

## SEMICONINUITY AND $X$ -RANK

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**ABSTRACT.** Let  $X \subset \mathbb{P}^r$  be an integral and non-degenerate variety. For each  $q \in \mathbb{P}^r$  the  $X$ -rank  $r_X(q)$  of  $q$  is the minimal cardinality of a subset of  $X$  whose linear span contains  $q$ . Obviously,  $r_X^{-1}(1) = X$ . In this note we give conditions which assure that  $r_X^{-1}(2) \cup X$  is not closed in  $\mathbb{P}^r$  and hence the function  $r_X$  is not semicontinuous.

### 1. INTRODUCTION

Let  $X \subset \mathbb{P}^r$  be an integral and non-degenerate variety defined over an algebraically closed field of characteristic 0. For each  $q \in \mathbb{P}^r$  the  $X$ -rank  $r_X(q)$  of  $q$  is the minimal cardinality of a set  $S \subset X$  such that  $q \in \langle S \rangle$ , where  $\langle \cdot \rangle$  denote the linear span. Thus the  $X$ -rank function is a function from  $\mathbb{P}^r$  to the set of positive integers. Obviously,  $r_X^{-1}(1) = X$ . The definition of secant varieties ([1]) and a theorem of Chevalley show that there is a finite partition  $F_i$ ,  $1 \leq i \leq e$ , of  $\mathbb{P}^r$  with each  $F_i$  a locally closed subset of  $\mathbb{P}^r$  for the Zariski topology such that the restriction of  $r_X$  to each  $F_i$  is constant.

There are two classical cases in which  $r_X$  is semicontinuous:

- (1) Quadratics forms in  $n$  variables with the rank of the quadratic form as the  $X$ -rank function;
- (2)  $m \times n$ -matrices with the rank of the matrix as the  $X$ -rank function.

Case (1) is in the set-up of the  $X$ -rank with  $r = -1 + \binom{n+1}{2}$  and  $X$  the image of the order 2 Veronese embedding of  $\mathbb{P}^{r-1}$ . Case 2 is in the set-up of  $X$ -rank with  $r = -1 + mn$  and  $X$  the image of the Segre embedding of  $\mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$ . It is well-known that very often the  $X$ -rank is not semicontinuous. A full classification of all homogeneous embeddings of projective homogeneous spaces by, say, semisimple algebraic groups with semicontinuous rank function is given in [7]. Their list is very short: semicontinuity is a rare phenomena. Of course, some non-homogeneous  $X$  has semicontinuous  $X$ -rank. We collect in Remarks 2.1 and 2.2 a few well-known examples. In this note we give 2 broad cases in which semicontinuity fails. In both cases it fails at the first step: it is the set  $r_X^{-1}(2) \cup X$  which is not closed.

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**Theorem 1.1.** *Let  $X \subset \mathbb{P}^r$ ,  $r \geq 4$ , be a smooth, connected and non-degenerate curve. Then:*

- (a) *We have  $\dim \sigma_2(X) = 3$ ,  $\dim \tau(X) = 2$  and  $r_X(p) > 2$  for a general  $p \in \tau(X)$ .*
- (b) *The  $X$ -rank function is not semicontinuous.*

Theorem 1.1 is not true (without adding other assumptions) in the cases  $r \leq 3$  (Remarks 2.1 and 2.2).

**Question 1:** Is Theorem 1.1 true for singular curves?

Theorem 1.1 is true for a singular curve such that the normalization map  $\tilde{X} \rightarrow X$  is unramified. Indeed, the proof we give in Section 2 works with the only modification of quoting [4, Remark 3.8] instead of [4, Th. 3.1]. The answer to Question 1 is yes for an arbitrary singular curve  $X$  (the closedness of  $r_X^{-1}(2) \cup X$ ) if and only if for every singular and non-degenerate curve  $X \subset \mathbb{P}^r$ ,  $r \geq 3$ ,  $T_p X \cap (X \setminus \{p\}) = \emptyset$  for a general  $p \in X_{\text{reg}}$ . This is a very difficult open problem, open since [4], published in 1986. See [2, 5] for more recent partial solutions of this problem.

In higher dimensions we give the following conditions which assure that  $r_X^{-1}(2) \cup X$  is not closed in  $\mathbb{P}^r$ .

**Theorem 1.2.** *Let  $X \subset \mathbb{P}^r$  be an integral and non-degenerate  $n$ -dimensional variety. Assume that the following conditions are satisfied:*

- (i)  *$T_p X \cap (X \setminus \{p\}) = \emptyset$  for a general  $p \in X_{\text{reg}}$ .*
- (ii) *Take a general  $u \in X_{\text{reg}}$ . Let  $\ell_u : \mathbb{P}^r \setminus \{u\} \rightarrow \mathbb{P}^{r-1}$  denote the linear projection from  $u$ . Let  $Y \subset \mathbb{P}^{r-1}$  be the closure of  $\ell_u(X \setminus \{u\})$  in  $\mathbb{P}^{r-1}$ . Assume  $T_a Y \cap (Y \setminus \{a\}) = \emptyset$  for a general  $a \in Y_{\text{reg}}$ .*

*Then:*

- (a) *We have  $\dim \sigma_2(X) = 2n + 1$  and  $r_X(q) > 2$  for a general  $q \in \tau(X)$ .*
- (b) *The  $X$ -rank function is not semicontinuous.*

The nice paper [6] shows the existence of “strange” not uniruled examples, but it is not related to our questions.

## 2. PROOFS AND RELATED EXAMPLES

*Remark 2.1.* Let  $X \subset \mathbb{P}^{n+1}$  be an integral and non-degenerate  $n$ -dimensional variety. Fix  $p \in \mathbb{P}^{n+1} \setminus X$ . The linear projection from  $p$  induces a finite morphism  $\mu : X \rightarrow \mathbb{P}^n$ . Since  $\mu$  is finite, it is surjective. Since we are in characteristic 0, the differential of  $\mu$  is not identically 0. Thus a general line  $L \subset \mathbb{P}^{n+1}$  containing  $p$  contains  $\deg(X)$  points of  $X$ . Since  $\deg(X) \geq 2$ ,  $r_X(p) = 2$ . Thus the  $X$ -rank function is semicontinuous. This is false in positive characteristic if and only if  $p$  is a strange point of  $X$ .

*Remark 2.2.* Let  $X \subset \mathbb{P}^3$  be an integral and non-degenerate curve. In characteristic 0 the  $X$ -rank function is semicontinuous if and only if no  $p \in \mathbb{P}^3$  has  $X$ -rank 3. Both cases occur ([8]).

*Proof of Theorem 1.1:* We have  $\dim \sigma_2(X) = 3$  ([1, Remark 1.6]) and obviously  $\dim \tau(X) = 2$ . All points of  $\sigma_2(X) \setminus \tau(X)$  have  $X$ -rank 2. Assume that a general

point of  $\tau(X)$  has rank 2. Thus for a general  $p \in X$  and a general  $q \in T_p X$  there are  $u, v \in X$  such that  $q \in \langle \{u, v\} \rangle$ . Since we are in characteristic 0, no point of  $\mathbb{P}^r$  is contained in infinitely many tangent lines of  $X$ . Thus (varying  $p$ ) to cover a non-empty Zariski open subset of  $\tau(X)$  for a fixed, but general,  $p$  we need to use infinitely many pairs  $\{u, v\}$ . Thus for a general  $(p, u) \in X$  there is  $v \in X$  such that  $(T_p X \setminus \{p\}) \cap \langle \{u, v\} \rangle \neq \emptyset$ . By [4, Th. 3.1] and the generality of  $u$  the line  $T_u X$  is the only tangent line of  $X$  containing  $u$ . Thus  $p \notin \langle \{u, v\} \rangle$ . Let  $\ell_u : \mathbb{P}^r \setminus \{u\} \rightarrow \mathbb{P}^{r-1}$  denote the linear projection from  $u$ . Since  $X$  is smooth at  $u$ , the morphism  $\ell_{u|X \setminus \{u\}} : X \setminus \{u\} \rightarrow \mathbb{P}^{r-1}$  extends to a morphism  $\mu : X \rightarrow \mathbb{P}^{r-1}$ . By [4, Th. 3.1] and the generality of  $u$  the line  $T_u X$  is the only tangent line of  $X$  containing  $u$ . Thus  $\mu$  is unramified. Since  $p \notin \langle \{u, v\} \rangle$ ,  $\mu(p) \neq \mu(v)$ . Since the pair  $(p, u)$  is general in  $X^2$ ,  $\mu(p)$  is a general element of  $\mu(X)$ . Note that the unramified map  $\mu : X \rightarrow \mu(X)$  is the normalization map. Since  $(T_p X \setminus \{p\}) \cap \langle \{u, v\} \rangle \neq \emptyset$ , we have  $\mu(v) \in T_{\mu(p)} \mu(X)$ , contradicting [4, Remark 3.8]), concluding the proof of part (a).

Part (b) follows from part (a), because  $\tau(X) \subseteq \sigma_2(X)$ .  $\square$

*Proof of Theorem 1.2:* Condition (ii) implies  $r \geq 2n + 2$ . By the discussion after Question 1 we may assume  $n \geq 2$ .

We first prove that  $\dim \sigma_2(X) = 2n + 1$ . Take a general  $(u, v) \in X_{\text{reg}}^2$ . By the Terracini Lemma ([1, Cor. 1.5]) it is sufficient to prove that  $\dim \langle T_u X \cup T_v X \rangle = 2n + 1$ , i.e.  $R \cap L = \emptyset$  for every tangent line  $R$  of  $X$  at  $u$  and every tangent line  $L$  of  $X$  at  $v$ . Take  $\ell_u$  and  $Y \subset \mathbb{P}^{r-1}$  as in Assumption (ii). Note that  $a := \ell_u(R \setminus \{u\})$  is a point of  $Y$ . Since  $(u, v)$  is general,  $\ell_u(v)$  is a general element of  $Y_{\text{reg}}$ . Hence  $a \neq \ell_u(v)$ . Note that  $\ell_u(L)$  is the tangent line of  $Y$  at  $\ell_u(v)$ . Assumption (ii) gives  $a \notin \ell_u(L)$ , i.e.  $R \cap L = \emptyset$ .

Now we prove that  $r_X(q) > 2$  for a general  $q \in \tau(X)$ . Assume  $r_X(q) = 2$  for a general  $q \in \tau(X)$ . Since  $q$  is general, there is  $p \in X_{\text{reg}}$  such that  $q \in T_p(X)$ . Since  $q$  is general in  $\tau(X)$ ,  $p$  is general in  $X_{\text{reg}}$ . Take  $u, v \in X$  such that  $u \neq v$  and  $q \in \langle \{u, v\} \rangle$ . Condition (i) and the generality of  $p$  gives  $u \neq p$  and  $v \neq p$ . By [9, Th. 1.4]  $\dim \tau(X) = 2n$ . Thus the generality of  $q$  implies that either the pair  $(u, v)$  is general in  $X \times X$  and  $\langle \{u, v\} \rangle \cap \tau(X) = \{q\}$  or the pair  $(u, v)$  varies in a codimension 1 algebraic subset of  $X \times X$  and  $\langle \{u, v\} \rangle \subset \tau(X)$ .

Assume that the pair  $(u, v)$  varies in a codimension 1 algebraic subset of  $X \times X$  and  $\langle \{u, v\} \rangle \subset \tau(X)$ . We get that the projection  $X \times \tau(X) \times X \times X \rightarrow X \times \tau(X) \times X$  maps the set of all quadruples  $(p, q, u, v) \in X \times \tau(X) \times X \times X$  just obtained onto a Zariski open subset of  $X \times \tau(X) \times X$  with fibers of dimension  $n - 1 > 0$ . We get that  $\ell_u(p)$  is general in  $Y$ ,  $\ell_u(p) \neq \ell_u(v)$  and that  $\ell_u(v) = \ell_u(q) \in (T_{\ell_u(p)} Y \setminus \ell_u(p))$ , contradicting assumption (ii).

Now assume that for a general  $q \in \tau(X)$  the pair  $(u, v)$  is general in  $q$ . We have  $(p, u, v, q) \in X^3 \times \tau(X)$  and the projection  $X^3 \times \tau(X) \rightarrow X^3$  send the set of all such quadruples into a dominant subset of  $X^3$ . We get that  $\ell_u(p)$  is a general point of  $Y_{\text{reg}}$ , that  $\ell_u(v) \neq \ell_u(p)$  and that  $\ell_u(v) = \ell_u(q) \in T_{\ell_u(p)} Y$ , contradicting assumption (ii).

Part (b) follows from part (a), because  $\tau(X) \subseteq \sigma_2(X)$ .  $\square$

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