d(x_1)y_1 + x_1h_d(y_1)]_{1/2} = 2h_d((x_1y_1)_1)t$;
(vi) $g_d((x_0y_0)_0) = [g_d(x_0)y_0 + x_0g_d(y_0)]_0$;
$f_d((x_0y_0)_{1/2}) = [g_d(x_0)y_0 + x_0g_d(y_0)]_{1/2} = 2g_d((x_0y_0)_0)t$;
(vii) $h_d((x_{1/2}y_{1/2})_1) = [f_d(x_{1/2})y_{1/2} + x_{1/2}f_d(y_{1/2})]_1$;
$g_d((x_{1/2}y_{1/2})_0) = [f_d(x_{1/2})y_{1/2} + x_{1/2}f_d(y_{1/2})]_0$;
$(viii)$ $f_d(x_1x_0) = h_d(x_1)x_0 + x_1g_d(x_0)$;
(ix) $f_d(x_1x_{1/2}) = h_d(x_1)x_{1/2} + x_1f_d(x_{1/2})$;
(x) $f_d(x_0x_{1/2}) = g_d(x_0)x_{1/2} + x_0f_d(x_{1/2})$.

Proof. Let d be a derivation of A and e a pseudo-idempotent of A. Since $d(e) \in A_e(1/2)$, we have (i). Let $x_1 \in A_e(1)$. We have $ex_1 = x_1$, and then $d(e)x_1 + ed(x_1) = d(x_1)$. Let's set $d(x_1) = a_1 + a_{1/2} + a_0$. Then $d(e)x_1 + a_1 + \frac{1}{2}a_{1/2} = a_1 + a_{1/2} + a_0$, and we have $a_{1/2} = 2d(e)x_1$ and $a_0 = 0$. Hence $d(x_1) = h_d(x_1) + 2d(e)x_1$ with h_d an endomorphism of $A_e(1)$ and (ii) is proved.

By similar calculations we have (iii) and (iv).

With x_1, $y_1 \in A_e(1)$, we have $(x_1y_1)_{1/2} = 4x_1(y_1t) = 4y_1(x_1t) = 2(x_1y_1)t$

$$\begin{aligned} d(x_1y_1) &= d[(x_1y_1)_1] + d[(x_1y_1)_{1/2}] = d[(x_1y_1)_1] + 2d((x_1y_1)t) \\ &= h_d[(x_1y_1)_1] + 2d(e)(x_1y_1) + 2d((x_1y_1))t + 2(x_1y_1)d(t) \\ &= h_d[(x_1y_1)_1] + 2h_d(x_1y_1)t + 2d(e)(x_1y_1), \end{aligned}$$

because $d((x_1y_1))t = h_d(x_1y_1)t$.

We also have,

$$\begin{aligned} d(x_1y_1) &= d(x_1)y_1 + x_1d(y_1) \\ &= x_1[h_d(y_1) + 2d(e)y_1] + [h_d(x_1) + 2d(e)x_1]y_1 \\ &= h_d(x_1)y_1 + x_1h_d(y_1) + 2[d(e)y_1]x_1 + 2[d(e)x_1]y_1. \\ &= h_d(x_1)y_1 + x_1h_d(y_1) + 2d(e)(x_1y_1). \end{aligned}$$

It follows that

$$h_d((x_1y_1)_1) = [h_d(x_1)y_1 + x_1h_d(y_1)]_1 \text{ and}$$

$$f_d((x_1y_1)_{1/2}) = [h_d(x_1)y_1 + x_1h_d(y_1)]_{1/2} = 2h_d(x_1y_1)t, \text{ and we have } (v).$$

We show by similar calculations that:

$$g_d((x_0y_0)_0) = [g_d(x_0)y_0 + x_0g_d(y_0)]_0 \text{ et}$$

$$f_d((x_0y_0)_{1/2}) = [g_d(x_0)y_0 + x_0g_d(y_0)]_{1/2} = 2g_d(x_0y_0)t, \text{and we have } (vi).$$

Let $x_{1/2}, y_{1/2} \in A_e(1/2)$, we have

$$\begin{aligned} d(x_{1/2}y_{1/2}) &= d((x_{1/2}y_{1/2})_1) + d((x_{1/2}y_{1/2})_0) \\ &= h_d((x_{1/2}y_{1/2})_1) + 2d(e)(x_{1/2}y_{1/2})_1 \\ &\quad + g_d((x_{1/2}y_{1/2})_0) - 2d(e)(x_{1/2}y_{1/2})_0, \end{aligned}$$

but

$$\begin{aligned} d(x_{1/2}y_{1/2}) &= d(x_{1/2})y_{1/2} + x_{1/2}d(y_{1/2}) \\ &= (f_d(x_{1/2}) + 2[d(e)x_{1/2}]_0 - 2[d(e)x_{1/2}]_1)y_{1/2} \\ &\quad + x_{1/2}(f_d(y_{1/2}) + 2[d(e)y_{1/2}]_0 - 2[d(e)y_{1/2}]_1) \\ &= f_d(x_{1/2})y_{1/2} + 2y_{1/2}[d(e)x_{1/2}]_0 - 2y_{1/2}[d(e)x_{1/2}]_1 \\ &\quad + x_{1/2}f_d(y_{1/2}) + 2x_{1/2}[d(e)y_{1/2}]_0 - 2x_{1/2}[d(e)y_{1/2}]_1 \\ &= f_d(x_{1/2})y_{1/2} + x_{1/2}f_d(y_{1/2}) + 2d(e)(x_{1/2}y_{1/2})_1 \\ &\quad - 2d(e)(x_{1/2}y_{1/2})_0 \end{aligned}$$

because of identity (x) of Lemma 3.4. It follows that:

$$\begin{aligned} d(x_{1/2}y_{1/2}) &= f_d(x_{1/2})y_{1/2} + x_{1/2}f_d(y_{1/2}) + 2d(e)(x_{1/2}y_{1/2})_1 \\ &\quad - 2d(e)(x_{1/2}y_{1/2})_0 \end{aligned}$$

and we have

$$h_d((x_{1/2}y_{1/2})_1) = [f_d(x_{1/2})y_{1/2} + x_{1/2}f_d(y_{1/2})]_1 \text{ and}$$

$$g_d((x_{1/2}y_{1/2})_0) = [f_d(x_{1/2})y_{1/2} + x_{1/2}f_d(y_{1/2})]_0, \text{ and we have } (vii).$$

We have

$$\begin{aligned} d(x_1x_0) &= f_d((x_1x_0)_{1/2}) \\ &= d(x_1)x_0 + x_1d(x_0) \\ &= [h_d(x_1) + 2d(e)x_1]x_0 + x_1[g_d(x_0) - 2d(e)x_0] \\ &= h_d(x_1)x_0 + x_1g_d(x_0), \end{aligned}$$

$$f_d((x_1x_0)_{1/2}) = h_d(x_1)x_0 + x_1g_d(x_0), \text{ and we have } (viii).$$

In a similar way

$$\begin{aligned} d(x_1x_{1/2}) &= f_d(x_1x_{1/2}) + 2[d(e)(x_1x_{1/2})]_0 - 2[d(e)(x_1x_{1/2})]_1 \\ &= d(x_1)x_{1/2} + x_1d(x_{1/2}) \\ &= [h_d(x_1) + 2d(e)x_1]x_{1/2} + x_1[f_d(x_{1/2}) + 2[d(e)x_{1/2}]_0 - 2[d(e)x_{1/2}]_1] \\ &= h_d(x_1)x_{1/2} + x_1f_d(x_{1/2}) + 2[d(e)x_1]x_{1/2} + 2x_1[d(e)x_{1/2}]_0 \\ &\quad - 2x_1[d(e)x_{1/2}]_1, \end{aligned}$$

$$f_d(x_1x_{1/2}) = h_d(x_1)x_{1/2} + x_1f_d(x_{1/2}), \text{ and we have } (ix).$$

So we have

$$\begin{aligned} d(x_0x_{1/2}) &= f_d(x_0x_{1/2}) \\ &= d(x_0)x_{1/2} + x_0d(x_{1/2}) \\ &= [g_d(x_0) - 2d(e)x_0]x_{1/2} + x_0[f_d(x_{1/2}) + 2[d(e)x_{1/2}]_0 - 2[d(e)x_{1/2}]_1] \\ &= g_d(x_0)x_{1/2} + x_0f_d(x_{1/2}) - 2[d(e)x_0]x_{1/2} + x_0[2[d(e)x_{1/2}]_0 \\ &\quad -2[d(e)x_{1/2}]_1], \end{aligned}$$

$$f_d(x_0x_{1/2}) = g_d(x_0)x_{1/2} + x_0f_d(x_{1/2}), \text{ and finally } (x).$$

Conversely, once we have identities (i) to (xi), setting $x = x_1 + x_{1/2} + x_0$ and $y = y_1 + y_{1/2} + y_0$, we show that $d(xy) = d(x)y + xd(y)$. □

Example 3.7. Let A be the four dimensional Lie triple K-algebra which multiplication table in the basis $\{e, t, u, r\}$ is given by : $e^2 = e + t$, $u^2 = u + r$, $et = \frac{1}{2}t$, $ur = \frac{1}{2}r$, all other products being zero. The Peirce decomposition of A with respect to pseudo-idempotent e gives $A_e(1) = K(e + 2t)$, $A_e(1/2) = Kt$, $A_e(0) =< u, r >$. Let d be a derivation of A. Since e and u are pseudo-idempotents, we have $d(t) = d(r) = 0$, $d(e) = \alpha t$, $d(u) = \beta r$. The derivation algebra is two dimensional.

Example 3.8. Let's consider the four dimensional Lie triple K-algebra A which multiplication table in the basis $\{e, t_1, t_2, v\}$ is given by : $e^2 = e + t_1$, $et_1 = \frac{1}{2}t_1$, $et_2 = \frac{1}{2}t_2$, $ev = v$ and $vt_1 = t_2$, all other products being zero. Then we have $A_e(1) =< (e + 2t), v >$, $A_e(1/2) =< t_1, t_2 >$, $A_e(0) = 0$. Let d be a derivation of A. We have $d(t_1) = 0$. Set $d(e) = \alpha_1t_1 + \beta_1t_2$. Thus $\alpha_1t_1 + \beta_1t_2 = d(e + 2t_1) = h_d(e+2t_1)+2d(e)(e+2t_1) = h_d(e+2t_1)+\alpha_1t_1+\beta_1t_2$. It follows that $h_d(e+2t_1) = 0$.

Setting $d(v) = \alpha_2(e+2t_1)+\beta_2v+\gamma_2t_1+\eta_2t_2$, relation $0 = d(v^2) = 2d(v)v$ gives $\alpha_2 = \gamma_2 = 0$. It follows that $h_d(v) = \beta_2v$. Furthermore, relation $f_d((vt_1)_{1/2}) = h_d(t_1) + f_d(t_1)$ gives $f_d(t_2) = \beta_2t_2$.

We have $d(e) = \alpha_1t_1 + \beta_1t_2, h_d(e + 2t_1) = 0, h_d(v) = \beta_2v, f_d(t_1) = 0, f_d(t_2) = \beta_2t_2$. The derivation algebra is three dimensional.

Proposition 3.9. *Let's consider a pseudo-idempotent* $e \neq 0$. *Subspace* $J_e(1/2) = \{x_{1/2} \in A_e(1/2) \mid x_{1/2}A_e(1/2) = 0\}$ *is a characteristic ideal and* $A/J_e(1/2)$ *is a Lie triple algebra with* $\overline{e}$ *as idempotent.*

Proof. Let $x_{1/2} \in J_e(1/2)$, $a_{1/2} \in A_e(1/2)$ and $y_\lambda \in A_e(\lambda)$ $(\lambda = 0, 1)$. We have $[(x_{1/2}y_\lambda)a_{1/2}]_\lambda = [(a_{1/2}y_\lambda)x_{1/2}]_\lambda = 0$ and $[(x_{1/2}y_\lambda)a_{1/2}]_{1-\lambda} = [(a_{1/2}y_\lambda)x_{1/2}]_{1-\lambda} = 0$, and then $(x_{1/2}y_\lambda)a_{1/2} = 0$. Hence $A_e(\lambda)J_e(1/2) \subseteq J_e(1/2)$, and it follows that $AJ_e(1/2) \subseteq J_e(1/2)$, then $J_e(1/2)$ is an ideal of A. Since $t \in L_e(1/2) \subseteq J_e(1/2)$, $\bar{e}$ is an idempotent of quotient algebra $A/J_e(1/2)$.

Let's consider now $d \in Der_K(A)$, $x_{1/2} \in J_e(1/2)$ and $a_{1/2} \in A_e(1/2)$. We have $0 = d(x_{1/2}a_{1/2}) = x_{1/2}d(a_{1/2}) + d(x_{1/2})a_{1/2}$. But $d(x_{1/2}) = f_d(x_{1/2}) \in A_e(1/2)$ and $d(a_{1/2}) = f_d(a_{1/2}) + 2(d(e)a_{1/2})_0 - 2(d(e)a_{1/2})_1$, it follows that $x_{1/2}d(a_{1/2}) = 0$ because $x_{1/2}(d(e)a_{1/2})_1 = x_{1/2}(d(e)a_{1/2})_0$. So $d(x_{1/2})a_{1/2} = 0$, and $d(x_{1/2}) \in J_e(1/2)$. We conclude that $d(J_e(1/2)) \subseteq J_e(1/2)$. □

4. Dimensionally nilpotent Lie triple algebras

Definition 4.1. Let A be a $n+1$ finite dimensional K-algebra. If there is a nilpotent K-derivation d of A such that $d^{n+1} = 0$ and $d^n \neq 0$, d is said to be dimensionally nilpotent, and so is the algebra A, though A is not necessarily nilpotent. If so, there is a basis $\{e_0, e_1, \ldots, e_n\}$ of A such that $d(e_i) = e_{i+1}$ $(i = 0, \ldots n-1)$ and $d(e_n) = 0$ and the basis $\{e_0, e_1, \ldots, e_n\}$ is said to be *adapted* to d.

Example 4.2. [6, Exemple 2.5] Let K be a commutative field of characteristic $\neq 2$ and $A = G(n+1, 2)$ the gametic diploid algebra with $n+1$ alleles. Its multiplication table in the natural basis $\{a_0, \ldots, a_n\}$ is given by $a_i a_j = \frac{1}{2}a_i + \frac{1}{2}a_j$. We know that the mapping $\omega : A \to K, a_i \mapsto 1$ is a weight function and if we set $e_i = a_0 - a_i$ $(i \neq 0)$ then $\{e_1, \ldots, e_n\}$ is a basis of the ideal $N = \ker\omega$ and $e_0 = a_0$ is an idempotent of A such that $\{e_0, e_1, \ldots, e_n\}$ is the canonical basis of A, so $e_0 e_i = \frac{1}{2}e_i$ $(i = 1, \ldots, n)$ and if d is a derivation of A, $e_0 d(e_i) = \frac{1}{2}d(e_i)$ $(i = 1, \ldots, n)$, because $d(e_0) \in N$ and N is a zero algebra. So we just need to define $d : A \to A$ by $d(e_i) = e_{i+1}$ $(i = 1, \ldots, n-1)$, $d(e_0) = e_1$ and $d(e_n) = 0$. It follows that $d^{n+1} = 0$ and $d^n \neq 0$, showing the gametic algebra $A = G(n+1, 2)$ is dimensionally nilpotent.

Example 4.3. Let K be a commutative field of characteristic $\neq 2$ and A the $n+1$ dimensional commutative K-algebra, which multiplication table in the basis $\{e_0, e_1, \ldots, e_n\}$ is given by $e_0 e_i = \frac{1}{2}e_i$ $(i = 1, \ldots, n)$, $e_0^2 = e_0 + e_n$, all other products being zero. If d is a derivation of A, $e_0 d(e_i) = \frac{1}{2}d(e_i)$ $(i = 1, \ldots, n)$ because $d(e_0) \in N =< e_1, \ldots, e_n >$ and N is a zero algebra. Here, we just need again to define $d : A \to A$ by $d(e_i) = e_{i+1}$ $(i = 1, \ldots, n-1)$, $d(e_0) = e_1$ and $d(e_n) = 0$. We have $d^{n+1} = 0$ and $d^n \neq 0$, that shows the algebra A is dimensionally nilpotent. Since Ke_n is an ideal, the quotient algebra A/Ke_n is isomorphic to $G(n, 2)$.

4.1. **Basic tools.**

Theorem 4.4 ([9])**.** *Let K be a perfect field of characteristic $\neq 2$ and 3 and A finite dimensional K-Jordan algebra, dimensionally nilpotent. Then either A is nilpotent or* $\dim_K(A/rad(A)) = 1$.

Remark 4.5. Let A be a dimensionally nilpotent Lie triple non nilalgebra. Because of Theorem 2.3 we consider two cases :

1) A has an idempotent e. Since the ideal J is caracteristic, the quotient algebra $\overline{A} = A/J$ is a dimensionally nilpotent Jordan algebra. Because of Theorem 4.4 we have $\dim_K(\overline{A}/rad(\overline{A})) = 1$ and since $rad(\overline{A}) \simeq rad(A)/J$, the first isomorphism theorem gives $A/rad(A) \simeq \overline{A}/rad(\overline{A})$ and $\dim_K(A/rad(A)) = 1$. Then we can write $A = Ke \oplus N$, with $N = rad(A)$.
2) A has a pseudo-idempotent e. Since the ideal $J_e(1/2)$ is characteristic, the quotient algebra $\overline{A} = A/J_e(1/2)$ is a dimensionally Lie triple algebra with $\overline{e}$ as idempotent. Because of 1) we can write $\overline{A} = K\overline{e} \oplus \overline{N}$ with $\overline{N} = rad(\overline{A})$. So we have $A = Ke \oplus N$ with $N = rad(A)$.

Lemma 4.6. *Let x, $y \in N$ such that $x \neq 0$ and $\alpha \in K$. If $xy = \alpha y$ then $\alpha = 0$ or $y = 0$.*

Proof. Since N is nilpotent, there is $m \in \mathbb{N}^*$ such that $L_x^m(y) = \alpha^m y = 0$, L_x being the multiplicative operator by x. Then $\alpha = 0$ or $y = 0$. □

From now, throughout the paper, A is a dimensionally nilpotent Lie triple non nil-algebra of dimension $n+1$, with $\{e_0, e_1, \ldots, e_n\}$ an adapted basis to the derivation d. We can consider e_0 either, as an idempotent, or a pseudo-idempotent. In the last case, $e_0^2 = e_0 + t$, $e_0 t = \frac{1}{2}t$ and $t^2 = 0$ implies $d(t) = 0$ (Lemma 3.5), that means $t = \alpha e_n$ with $\alpha \in K$. Since $t \in A_{e_0}(1/2)$, if $\alpha \neq 0$, then $e_n \in A_{e_0}(1/2)$.

Lemma 4.7. *We have:*
(i) $e_0 e_n = \lambda_n e_n$
(ii) $e_k e_n = 0$ *with* $1 \leq k \leq n$

Proof. Let's write $e_0 e_n = \sum_{i=0}^{n} \lambda_i e_i$. By derivating k times successively, we have $e_k e_n = \sum_{i=0}^{n-k} \lambda_i e_{i+k}$. With $k = n$, it follows that $e_n^2 = \lambda_0 e_n$ and because of Lemma 4.6 we have $\lambda_0 = 0$. Set $k = n-1$, we have $e_{n-1}e_n = \lambda_1 e_n$. That implies $\lambda_1 = 0$. And so on, we get $\lambda_0 = \lambda_1 = \cdots = \lambda_{n-1} = 0$, $e_0 e_n = \lambda_n e_n$. By successively derivation $e_0 e_n$ it follows that $e_k e_n = 0$ with $1 \leq k \leq n$. □

Lemma 4.8. *We have :*

(i) $e_0 e_k = \lambda_k e_k + \sum_{i=k+1}^{n} a_{k,i} e_i$ *with* $1 \leq k \leq n-1$;
(ii) $e_i e_k = \sum_{j=k+1}^{n} \gamma_{ikj} e_j$ *with* $1 \leq i \leq k \leq n-1$;
(iii) $\lambda_k \in \{0, \frac{1}{2}, 1\}$ *with* $1 \leq k \leq n$.

Proof. Let's reason by induction with respect to n. For $n = 1$ the multiplication table of the algebra A is given by $e_0^2 = e_0$, $e_0 e_1 = \frac{1}{2} e_1$, $e_1^2 = 0$ and the lemma is satisfied. Let's assume the lemma is true until an order n. Because of Lemma 4.7 the subspace $I_{n+1} = Ke_{n+1}$ is a d-invariant ideal of A. The quotient algebra A/I_{n+1} is dimensionally nilpotent of dimension $n+1$. By the hypothesis, we have $\overline{e}_0 \overline{e}_k = \lambda_k \overline{e}_k + \sum_{i=k+1}^{n} a_{k,i} \overline{e}_i$ and $\overline{e}_i \overline{e}_k = \sum_{j=k+1}^{n} \gamma_{ikj} \overline{e}_j$, with $1 \leq i \leq k \leq n$. Otherwise $e_0 e_k = \lambda_k e_k + \sum_{i=k+1}^{n} a_{k,i} e_i + a_{k,n+1} e_{n+1}$ and $e_i e_k = \sum_{j=k+1}^{n} \gamma_{ikj} e_j + \gamma_{ik,n+1} e_{n+1}$; and results (i) and (ii) follow.

Now we just need to show (iii). Since $2L_{e_0}^3 - 3L_{e_0}^2 + L_{e_0} = 0$, with L_{e_0} being the multiplicative operator by e_0, applying it to e_k we have $2\lambda_k^3 - 3\lambda_k^2 + \lambda_k = 0$, then $\lambda_k \in \{0, \frac{1}{2}, 1\}$. □

4.2. **Example of low dimensions.** Here we deal with cases $1 \leq n \leq 4$. Let A be a dimensionally nilpotent Lie triple algebra, of dimension $n+1$ and $\{e_0, e_1, \ldots, e_n\}$ be a basis adapted to d. We have $\ker d = Ke_n$. Since e_0 is an idempotent or a pseudo-idempotent, $e_1 = d(e_0) \in A_{e_0}(1/2)$, i.e $e_0e_1 = \frac{1}{2}e_1$. By derivation this we get $e_0e_2 + e_1^2 = \frac{1}{2}e_2$, that means

$$\lambda_2 + \gamma_{112} = \frac{1}{2} \text{ et } a_{2,k} + \gamma_{11k} = 0 \quad (3 \leq k \leq n). \tag{4.1}$$

We also have $e_1^2 = \sum_{k=2}^n \gamma_{11k}e_k$ which derivative is $2e_1e_2 = \sum_{k=2}^{n-1} \gamma_{11k}e_{k+1} = \sum_{k=3}^n \gamma_{11,k-1}e_k$, that means $2\gamma_{12k} = \gamma_{11,k-1}$ $(3 \leq k \leq n)$. Let's derivate for the second time $e_0e_1 = \frac{1}{2}e_1$. We have $e_0e_3 + 3e_1e_2 = \frac{1}{2}e_3$, that means

$$\lambda_3 + 3\gamma_{123} = \frac{1}{2} \text{ et } a_{3,k} + 3\gamma_{1,2,k} = 0 \quad (4 \leq k \leq n). \tag{4.2}$$

However we have, $d(e_0e_2) = e_0e_3 + e_1e_2 = \lambda_2e_3 + \sum_4^n a_{2,k-1}e_k$, that implies

$$\lambda_3 + \gamma_{123} = \lambda_2 \text{ et } a_{3,k} + \gamma_{12k} = a_{2,k-1} \quad (4 \leq k \leq n). \tag{4.3}$$

So (4.3) implies $\gamma_{123} \in \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$. So it is necessary to take $\lambda_3 = \frac{1}{2}$ in (4.2). Whence $\lambda_3 = \lambda_2 = \frac{1}{2}$ and $\gamma_{123} = \gamma_{112} = 0$ if $3 \leq n$.

Derivating $e_0e_3 + 3e_1e_2 = \frac{1}{2}e_3$, one has $e_0e_4 + 4e_1e_3 + 3e_2^2 = \frac{1}{2}e_4$, that means

$$\lambda_4 + 4\gamma_{134} + 3\gamma_{224} = \frac{1}{2} \tag{4.4}$$

However, $d(e_0e_3) = e_0e_4 + e_1e_3 = \lambda_3e_4 + \sum_{k=5}^n a_{3,k-1}e_k$, which implies

$$\lambda_4 + \gamma_{134} = \lambda_3 \text{ et } a_{4,k} + \gamma_{13k} = a_{3,k-1} \quad (5 \leq k \leq n) \tag{4.5}$$

We also have $e_1e_2 = \sum_{k=4}^n \gamma_{12k}e_k$ because $\gamma_{123} = 0$ and $d(e_1e_2) = e_1e_3 + e_2^2 = \sum_{k=5}^n \gamma_{12,k-1}e_k$, which implies $\gamma_{223} = 0$ and $\gamma_{134} + \gamma_{224} = 0$.

Case $\dim_K A = 2$ **i.e** $n = 1$**.**

We obviously have $e_0e_1 = \frac{1}{2}e_1$, $e_1^2 = 0$, $e_0^2 = e_0$ or $e_0^2 = e_0 + e_1$ all other products being zero.

	e_0	e_1
e_0	e_0	$\frac{1}{2}e_1$
e_1		0

	e_0	e_1
e_0	$e_0 + e_1$	$\frac{1}{2}e_1$
e_1		0

Case $\dim_K A = 3$ **i.e** $n = 2$**.**

Because of (4.1) we have $\lambda_2 + \gamma_{112} = \frac{1}{2}$. Let's discuss about the possible values of λ_2.

$*$ $\lambda_2 = 0 \Rightarrow \gamma_{112} = \frac{1}{2}$, so $e_0^2 = e_0$, $e_0e_1 = \frac{1}{2}e_1$, $e_1^2 = \frac{1}{2}e_2$ all other products being zero.

	e_0	e_1	e_2
e_0	e_0	$\frac{1}{2}e_1$	0
e_1		$\frac{1}{2}e_2$	0
e_2			0

$*$ $\lambda_2 = \frac{1}{2} \Rightarrow \gamma_{112} = 0$, so $e_0^2 = e_0$ or $e_0^2 = e_0 + e_2$, $e_0e_1 = \frac{1}{2}e_1$, $e_0e_2 = \frac{1}{2}e_2$ all other products being zero.

	e_0	e_1	e_2
e_0	e_0	$\frac{1}{2}e_1$	$\frac{1}{2}e_2$
e_1		0	0
e_2			0

	e_0	e_1	e_2
e_0	$e_0 + e_2$	$\frac{1}{2}e_1$	$\frac{1}{2}e_2$
e_1		0	0
e_2			0

$*$ $\lambda_2 = 1 \Rightarrow \gamma_{112} = -\frac{1}{2}$, so $e_0^2 = e_0$, $e_0e_1 = \frac{1}{2}e_1$, $e_0e_2 = e_2$, $e_1^2 = -\frac{1}{2}e_2$ all other products being zero.

	e_0	e_1	e_2
e_0	e_0	$\frac{1}{2}e_1$	e_2
e_1		$-\frac{1}{2}e_2$	0
e_2			0

Case $\dim_K A = 4$ **i.e** $n = 3$**.**

Because of the preliminary calculations, $\lambda_3 = \lambda_2 = \lambda_1 = \frac{1}{2}$, $\gamma_{112} = \gamma_{123} = 0$ and $a_{2,3} + \gamma_{113} = 0$. So $e_0e_3 = \frac{1}{2}e_3 \Rightarrow e_3 \in A_{e_0}(1/2)$ and $e_1^2 = \gamma_{113}e_3$. Since $A_{e_0}(1/2)^2 \subseteq A_{e_0}(0) + A_{e_0}(1)$ we have $\gamma_{113} = 0$ implying $a_{2,3} = 0$ and finally $e_0e_2 = \frac{1}{2}e_2$. So we have the following multiplication table : $e_0e_1 = \frac{1}{2}e_1$, $e_0e_2 = \frac{1}{2}e_2$, $e_0e_3 = \frac{1}{2}e_3$, $e_0^2 = e_0$ or $e_0^2 = e_0 + e_3$, all other products being zero.

	e_0	e_1	e_2	e_3
e_0	e_0	$\frac{1}{2}e_1$	$\frac{1}{2}e_2$	$\frac{1}{2}e_3$
e_1		0	0	0
e_2			0	0
e_3				0

	e_0	e_1	e_2	e_3
e_0	$e_0 + e_3$	$\frac{1}{2}e_1$	$\frac{1}{2}e_2$	$\frac{1}{2}e_3$
e_1		0	0	0
e_2			0	0
e_3				0

Case $\dim_K A = 5$ **i.e** $n = 4$**.**

$*$ $\lambda_4 = 0 \Rightarrow \gamma_{134} = \frac{1}{2}$ because of (4.5). Since $e_1e_3 = \gamma_{134}e_4 = \frac{1}{2}e_4$ we have $e_3 \in A_{e_0}(1/2)$ because $A_{e_0}(1/2)^2 \subseteq A_{e_0}(0) + A_{e_0}(1)$, $A_{e_0}(1/2)A_{e_0}(1) \subseteq A_{e_0}(1/2)$, $A_{e_0}(1/2)A_{e_0}(1) \subseteq A_{e_0}(1/2)$. So $e_0e_3 = \frac{1}{2}e_3$ and $a_{3,4} = 0 \Rightarrow \gamma_{124} = 0$ because of (4.3) and finally $\gamma_{113} = a_{2,3} = 0$. In the same way $e_2^2 = -\frac{1}{2}e_4 \Rightarrow e_2 \in A_{e_0}(0)$ or $e_2 \in A_{e_0}(1/2)$, because $A_{e_0}(1/2)^2 \subseteq A_{e_0}(0) + A_{e_0}(1)$ and $A_{e_0}(0)^2 \subseteq A_{e_0}(0)$. But $e_0e_2 = \frac{1}{2}e_2 + a_{2,4}e_4 \Rightarrow e_2 \in A_{e_0}(1/2)$ so $a_{2,4} = 0 = \gamma_{114}$ because of (4.1). Whence the following multiplication table

	e_0	e_1	e_2	e_3	e_4
e_0	e_0	$\frac{1}{2}e_1$	$\frac{1}{2}e_2$	$\frac{1}{2}e_3$	0
e_1		0	0	$\frac{1}{2}e_4$	0
e_2			$-\frac{1}{2}e_4$	0	0
e_3				0	0
e_4					0

$*\ \lambda_4 = \frac{1}{2} \Rightarrow \gamma_{134} = \gamma_{224} = 0$ because of (4.4) and (4.5). We have $e_1e_2 = \gamma_{1,2,4}e_4 \in A_{e_0}(1/2) \Rightarrow e_2 \in A_{e_0}(0)$ or $e_2 \in A_{e_0}(1)$ this is a contradiction (because $e_0e_2 = \frac{1}{2}e_2 + a_{2,3}e_3$) so $\gamma_{124} = 0$ and then $a_{3,4} = \gamma_{113} = 0$. Whence the following multiplication table:

	e_0	e_1	e_2	e_3	e_4
e_0	e_0	$\frac{1}{2}e_1$	$\frac{1}{2}e_2$	$\frac{1}{2}e_3$	$\frac{1}{2}e_4$
e_1		0	0	0	0
e_2			0	0	0
e_3				0	0
e_4					0

	e_0	e_1	e_2	e_3	e_4
e_0	$e_0 + e_4$	$\frac{1}{2}e_1$	$\frac{1}{2}e_2$	$\frac{1}{2}e_3$	$\frac{1}{2}e_4$
e_1		0	0	0	0
e_2			0	0	0
e_3				0	0
e_4					0

$*\ \lambda_4 = 1 \Rightarrow e_4 \in A_{e_0}(1) \Rightarrow \gamma_{134} = -\frac{1}{2}$

We have $e_1e_3 = -\frac{1}{2}e_4 \Rightarrow e_3 \in A_{e_0}(1/2) \Rightarrow e_0e_3 = \frac{1}{2}e_3 \Rightarrow a_{3,4} = 0$. So $\gamma_{124} = \gamma_{113} = a_{2,3} = 0$. In the same way $e_2^2 = \frac{1}{2}e_4 \Rightarrow e_2 \in A_{e_0}(1/2)$ (because e_2 can not be in $A_{e_0}(1)$), $e_0e_2 = \frac{1}{2}e_2 \Rightarrow a_{2,4} = \gamma_{114} = 0$. Whence this table

	e_0	e_1	e_2	e_3	e_4
e_0	e_0	$\frac{1}{2}e_1$	$\frac{1}{2}e_2$	$\frac{1}{2}e_3$	e_4
e_1		0	0	$-\frac{1}{2}e_4$	0
e_2			$\frac{1}{2}e_4$	0	0
e_3				0	0
e_4					0

4.3. **Main results in general case.**

Theorem 4.9 (Main theorem)**.** *Let A be dimensionally nilpotent Lie triple non nilalgebra and $\{e_0, e_1, \ldots, e_n\}$ be an adapted basis of A. Then:*

1) *If $n = 2p+1$, the multiplication table of A is one of the two following:*

(*i*) $e_0^2 = e_0$, $e_0e_i = \frac{1}{2}e_i$ $(1 \leq i \leq 2p+1)$, *all other products being zero.*

(*i'*) $e_0^2 = e_0 + e_{2p+1}$, $e_0e_i = \frac{1}{2}e_i$ $(1 \leq i \leq 2p+1)$, *all other products being zero.*

2) *If $n = 2p$, the multiplication table of A is one of the four following :*

(*i*) $e_0^2 = e_0$, $e_0e_i = \frac{1}{2}e_i$ $(1 \leq i \leq 2p)$, *all other products being zero.*

(*i'*) $e_0^2 = e_0 + e_{2p}$, $e_0e_i = \frac{1}{2}e_i$ $(1 \leq i \leq 2p)$, *all other products being zero.*

(*ii*) $e_0^2 = e_0$, $e_0e_i = \frac{1}{2}e_i$ $(1 \leq i \leq 2p-1)$, $e_0e_{2p} = 0$, $e_ie_{2p-i} = \frac{1}{2}(-1)^{i-1}e_{2p}$, $(1 \leq i \leq p)$, *all other products being zero.*

(*iii*) $e_0^2 = e_0$, $e_0e_i = \frac{1}{2}e_i$ $(1 \leq i \leq 2p-1)$, $e_0e_{2p} = e_{2p}$, $e_ie_{2p-i} = \frac{1}{2}(-1)^{i}e_{2p}$ $(1 \leq i \leq p)$, *all other products being zero.*

Proof. Reasoning by induction with respect on n. Subsection 4.2 shows that the theorem is true when $n \leq 4$. Let's assume that it is true until an order $n > 4$ and let's show it remains true for $n+1$. Integer n being either even or odd, we consider two cases :

1) $n = 2p$ is even. The multiplication table of A is like the following $e_0e_k = \frac{1}{2}e_k + a_{k,2p+1}e_{2p+1}$ $(1 \leq k \leq 2p-1)$, $e_0e_{2p} = \lambda_{2p}e_{2p} + a_{2p,2p+1}e_{2p+1}$, $e_0e_{2p+1} =$

$\lambda_{2p+1}e_{2p+1}$ and $e_ie_{2p-i} = \varepsilon_i e_{2p} + \gamma_{i,2p-i,2p+1}e_{2p+1}$, with $\varepsilon_i = 0$, $\varepsilon_i = \frac{1}{2}(-1)^{i-1}$ or $\varepsilon_i = \frac{1}{2}(-1)^i$ $(i = 1, \dots, p)$ according to $\lambda_{2p} = \frac{1}{2}$, $\lambda_{2p} = 0$ or $\lambda_{2p} = 1$, respectively. We have $d(e_ie_{2p-i}) = e_ie_{2p+1-i} + e_{i+1}e_{2p-i} = \varepsilon_i e_{2p+1}$, and then the following system

$$\begin{cases} e_1e_{2p} + e_2e_{2p-1} = \varepsilon_1 e_{2p+1}, \\ \cdots\cdots, \\ e_ie_{2p+1-i} + e_{i+1}e_{2p-i} = \varepsilon_i e_{2p+1}, \\ \cdots\cdots, \\ e_pe_{p+1} + e_{p+1}e_p = 2e_pe_{p+1} = \varepsilon_p e_{2p+1}. \end{cases} \tag{S}$$

We see that $e_pe_{p+1} = \frac{1}{2}\varepsilon_p e_{2p+1}$, $e_{p-1}e_{p+2} = (\varepsilon_{p-1} - \frac{1}{2}\varepsilon_p)e_{2p+1} = \frac{3}{2}\varepsilon_{p-1}e_{2p+1}$, $e_{p-i}e_{p+1+i} = \frac{2i+1}{2}\varepsilon_{p-i}e_{2p+1}$, $e_1e_{2p} = \frac{2p-1}{2}\varepsilon_1 e_{2p+1}$. But $d(e_0e_{2p}) = e_0e_{2p+1} + e_1e_{2p} = \lambda_{2p}e_{2p+1}$, that means $e_1e_{2p} = (\lambda_{2p} - \lambda_{2p+1})e_{2p+1}$, and also $\lambda_{2p} - \lambda_{2p+1} = \frac{2p-1}{2}\varepsilon_1$.

If $\lambda_{2p} = 0$, we have $\varepsilon_1 = \frac{1}{2}$ and $\lambda_{2p+1} = -\frac{2p-1}{4} \notin \{0, \frac{1}{2}, 1\}$, that is impossible.

If $\lambda_{2p} = 1$, we have $\varepsilon_1 = -\frac{1}{2}$ and $\lambda_{2p+1} = 1 + \frac{2p-1}{4} = \frac{2p+3}{4} \notin \{0, \frac{1}{2}, 1\}$, which impossible.

Hence $\lambda_{2p} = \frac{1}{2}$, $\varepsilon_1 = 0$ and $\lambda_{2p+1} = \frac{1}{2}$.

Since all the λ_k are equal to $\frac{1}{2}$ $(k \neq 2p+1)$, applying $2L^3_{e_0} - 3L^2_{e_0} + L_{e_0} = 0$ to e_k, it follows that $2a_{k,2p+1}\lambda_{2p+1}(\lambda_{2p+1} - 1) = 0$. If $\lambda_{2p+1} = \frac{1}{2}$, then we have $a_{k,2p+1} = 0$, which means $e_0e_k = \frac{1}{2}e_k$ for all k. Hence $e_ie_j = 0$, with $1 \leq i \leq j \leq n$.

2) $n = 2p - 1$ is odd. The multiplication table of A is like the following $e_0e_k = \frac{1}{2}e_k + a_{k,2p}e_{2p}$ $(k = 1, \dots, 2p-1)$, $e_0e_{2p} = \lambda_{2p}e_{2p}$ and $e_ie_{2p-1-i} = \gamma_{i,2p-1-i,2p}e_{2p}$ $(i = 1, \dots, p-1)$. By derivating this last relation, we have $e_ie_{2p-i} + e_{i+1}e_{2p-1-i} = 0$, and the following system

$$\begin{cases} e_1e_{2p-1} + e_2e_{2p-2} = 0, \\ e_2e_{2p-2} + e_3e_{2p-3} = 0, \\ \cdots\cdots, \\ e_{p-1}e_{p+1} + e_p^2 = 0. \end{cases} \tag{S'_p}$$

So $e_1e_{2p-1} = -e_2e_{2p-2} = e_3e_{2p-3} = \cdots = (-1)^{i-1}e_ie_{2p-i} = \cdots = (-1)^{p-1}e_p^2$, that means $e_ie_{2p-i} = (-1)^{i-1}e_1e_{2p-1}$. However, since $d(e_0e_{2p-1}) = e_0e_{2p} + e_1e_{2p-1} = \frac{1}{2}e_{2p}$, we have $e_1e_{2p-1} = (\frac{1}{2} - \lambda_{2p})e_{2p}$. Because $\lambda_{2p} \in \{0, \frac{1}{2}, 1\}$, we consider three situations :

If $\lambda_{2p} = 0$, we have $e_1e_{2p-1} = \frac{1}{2}e_{2p}$, $e_ie_{2p-i} = \frac{1}{2}(-1)^{i-1}e_{2p}$ $(i = 1, \dots, p)$ and $e_0^2 = e_0$.

If $\lambda_{2p} = 1$, we have $e_1e_{2p-1} = -\frac{1}{2}e_{2p}$, $e_ie_{2p-i} = \frac{1}{2}(-1)^i e_{2p}$ $(i = 1, \dots, p)$ and $e_0^2 = e_0$.

If $\lambda_{2p} = \frac{1}{2}$, we have $e_1e_{2p-1} = e_ie_{2p-i} = 0$ $(i = 1 \dots p)$ and $2a_{i,2p}\lambda_{2p}(\lambda_{2p} - 1) = 0$ shows that $a_{i,2p-i} = 0$. Hence $e_0e_i = \frac{1}{2}e_i$ $(i = 1, \dots, p)$. Furthermore we have, either $e_0^2 = e_0$, or $e_0^2 = e_0 + e_{2p}$.

For cases $\lambda_{2p} = 0$ and $\lambda_{2p} = 1$, we just need to show $e_ie_j = 0$ for $i + j < 2p$. The following lemma completes the proof of the theorem. And Note 4.12 shows that all algebras defined in this theorem are Lie triple. □

Lemma 4.10. *$e_0e_i = \frac{1}{2}e_i$ for $i = 1, \dots, n-1$.*

Proof. We have $e_ie_{2k-i} = \gamma_{i,2k-i,n}e_n$ for $i = 1, \dots, k-1$. By derivating this we have $e_ie_{2k-i+1} + e_{i+1}e_{2k-i} = 0$. By varying i we have the following system

$$\begin{cases} e_1e_{2k} + e_2e_{2k-1} = 0, \\ e_2e_{2k-1} + e_3e_{2k-2} = 0, \\ \cdots\cdots, \\ e_{k-1}e_{k+2} + e_ke_{k+1} = 0, \\ e_ke_{k+1} + e_{k+1}e_k = 2e_ke_{k+1} = 0. \end{cases} \qquad (S_k)$$

Going up the lines of this system we see that $e_ie_{2k+1-i} = 0$ for $i = 1, \dots, k$ in particular $e_1e_{2k} = 0$, so $e_0e_{2k+1} + e_1e_{2k} = \frac{1}{2}e_{2k+1}$ and $e_0e_{2k+1} = \frac{1}{2}e_{2k+1}$, $k = 0, \dots, p-1$.

Now we reason by induction with respect to n. Let's assume that it is true until an order n. We distinguish two cases :

- $n+1 = 2p+1$ is odd. We have $e_0e_{n+1} = \frac{1}{2}e_{n+1}$, which imposes $e_0e_n = \frac{1}{2}e_n$.
- $n+1 = 2p$ is even. Since $n = 2p-1$ is odd, we have $e_0e_n = \frac{1}{2}e_n$.

□

Since every commutative Jordan algebra is a Lie triple algebra, we have the following result:

Corollary 4.11. *Let A be a commutative Jordan non nil-algebra, dimensionally nilpotent. Let $\{e_0, e_1, \dots, e_n\}$ be an adapted basis of A. Then:*
1) If $n = 2p+1$, the multiplication table of A is:
$e_0^2 = e_0$, $e_0e_i = \frac{1}{2}e_i$ $(1 \le i \le 2p+1)$, all other products being zero.
2) If $n = 2p$, the multiplication table of A is one of the following three tables :
(i) $e_0^2 = e_0$, $e_0e_i = \frac{1}{2}e_i$ $(1 \le i \le 2p)$, all other products being zero.
(ii) $e_0^2 = e_0$, $e_0e_i = \frac{1}{2}e_i$ $(1 \le i \le 2p-1)$, $e_0e_{2p} = 0$, $e_ie_{2p-i} = \frac{1}{2}(-1)^{i-1}e_{2p}$, $(1 \le i \le p)$, all other products being zero.
(iii) $e_0^2 = e_0$, $e_0e_i = \frac{1}{2}e_i$ $(1 \le i \le 2p-1)$, $e_0e_{2p} = e_{2p}$, $e_ie_{2p-i} = \frac{1}{2}(-1)^{i}e_{2p}$ $(1 \le i \le p)$, all other products being zero.

Proof. The Corollary follows from Theorem 4.9 knowing that Jordan algebras do not admit pseudo-idempotent. □

Note 4.12. 1) Multiplication tables in Theorem 4.9 1) (i) and in Corollary 4.11 1) when $n = 2p+1$ is odd are those of *gametic algebras* $G(2p+2, 2)$ (Example 4.2). It is the same for those defined in Theorem 4.9 2) (i) and in Corollary 4.11 2) (i) when $n = 2p$ is even. These are gametic algebras $G(2p+1, 2)$. They are characterized as elementary train algebras with equation $x^2 - \omega(x)x = 0$, in which $\omega : A \to K$, $e_0 \mapsto 1$, $e_i \mapsto 0$ is an homomorphism of algebras [11].

2) Multiplication tables in Theorem 4.9 2) (ii) and in Corollary 4.11 2) (ii), when $n = 2p$ is even, are those of *normal Bernstein algebras* of type $(2p, 1)$. Normal Bernstein algebras are defined by equation $x^2y = \omega(x)xy$. These are Bernstein-Jordan algebras, characterized by the train equation $x^3 - \omega(x)x^2 = 0$ [10, 11].

3) Multiplication tables in Theorem 4.9 2) (*iii*) and in Corollary 4.11 2) (*iii*), when $n = 2p$ is even, are those of the other class of *train algebras of rank* 3 *which are Jordan algebras* of type $(2p, 1)$. They are defined by equation $x^3 - 2\omega(x)x^2 + \omega(x)^2x = 0$ [10, Theorem 2.1].

4) Multiplication tables in Theorem 4.9 1) (i') and 2) (i') are those of *train algebras* satisfying $x^3 - \frac{3}{2}\omega(x)x^2 + \frac{1}{2}\omega(x)^2x = 0$. These are Lie triple algebras because of [2, Proposition 5.2, (iii)].

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