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LIGHTLIKE HYPERSURFACES OF A PARA-SASAKIAN SPACE FORM

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Abstract. In this paper, lightlike hypersurfaces of a para-Sasakian space form are studied. We obtain some non-existence results for screen semi-invariant lightlike hypersurfaces of a para-Sasakian space form. It is shown that a lightlike hypersurface is totally geodesic if and only if it is locally symmetric.

1. Introduction

It is well known that in case the induced metric on the submanifold of semi-Riemannian manifold is degenerate, the study becomes more different from the study of non-degenerate submanifolds. The primary difference between the lightlike submanifolds and non-degenerate submanifolds arises due to the fact in the first case the normal vector bundle has non-trivial intersection with the tangent vector bundle and moreover in a lightlike hypersurface the normal vector bundle is contained in the tangent vector bundle.

Lightlike submanifolds of semi-Riemannian manifolds were introduced by K. L. Duggal and A. Bejancu in [1]. Since then many authors studied lightlike hypersurfaces of semi-Riemannian manifolds and especially of indefinite Sasakian manifolds (for differential geometry of lightlike submanifolds we refer the book [6]).

In 1985, S. Kaneyuki and M. Konzai [2] defined the almost paracontact structure on pseudo-Riemannian manifold M of dimension $(2n+1)$ and constructed the almost paracomplex structure on $M^{2n+1} \times \mathbb{R}$. Recently, S. Zamkovoy [3] studied paracontact metric manifolds and some remarkable subclasses like para-Sasakian manifolds. Especially, in the recent years, many authors ([9], [10], [11], [12], [13]) have pointed out the importance of paracontact geometry, and in particular of para-Sasakian geometry, by several papers giving the relationships with the theory of para-Kähler manifolds and its role in pseudo-Riemannian geometry and mathematical physics.

In the present paper we study lightlike geometry of a para-Sasakian space form. The paper is organized as follows. Section 2 is devoted to preliminaries. In section 3, we introduce screen semi-invariant lightlike hypersurface of a para-Sasakian

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space form and show that there is no screen semi-invariant lightlike hypersurface of para-Sasakian space form $\bar{M}(k)$ $(k \neq -1)$ with parallel second fundamental form. Moreover, nonexistence of screen semi-invariant hypersurfaces of a para-Sasakian space form $\bar{M}(k)$ $(k \neq \frac{1}{3})$ with parallel screen distribution is proved. In section 4, we prove that a lightlike hypersurface of a para-Sasakian space form $\bar{M}(k)$ of constant curvature $k = -1$ with $\xi \in \Gamma(TM)$ is locally symmetric if and only if it is totally geodesic.

2. PRELIMINARIES

A $(2n+1)-$dimension pseudo-Riemannian manifold $\bar{M}^{2n+1}$ has an almost paracontact structure $(\bar{\phi}, \xi, \eta)$ if it equipped a tensor field $\bar{\phi}$ of type $(1,1)$, a vector field ξ, a $1-$form η satisfying the following conditions [2]:

$$\bar{\phi}^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \bar{\phi}\xi = 0, \quad \eta \circ \bar{\phi} = 0. \tag{2.1}$$

If we set $D = Ker\ \eta = \left\{X \in \Gamma\left(T\bar{M}\right) : \eta\left(X\right) = 0\right\}$, then the tensor field $\bar{\phi}$ induces an almost paracomplex structure (see[2]) on each fibre on D.

If a manifold $\bar{M}$ with an almost paracontact structure $(\bar{\phi}, \xi, \eta)$ admits a pseudo-Riemannian metric $\bar{g}$ such that [3]

$$\bar{g}(\bar{\phi}X, \bar{\phi}Y) = -\bar{g}(X,Y) + \eta(X)\eta(Y), \quad X,\ Y \in \Gamma(T\bar{M}) \tag{2.2}$$

then we say that $\bar{M}$ has an almost paracontact metric structure and $\bar{g}$ is called compatible metric. Any compatible metric $\bar{g}$ with a given almost paracontact structure is necessarily of signature $(n+1, n)$.

Setting $Y = \xi$, we have

$$\eta(X) = \bar{g}(X, \xi). \tag{2.3}$$

The fundamental $2-$form of $\bar{M}$ is defined by

$$\Phi(X,Y) = \bar{g}(X, \bar{\phi}Y). \tag{2.4}$$

Definition 2.1. [3] If $\bar{g}(X, \bar{\phi}Y) = d\eta(X,Y)$ (where $d\eta(X,Y) = \frac{1}{2}\{X\eta(Y) - Y\eta(X) - \eta([X,Y])\}$) then η is a paracontact form and the almost paracontact metric manifold $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ is said to be paracontact metric manifold.

An almost paracontact metric structure $(\bar{\phi}, \xi, \eta, \bar{g})$ is a para-Sasakian manifold if and only if [3]

$$(\bar{\nabla}_X \bar{\phi})Y = -\bar{g}(X,Y)\xi + \eta(Y)X, \tag{2.5}$$

where $X,\ Y \in \Gamma(T\bar{M})$ and $\bar{\nabla}$ is a Levi-Civita connection on $\bar{M}$.

From (2.5), it can be seen that

$$\bar{\nabla}_X \xi = -\bar{\phi}X.$$

Theorem 2.2. [4] *Let $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ be a para-Sasakian manifold of dimension $(2n+1)$, $n > 1$. If the paraholomorphic sectional curvature is independent of the paraholomorphic section at a point, the curvature tensor has form*

$$\begin{aligned} \bar{g}(\bar{R}(X,Y)Z,W) &= \frac{k-3}{4}\left(\bar{g}(Y,Z)\bar{g}(X,W) - \bar{g}(X,Z)\bar{g}(Y,W)\right) \\ &\quad + \frac{k+1}{4}\begin{pmatrix} \bar{g}(X,Z)\eta(Y)\eta(W) + \bar{g}(Y,W)\eta(X)\eta(Z) \\ -\bar{g}(X,W)\eta(Y)\eta(Z) - \bar{g}(Y,Z)\eta(X)\eta(W) \\ +\bar{g}(Y,\bar{\phi}Z)\bar{g}(\bar{\phi}X,W) - \bar{g}(X,\bar{\phi}Z)\bar{g}(\bar{\phi}Y,W) \\ +2\bar{g}(\bar{\phi}X,Y)\bar{g}(\bar{\phi}Z,W) \end{pmatrix}. \end{aligned} \tag{2.6}$$

A para-Sasakian manifold $\bar{M}$ is said to be of constant paraholomorphic curvature or para-Sasakian space form if the paraholomorphic sectional curvature is constant on the manifold. We denote a para-Sasakian space form by $\bar{M}(k)$.

Let $(\bar{M}, \bar{g})$ be a $(2n+1)$-dimensional semi-Riemannian manifold with index q, $0 < q < 2n+1$ and (M, g) is a hypersurface of $\bar{M}$, with $g = \bar{g} \mid_M$. M is a lightlike hypersurface of $\bar{M}$ if the metric g is of constant rank $2n-1$. The orthogonal complement $TM^{\perp}$ of tangent space TM, defined by

$$TM^{\perp} = \bigcup_{p \in M} \{Y_p \in T_p\bar{M} : g_p(X_p, Y_p) = 0, \forall X_p \in \Gamma(T_pM)\},$$

is a distribution of rank 1 on M [1]. $TM^{\perp} \subset TM$ and then it coincides with the radical distribution $RadTM = TM^{\perp} \cap TM$. A complementary bundle of $TM^{\perp}$ in TM called a screen distrubution and denoted by $S(TM)$ is a nondegenerate distribution of constant rank $2n-1$ over M [1].

A lightlike hypersurface endowed with a specific screen distribution is denoted by the triple $(M, g, S(TM))$. As $TM^{\perp}$ lies in the tangent bundle, the following results has an important role in studying the geometry of a lightlike hypersurface [1].

Theorem 2.3. [1] *Let $(M, g, S(TM))$ be a lightlike hypersurface of a almost paracontact manifold $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$. Then there exists a unique rank one vector subbundle $ltr(TM)$ of $T\bar{M}$, with base space M, such that for any non-zero section E of $RadTM$ on a coordinate neighborhood $U \subset M$, there exists a unique section N of $ltr(TM)$ on U satisfying:*

$$\bar{g}(N, E) = 1, \quad \bar{g}(N, N) = 0, \quad \bar{g}(N, W) = 0, \quad \text{for all } W \in \Gamma(S(TM) \mid_U .$$

$ltr(TM)$ is called the lightlike transversal vector bundle of M with respect to $S(TM)$.

By Theorem 2.3 one can state the following decompositions [1]:

$$TM = S(TM) \perp RadTM, \tag{2.7}$$

$$\begin{aligned} T\bar{M} &= TM \oplus ltr(TM) \\ &= S(TM) \perp \{RadTM \oplus ltr(TM)\}, \end{aligned} \tag{2.8}$$

where $\perp$ and $\oplus$ denote the orthogonal and non-orthogonal direct sum. Let $\bar{\nabla}$ be the Levi-Civita connection on $(\bar{M}, \bar{g})$. Then by using decomposition given by (2.8) and considering a normalizing pair $\{E, N\}$ as in Theorem 2.3, we have Gauss and Weingarten formulas respectively

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.9}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{lt}_X N, \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(ltr(TM))$,where $\nabla_X Y$, $A_N X \in \Gamma(TM)$ and $h(X, Y)$, $\nabla^{lt}_X N \in \Gamma(ltr(TM))$. Here ∇ is an induced symmetric linear connection on the vector subbundle $ltr(TM)$, h is a $ltr(TM)$-valued symmetric bilinear form and A_N is shape operator of M .

Locally let E, N and U be as in Theorem 2.3, for any $X, Y \in \Gamma(TM\ |_U)$ by putting

$$B(X,Y) = \bar{g}(h(X,Y),E) \quad \text{and} \quad \tau(X) = \bar{g}(\nabla^{lt}_X N, E),$$

we can write

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \tag{2.11}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.12}$$

where B, A_N, τ and ∇ are called the local second fundamental form, the local shape operator, the transversal differential 1-form and induced linear torsion free connection, respectively, on $TM\ |_U$. It is important to mention that the second fundamental form B of M is independent of the choice of screen distribution and

$$B(\cdot, E) = 0. \tag{2.13}$$

Let $P : \Gamma(TM) \to \Gamma(S(TM))$ be the projection morphism with respect to the orthogonal decomposition of TM. We have, for any $X, Y \in \Gamma(TM)$ and $E \in \Gamma(TM^{\perp})$

$$\nabla_X PY = \nabla^*_X PY + C(X,PY)E, \tag{2.14}$$

$$\nabla_X E = -A^*_E X - \tau(X)E, \tag{2.15}$$

where $\nabla^*_X PY$, $A^*_E X$ belong to $\Gamma(S(TM))$. C, A^*_E and $\nabla^*$are called the second fundamental form, the local shape operator and induced connection on $S(TM)$, respectively. The induced linear connection ∇ is not a metric connection and we have

$$(\nabla_X g)(Y,Z) = B(X,Y)\theta(Z) + B(X,Z)\theta(Y), \tag{2.16}$$

for any $X, Y \in \Gamma(TM\ |_U)$, where θ is a differential 1-form locally defined on M by $\theta(\cdot) = \bar{g}(N, \cdot)$.

The local second fundamental forms B and C, respectively, on M and $S(TM)$ are related to their shape operators by

$$g(A^*_E X, PY) = B(X,PY), \quad g\ (A^*_E X, N) = 0, \tag{2.17}$$

$$g(A_N X, PY) = C(X,PY), \quad g(A_N X, N) = 0. \tag{2.18}$$

We denote by $\bar{R}$, R and R^* the curvature tensor of $\bar{\nabla}$, ∇ and ∇^*, respectively. Using the Gauss and Weingarten formulas for M and $S(TM)$, we obtain the Gauss and Codazzi equations for M and $S(TM)$ (see [6]),

$$\begin{aligned} \bar{g}(\bar{R}(X,Y)Z,PW) &= g(R(X,Y)Z,PW) + B(X,Z)C(Y,PW) \\ &\quad -B(Y,Z)C(X,PW), \end{aligned} \tag{2.19}$$

$$\begin{aligned} \bar{g}(\bar{R}(X,Y)Z,E) &= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ &\quad +\tau(X)B(Y,Z) - \tau(Y)B(X,Z), \end{aligned} \tag{2.20}$$

$$\begin{aligned} g(R(X,Y)PZ,N) &= (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\ &\quad +\tau(Y)C(X,PZ) - \tau(X)C(Y,PZ), \end{aligned} \tag{2.21}$$

$$\bar{g}(\bar{R}(X,Y)Z,N) = g(R(X,Y)Z,N), \tag{2.22}$$

for any $X, Y, Z, W \in \Gamma(TM)$. Also we have

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z + B(X,Z)A_N Y - B(Y,Z)A_N X \\ &\quad +\{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ &\quad +\tau(X)B(Y,Z) - \tau(Y)B(X,Z)\}N.\end{aligned} \tag{2.23}$$

3. SCREEN SEMI-INVARIANT LIGHTLIKE HYPERSURFACES OF PARA-SASAKIAN MANIFOLDS

Let (M, g) be a lightlike hypersurface of a para-Sasakian manifold $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ of dimension $(2n+1)$, tangent to the structure vector field ξ (that is, $\xi \in \Gamma(TM)$). If E is a local section of $\Gamma(Rad(TM))$, then $\bar{g}(\bar{\phi}E, E) = 0$ and $\bar{\phi}E$ is tangent to M. Therefore we get a 1-dimensional distribution $\bar{\phi}(Rad(TM))$ on M.

Definition 3.1. Let $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ be a $(2n+1)$-dimensional para-Sasakian manifold and M be a lightlike hypersurface of $\bar{M}$. If

$$\bar{\phi}(Rad(TM)) \subset S(TM)$$

and

$$\bar{\phi}(ltr(TM)) \subset S(TM),$$

then M will be called a screen semi-invariant lightlike hypersurface.

Thus we can state

$$\bar{g}(\bar{\phi}N, E) = -\bar{g}(N, \bar{\phi}E) = 0, \qquad \bar{g}(\bar{\phi}N, N) = 0. \tag{3.1}$$

Also, from (2.2) we obtain

$$\bar{g}(\bar{\phi}N, \bar{\phi}E) = -1. \tag{3.2}$$

Therefore $\bar{\phi}(Rad(TM)) \oplus \bar{\phi}(ltr(TM))$ is a non-degenerate vector subbundle of $S(TM)$ of rank 2.

Following, being $S(TM)$ and $\bar{\phi}(Rad(TM)) \oplus \bar{\phi}(ltr(TM))$ non-degenerate, we can define the unique non-degenerate distribution D_0 such that [6]

$$S(TM) = \{\bar{\phi}(Rad(TM)) \oplus \bar{\phi}(ltr(TM))\} \perp D_0. \tag{3.3}$$

Then $\xi \in D_0$ and D_0 is invariant under $\bar{\phi}$, i.e. $\bar{\phi}(D_0) = D_0$. Moreover, from (2.7), (2.8) and (3.3)

$$TM = \{\bar{\phi}(Rad(TM)) \oplus \bar{\phi}(ltr(TM))\} \perp D_0 \perp Rad(TM), \tag{3.4}$$

$$T\bar{M} = \{\bar{\phi}(Rad(TM)) \oplus \bar{\phi}(ltr(TM))\} \perp D_0 \perp \{Rad(TM) \oplus ltr(TM)\}. \tag{3.5}$$

Now, we consider the distributions $D = Rad(TM) \perp \bar{\phi}(Rad(TM)) \perp D_0$, $D' = \bar{\phi}(ltr(TM))$ on M. Then D is a $\bar{\phi}$-invariant distribution and

$$TM = D \oplus D'. \tag{3.6}$$

Let us consider the local lightlike vector fields $U = \bar{\phi}N \in \bar{\phi}(ltr(TM))$ and $V = \bar{\phi}E \in D$. From (3.6) any $X \in \Gamma(TM)$ can be written as

$$X = SX + QX, \qquad QX = u(X)U,$$

where S and Q are the projection morphism of TM into D and $D^{'}$, respectively, and u is a differential 1-form locally defined on M by

$$u(X) = -g(X, \bar{\phi}E). \tag{3.7}$$

We note that

$$u(U) = 1, \tag{3.8}$$
$$u(Y) = 0, \quad \forall Y \in \Gamma(D), \tag{3.9}$$
$$\bar{\phi}^2 N = N. \tag{3.10}$$

Then applying $\bar{\phi}$ to X, one obtains

$$\begin{aligned} \bar{\phi}X &= \bar{\phi}(SX) + u(X)\bar{\phi}U \\ &= \bar{\phi}(SX) + u(X)N. \end{aligned}$$

For any $X \in \Gamma(TM)$, if we put $\phi X = \bar{\phi}(SX)$ in the last equation we have

$$\bar{\phi}X = \phi X + u(X)N, \tag{3.11}$$

where ϕ is a tensor field of type $(1,1)$ on M. Applying $\bar{\phi}$ again, we get $X - \eta(X)\xi = \bar{\phi}\phi X + u(X)U$. We note that if $X \in \Gamma(TM)$, then $SX \in D$, $\phi X = \bar{\phi}(SX) \in D$, so that $S(\phi X) = \phi X$. Furthermore, since $\bar{\phi}\phi X = \bar{\phi}(S\phi X) = \bar{\phi}S(\phi X) = \phi^2 X$, we can write

$$\phi^2 X = X - \eta(X)\xi - u(X)U. \tag{3.12}$$

Using (2.11), (2.12) and (3.11), we get

$$\begin{aligned} (\bar{\nabla}_X \bar{\phi})Y &= (\nabla_X \phi)Y - u(Y)A_N X - B(X,Y)U \\ &\quad -(B(X,\phi Y) + (\nabla_X u)Y + u(Y)\tau(X))N, \end{aligned}$$

for any $X, Y \in \Gamma(TM)$. Then from (2.5), comparing the components along TM and $ltr(TM)$, we have

$$(\nabla_X \phi)Y = -g(X,Y)\xi + \eta(Y)X + u(Y)A_N X + B(X,Y)U, \tag{3.13}$$
$$(\nabla_X u)Y = -B(X,\phi Y) - u(Y)\tau(X). \tag{3.14}$$

Definition 3.2. Let $(M, g, S(TM))$ be a lightlike hypersurface of a para-Sasakian manifold $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$. Then M is called totally umbilical lightlike hypersurface if there exists a smooth function λ such that

$$B(X,Y) = \lambda g(X,Y),$$

for any coordinate neighborhood U and $X, Y \in \Gamma(TM\,|_U)$.

Theorem 3.3. *Let $(\bar{M}(k), \bar{\phi}, \xi, \eta, \bar{g})$ be a para-Sasakian space form. If a screen semi-invariant lightlike hypersurface M of $\bar{M}(k)$ is totally umbilical then $k = -1$.*

Proof. From (2.6) we get,

$$\bar{g}(\bar{R}(X,Y)Z,E) = \frac{k+1}{4}\begin{pmatrix} -\bar{g}(\bar{\phi}X,Z)\bar{g}(Y,V) + \bar{g}(\bar{\phi}Y,Z)\bar{g}(X,V) \\ +2\bar{g}(X,\bar{\phi}Y)\bar{g}(\bar{\phi}Z,V) \end{pmatrix}, \tag{3.15}$$

for any $X, Y, Z \in \Gamma(TM)$. On the other hand, replacing X, Y, Z by PX, E, PZ in (3.15) and (2.20) give

$$\bar{g}(\bar{R}(X,Y)Z,E) = \frac{3(k+1)}{4}u(PX)u(PZ)$$

and

$$\begin{aligned}\bar{g}(\bar{R}(X,Y)Z,E) &= -B(\nabla_{PX}E,PZ)-EB(PX,PZ)+B(\nabla_E PX,PZ)\\ &\quad +B(PX,\nabla_E PZ)-\tau(E)B(PX,PZ),\end{aligned}$$

respectively. By using Gauss and Weingarten formulas and (2.13) then comparing the two corresponding equations, we get

$$\frac{3(k+1)}{4}u(PX)u(PZ)=(\lambda^2-E(\lambda)-\lambda\tau(E))\bar{g}(PX,PZ).$$

Choosing $X=Z=U\in S(TM)$, we have $PX=PZ=U$ and being $u(U)=1$ and $g(U,U)=0$, from previous equation we obtain

$$k=-1.$$

□

Definition 3.4. Let $(M,g,S(TM))$ be a screen semi-invariant lightlike hypersurface of a para-Sasakian manifold $(\bar{M},\bar{\phi},\xi,\eta,\bar{g})$. The screen distribution $S(TM)$ is called totally umbilical if for any coordinate neighborhood U there exists a smooth function λ such that

$$C(X,PY)=\lambda g(X,PY),$$

for any $X,Y\in\Gamma(TM\mid_U)$.

Theorem 3.5. *Let $(M,g,S(TM))$ be a screen semi-invariant lightlike hypersurface of a para-Sasakian manifold $(\bar{M},\bar{\phi},\xi,\eta,\bar{g})$ with totally umbilical screen distribution $S(TM)$. Then we have*

$$C(X,PY)=0,$$

for any $X,Y\in\Gamma(TM)$, i.e. $S(TM)$ is totally geodesic.

Proof. Putting $Z=PZ$ in (2.22) and using (2.14) we get

$$\begin{aligned}\bar{g}(\bar{R}(X,Y)PZ,N) &= X(C(Y,PZ))-C(Y,\nabla^*_X PZ)-C(\nabla_X Y,PZ)\\ &\quad -\tau(X)C(Y,PZ)-Y(C(X,PZ))+C(X,\nabla^*_Y PZ)\\ &\quad +C(\nabla_Y X,PZ)+\tau(Y)C(X,PZ),\end{aligned}$$

for any $X,Y,Z\in\Gamma(TM)$. Now replacing X, Y, Z by E, U, U, respectively in last equation we obtain

$$\bar{g}(\bar{R}(E,U)U,N)=-\lambda(g(U,\nabla^*_E U)+g(\nabla_E U,U)+g(\nabla_U E,U)).$$

Since $\nabla^*_E U=\nabla_E U-C(E,U)U=\nabla_E U$ and $\nabla^*_U U=\nabla_U U-C(U,U)E=\nabla_U U$ we find

$$\begin{aligned}\bar{g}(\bar{R}(E,U)U,N) &= \lambda(g(\nabla_U E,U)-2g(\nabla_E U,U))\\ &= \lambda(\bar{g}(U,\bar{\nabla}_U E))\\ &= \lambda(\bar{g}(\bar{\nabla}_U E,\bar{\phi}N))\\ &= \lambda(\bar{g}(\bar{\nabla}_U\bar{\phi})E,N)-\bar{g}(\bar{\nabla}_U\bar{\phi}E,N))\\ &= -\lambda(\bar{g}(\bar{\nabla}_U N,\bar{\phi}E))\\ &= \lambda g(A_N U,\bar{\phi}E)=\lambda C(U,\bar{\phi}E)=\lambda^2 g(U,V)=-\lambda^2.\end{aligned}$$

From (2.6) we have

$$\begin{aligned}\bar{g}(\bar{R}(E,U)U,N) &= \frac{(k-3)}{4}\left(\bar{g}(U,U)\bar{g}(E,N)-\bar{g}(E,U)\bar{g}(U,N)\right)\\ &\quad+\frac{(k+1)}{4}\begin{pmatrix}\bar{g}(E,U)\eta(U)\eta(N)-\bar{g}(U,U)\eta(E)\eta(N)\\ +\bar{g}(N,U)\eta(U)\eta(E)-\bar{g}(E,N)\eta(U)\eta(U)\\ -\bar{g}(\bar{\phi}E,U)\bar{g}(\bar{\phi}U,N)-\bar{g}(\bar{\phi}U,U)\bar{g}(\bar{\phi}E,N)\\ -2\bar{g}(E,\bar{\phi}U)\bar{g}(\bar{\phi}U,N)\end{pmatrix}\\ &= 0.\end{aligned}$$

Thus, we find $\lambda = 0$ and $S(TM)$ is totally geodesic. □

Lemma 3.6. *Let $(M, g, S(TM))$ be a screen semi-invariant lightlike hypersurface of para-Sasakian space form $(\bar{M}(k), \bar{\phi}, \xi, \eta, \bar{g})$. Then, for any $X, Y, Z \in \Gamma(TM)$ i) The equation of Gauss of M is given by the following equation:*

$$\begin{aligned}R(X,Y)Z &= \frac{k-3}{4}\left(\bar{g}(Y,Z)X-\bar{g}(X,Z)Y\right)\\ &\quad+\frac{k+1}{4}\begin{pmatrix}\bar{g}(X,Z)\eta(Y)\xi-\bar{g}(Y,Z)\eta(X)\xi\\ +\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X\\ +\bar{g}(\bar{\phi}X,Z)\phi Y-\bar{g}(\bar{\phi}Y,Z)\phi X\\ -2\bar{g}(X,\bar{\phi}Y)\phi Z\end{pmatrix}\\ &\quad-B(X,Z)A_NY+B(Y,Z)A_NX.\end{aligned}\tag{3.16}$$

ii) The equation of Codazzi of M is given by the following equation:

$$(\nabla_X h)(Y,Z)-(\nabla_Y h)(X,Z)=\frac{k+1}{4}\begin{pmatrix}\bar{g}(\bar{\phi}Y,Z)u(X)\\ -\bar{g}(\bar{\phi}X,Z)u(Y)\\ +2\bar{g}(X,\bar{\phi}Y)u(Z)\end{pmatrix}N.\tag{3.17}$$

Proof. Since $(\bar{M}(k), \bar{\phi}, \xi, \eta, \bar{g})$ is a para-Sasakian space form, from (2.6) and (2.23) we obtain

$$\begin{aligned}R(X,Y)Z &= \frac{k-3}{4}\left(\bar{g}(Y,Z)X-\bar{g}(X,Z)Y\right)\\ &\quad+\frac{k+1}{4}\begin{pmatrix}\bar{g}(X,Z)\eta(Y)\xi-\bar{g}(Y,Z)\eta(X)\xi\\ +\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X\\ +\bar{g}(\bar{\phi}X,Z)\phi Y-\bar{g}(\bar{\phi}Y,Z)\phi X\\ -2\bar{g}(X,\bar{\phi}Y)\phi Z\end{pmatrix}\\ &\quad-A_{h(X,Z)}Y+A_{h(Y,Z)}X-(\nabla_X h)(Y,Z)+(\nabla_Y h)(X,Z).\end{aligned}$$

Using (3.11) in the last equation and comparing the tangential and transversal vector bundle parts of the both sides imply (3.16) and (3.17), respectively. □

Hence we have.

Lemma 3.7. *Let $(M, g, S(TM))$ be a screen semi-invariant lightlike hypersurface of para-Sasakian space form $\bar{M}(k)$. Then for any $X, Z \in \Gamma(TM)$*

$$\begin{aligned}\bar{g}(R(X,E)Z,N) &= \frac{k-3}{4}(-\bar{g}(X,Z)) \\ &\quad + \frac{k+1}{4}\left(\eta(X)\eta(Z) + u(Z)\theta(\phi X) + 2u(X)\theta(\phi Z)\right).\end{aligned}$$

Lemma 3.8. *Let $(M, g, S(TM))$ be a screen semi-invariant lightlike hypersurface of para-Sasakian space form $\bar{M}(k)$. Then, we have*

$$B(Y,U) = C(Y,V),$$

for any $X, Y, Z \in \Gamma(TM)$.

Proof. From the definition of B, we have

$$\begin{aligned}B(Y,\bar{\phi}N) &= \bar{g}(h(Y,\bar{\phi}N),E) \\ &= \bar{g}(\bar{\nabla}_Y\bar{\phi}N,E) = \bar{g}((\bar{\nabla}_Y\bar{\phi})N,E) - \bar{g}(\bar{\nabla}_Y N,\bar{\phi}E).\end{aligned}$$

By using (2.10) and (2.18) we get

$$B(Y,\bar{\phi}N) = -\bar{g}(\bar{\nabla}_Y N,\bar{\phi}E) = g(A_N Y,\bar{\phi}E) = C(Y,\bar{\phi}E),$$

which completes the proof. □

Theorem 3.9. *There is no screen semi-invariant lightlike hypersurface of para-Sasakian space form $\bar{M}(k)$ $(k \neq -1)$ with parallel second fundamental form.*

Proof. Assume that there exists a screen semi-invariant lightlike hypersurface of $\bar{M}(k)$ with $(k \neq -1)$ and a parallel second fundamental form. If we put $Y = E$, $Z = \bar{\phi}N$ in (3.17), we have

$$0 = \frac{k+1}{4}\begin{pmatrix}\bar{g}(\bar{\phi}E,\bar{\phi}N)u(X) - \bar{g}(\bar{\phi}X,\bar{\phi}N)u(E) \\ +2\bar{g}(X,\bar{\phi}E)u(\bar{\phi}N)\end{pmatrix}.$$

By using (3.1) and (3.2) in the last equation we get

$$\frac{-3(k+1)}{4}u(X) = 0.$$

Then putting $X = \bar{\phi}N$ in the last equations gives

$$k = -1,$$

which is a contradiction. Hence, the claims holds. □

Theorem 3.10. [7] *A screen distribution $S(TM)$ is parallel with respect to ∇ if and only if on each $U \subset M$ we have $C = 0$.*

Theorem 3.11. *There is no screen semi-invariant lightlike hypersurface of para-Sasakian space form $\bar{M}(k)$ $(k \neq \frac{1}{3})$ with parallel screen distribution.*

Proof. Suppose on contrary that $k \neq \frac{1}{3}$ and screen distribution is parallel. Then from (2.6) we obtain

$$\bar{g}(\bar{R}(E,\bar{\phi}N)\bar{\phi}E,N) = \frac{1-3k}{4}. \tag{3.18}$$

On the other hand, from (2.21) we have

$$\begin{aligned}\bar{g}(\bar{R}(X,Y)PZ,N) &= \bar{g}(R(X,Y)PZ,N)\\ &= (\nabla_X C)(Y,PZ)-(\nabla_Y C)(X,PZ)+\tau(Y)C(X,PZ)\\ &\quad -\tau(X)C(Y,PZ).\end{aligned}$$

So, we obtain

$$\bar{g}(\bar{R}(E,\bar{\phi}N)\bar{\phi}E,N)=0.$$

Using last equality together with (3.18) we obtain

$$k=\frac{1}{3},$$

which is a contradiction. This completes the proof. □

Definition 3.12. A lightlike hypersurface M is $D^{'}$-totally geodesic if the local second fundamental form B satisfies $B(X,Y)=0$, for any X, $Y\in D^{'}$.

Theorem 3.13. *Let $(M,g,S(TM))$ be a screen semi-invariant lightlike hypersurface of para-Sasakian space form $\bar{M}(k)$ such that its local second fundamental form B is parallel. If $\tau(E)\neq 0$, then $k=-1$ if and only if $D^{'}$-totally geodesic.*

Proof. Suppose that B is parallel. Then taking $Y=E$ in (2.20) and by using (2.6) we obtain

$$\tau(E)B(X,Z)=\frac{3(k+1)}{4}u(X)u(Z).$$

Putting $X=Z=U$, we have $\frac{3(k+1)}{4}=\tau(E)B(U,U)$ and if $\tau(E)\neq 0$, the equivalence follows. □

4. LOCALLY SYMMETRIC LIGHTLIKE HYPERSURFACES IN A PARA-SASAKIAN SPACE FORM

Let M be a lightlike hypersurface of a para-Sasakian space form $\bar{M}(k)$ with $\xi\in\Gamma(TM)$. Let us consider the pair $\{E,N\}$ on $U\subset M$. By using (2.6), (2.23), (3.11) and comparing the tangential and transversal parts of the both sides, we have for any $X,Y,Z\in\Gamma(TM)$,

$$\begin{aligned}R(X,Y)Z &= \frac{k-3}{4}\left(\bar{g}(Y,Z)X-\bar{g}(X,Z)Y\right)\\ &\quad +\frac{k+1}{4}\begin{pmatrix}\bar{g}(X,Z)\eta(Y)\xi+\eta(X)\eta(Z)Y\\ -\eta(Y)\eta(Z)X-\bar{g}(Y,Z)\eta(X)\xi\\ +\bar{g}(Y,\bar{\phi}Z)\phi X-\bar{g}(X,\bar{\phi}Z)\phi Y\\ +2\bar{g}(\bar{\phi}X,Y)\phi Z\end{pmatrix}\\ &\quad -B(X,Z)A_NY+B(Y,Z)A_NX,\end{aligned}\tag{4.1}$$

and

$$\begin{aligned}(\nabla_X B)(Y,Z)-(\nabla_Y B)(X,Z) &= \tau(Y)B(X,Z)-\tau(X)B(Y,Z)\\ &\quad +\frac{k+1}{4}\begin{pmatrix}-\bar{g}(X,\bar{\phi}Z)u(Y)\\ +\bar{g}(Y,\bar{\phi}Z)u(X)\\ +2\bar{g}(\bar{\phi}X,Y)u(Z)\end{pmatrix}.\end{aligned}\tag{4.2}$$

A lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be locally symmetric, if and only if for any $X, Y, Z, T, W \in \Gamma(TM)$ and $N \in \Gamma(ltr(TM))$, the following holds [8]:

$$g((\nabla_W R)(X,Y)Z, PT) = 0 \quad \text{and} \quad \bar{g}((\nabla_W R)(X,Y)Z, N) = 0. \tag{4.3}$$

That is,

$$(\nabla_W R)(X,Y)Z = 0.$$

Using Lemma 3.2 in [8], for any $X, Y, Z, W \in \Gamma(TM)$, $T \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$, we have,

$$\begin{aligned} \bar{g}((\bar{\nabla}_W \bar{R})(X,Y)Z, T) &= g((\nabla_W R)(X,Y)Z, T) + (\nabla_W B)(X,Z)C(Y,T) \\ &\quad + B(X,Z)g((\nabla_W A_N)Y, T) - (\nabla_W B)(Y,Z)C(X,T) \\ &\quad - B(Y,Z)g((\nabla_W A_N)X, T) - B(Y,Z)\tau(X)C(W,T) \\ &\quad + (\nabla_Y B)(X,Z)C(W,T) - (\nabla_X B)(Y,Z)C(W,T) \\ &\quad + B(X,Z)\tau(Y)C(W,T) - B(W,X)\bar{R}(N,Y,Z,T) \\ &\quad - B(W,Y)\bar{R}(X,N,Z,T) - B(W,Z)\bar{R}(X,Y,N,T), \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} \bar{g}((\bar{\nabla}_W \bar{R})(X,Y)Z, N) &= g((\nabla_W R)(X,Y)Z, N) + B(X,Z)g((\nabla_W (A_N Y), N) \\ &\quad - B(Y,Z)g((\nabla_W (A_N X), N) - B(W,X)\bar{R}(N,Y,Z,N) \\ &\quad - B(W,Y)\bar{R}(X,N,Z,N). \end{aligned} \tag{4.5}$$

Lemma 4.1. *Let $\bar{M}(k)$ be a para-Sasakian space form. Then we have, for any $X, Y, Z, W \in \Gamma(TM)$,*

$$(\bar{\nabla}_W \bar{R})(X,Y)Z = \frac{k+1}{4} \begin{pmatrix} \bar{g}(Y,Z)\bar{g}(X,\bar{\phi}W)\xi - \bar{g}(X,Z)\bar{g}(Y,\bar{\phi}W)\xi \\ +\bar{g}(Y,Z)\eta(X)\bar{\phi}W - \bar{g}(X,Z)\eta(Y)\bar{\phi}W \\ +\bar{g}(Y,\bar{\phi}W)\eta(Z)X - \bar{g}(X,\bar{\phi}W)\eta(Z)Y \\ +\bar{g}(Z,\bar{\phi}W)\eta(Y)X - \bar{g}(Z,\bar{\phi}W)\eta(X)Y \end{pmatrix}, \tag{4.6}$$

where $\bar{R}$ is the Riemannian curvature tensor of Levi-Civita connection $\bar{\nabla}$.

Proof. By using (2.6) and the definition of covariant derivative of differential forms, we obtain the result. □

Theorem 4.2. *There are no locally symmetric lightlike hypersurfaces of a para-Sasakian space form $\bar{M}(k)$ $(k \neq -1)$, tangent to structure vector field ξ. Moreover, if M is a lightlike hypersurface of a para-Sasakian space form $\bar{M}(k)$ of constant curvature $k = -1$ with $\xi \in \Gamma(TM)$, then M is locally symmetric if and only if it is totally geodesic.*

Proof. Let M be locally symmetric lightlike hypersurface of a para-Sasakian space form $\bar{M}(k)$. Assume that $k \neq -1$. From (4.6), we have

$$\bar{g}((\bar{\nabla}_W \bar{R})(X,Y)Z, N) = \frac{k+1}{4} \begin{pmatrix} \bar{g}(Y,Z)\eta(X)\nu(W) - \bar{g}(X,Z)\eta(Y)\nu(W) \\ +\bar{g}(Y,\bar{\phi}W)\eta(Z)\theta(X) - \bar{g}(X,\bar{\phi}W)\eta(Z)\theta(Y) \\ +\bar{g}(Z,\bar{\phi}W)\eta(Y)\theta(X) - \bar{g}(Z,\bar{\phi}W)\eta(X)\theta(Y) \end{pmatrix}, \tag{4.7}$$

for any $X, Y, Z, W \in \Gamma(TM)$, where $\nu(W) = \bar{g}(W, U)$.

From (2.6), we have $\bar{R}(E,N,E,N) = \frac{k-3}{4}$. By taking $X = E$ and $Z = E$ in (4.5) and (4.7), we obtain

$$-\frac{k-3}{4}B(W,Y) = \bar{g}((\bar{\nabla}_W \bar{R})(E,Y)E, N) = \frac{k+1}{4}\bar{g}(E,\bar{\phi}W)\eta(Y), \tag{4.8}$$

for any $Y, W \in \Gamma(TM)$, which implies

$$-\frac{k-3}{4}B(W,Y) = \frac{k+1}{4}u(W)\eta(Y). \tag{4.9}$$

Since the local second fundamental form B is symmetric, the relation (4.9) leads to

$$(k+1)\{u(W)\eta(Y) - u(Y)\eta(W)\} = 0.$$

Taking $Y = \xi$ and $W = U$, we have $k = -1$ which is a contradiction. Hence, the claim holds.

On the other hand, let M is a lightlike hypersurface of a para-Sasakian space form $\bar{M}(k)$ of constant curvature $k = -1$ with $\xi \in \Gamma(TM)$. If M is locally symmetric, we have $B = 0$, by virtue of (4.9). Conversely if (M, g) is totally geodesic, using (4.4), (4.5) and (4.6), we get $g((\nabla_W R)(X,Y)Z, PT) = 0$ and $\bar{g}((\nabla_W R)(X,Y)Z, N) = 0$, that is, M is locally symmetric. This completes the proof. □

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