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EXISTENCE AND EXPONENTIAL STABILITY OF STEPANOV-LIKE ALMOST AUTOMORPHIC MILD SOLUTIONS FOR SEMILINEAR EVOLUTION EQUATIONS

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Abstract. This work is concerned with the existence and exponential stability of Stepanov-like almost automorphic mild solutions for the following semilinear evolution equations

$$x'(t) = Ax(t) + F(t, x(t)), \quad t \in \mathbb{R},$$

where A is the infinitesimal generator of a C_0-semigroup of bounded linear operator on a Banach space X and $F : \mathbb{R} \times X \to X$ is a Stepanov-like almost automorphic function in t uniformly with respect to the second argument x. By applying the Banach contraction mapping principle (when F satisfies Lipschitz type conditions), and the Schauder's fixed point theorem (when F does not necessarily satisfy Lipschitz type conditions), we obtain the existence and exponential stability of Stepanov-like almost automorphic mild solutions for the semilinear evolution equations. Moreover, as application, two examples are given to illustrate our abstract results.

1. Introduction and preliminaries

In this paper, we are concerned with the existence and exponential stability of Stepanov-like almost automorphic mild solutions for the following semilinear evolution equations

$$x'(t) = Ax(t) + F(t, x(t)), \ t \in \mathbb{R}, \tag{1.1}$$

where A is the infinitesimal generator of a C_0-semigroup of bounded linear operator on a Banach space X and $F : \mathbb{R} \times X \to X$ is a Stepanov-like almost automorphic function in t uniformly with respect to the second argument x.

In the earlier sixties, Bochner introduced the concept of almost automorphic function in his papers [1, 2, 3] in relation to some aspects of differential geometry. The notion of almost automorphic function was introduced to avoid some assumptions of uniform convergence that arise when using almost periodic function, it is an important generalization of the classical almost periodic function which is one of the most attractive topics in the qualitative theory of differential equations because of its significance and applications in physics, mathematical

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biology, control theory, and other related fields. From that time the theory of almost automorphic function has been studied by numerous authors, and it also becomes one of the most attractive topics in the qualitative theory of differential equations because of its significance and applications. Meanwhile, stimulated by the work [1, 2, 3], many interesting generalizations of the almost automorphic function have been introduced including asymptotic almost automorphy by N'Guérékata [4], p-almost automorphy by Diagana [5], pseudo almost automorphy by Liang et al.[6], and so on. These concepts are important and of interest because of their significance and applications in physics, mechanics, mathematical biology, and many others. Stepanov-like almost automorphy is also one of the most important generalizations. The concept of Stepanov-like almost automorphy was introduced by N'Guérékata and Pankov in [7], and subsequently was applied to study the existence of Stepanov-like almost automorphic solutions to some parabolic evolution equations. In connection with differential equations, the great importance from both the applied and theoretical points of view of the existence of periodic solutions is well known. However, either because models are only an approximation of reality or due to numerical errors, in practice it is impossible to verify whether a solution is exactly periodic. The concept of Stepanov-like almost automorphic function allows relaxing some assumptions to obtain solutions that have properties similar to those of a periodic function. Meanwhile, the applications of the new theory for these generalized functions, especially the Stepanov-like almost automorphic function, to various types of linear, semilinear as well as nonlinear differential equations were studied extensively (see, e.g.[8, 9, 10, 11, 12, 13, 14, 15, 16, 17] and references therein).

The existence of almost automorphic solution for Eq.(1.1) has been extensively studied when the semigroup generated by A is exponential stability or has an exponential dichotomy, since in those case Eq.(1.1) has a unique bounded solution on $\mathbb{R}$ which is almost automorphic when F is almost automorphic in t and Lipschitzian with respect to the second argument x, and the Lipschitz constant is sufficiently small (depending on the constant governing the exponential stability or the exponential dichotomy), more details can be found in [13]. In [18], the authors studied the existence of almost automorphic solutions for the following ordinary differential equations

$$x'(t) = Gx(t) + e(t), \text{ for } t \in \mathbb{R}, \tag{1.2}$$

where G is a constant $n \times n$ matrix and $e : \mathbb{R} \to \mathbb{R}^n$ is almost automorphic. They proved that the existence of a bounded solution for Eq.(1.2) on $\mathbb{R}^+$ implies the existence of an almost automorphic solution. In [19], the author studied the existence of almost automorphic solutions for the following semilinear abstract differential equations

$$x'(t) = \mathbb{C}x(t) + \theta(t), \text{ for } t \geq 0, \tag{1.3}$$

where $\mathbb{C}$ generates an exponentially stable semigroup on a Banach space Y and θ is an almost automorphic function from $\mathbb{R}$ to Y. The author proved that the only bounded mild solution of Eq.(1.3) on $\mathbb{R}$ is almost automorphic. In [20] and [21], the authors investigated the existence and uniqueness of an almost periodic

solution for Eq.(1.1) when $A = 0$ and F is dissipative with respect to the second argument x and they proposed as application, the following ordinary differential equations in a Banach space $\mathbb{E}$

$$x'(t) = -|x(t)|^{\alpha} + h(t), \text{ for } t \in \mathbb{R}, \tag{1.4}$$

where $\alpha \geq 0$ and $h : \mathbb{R} \to \mathbb{E}$ is a continuous function, they showed if the input function h is almost periodic then Eq.(1.4) has a unique bounded solution on $\mathbb{R}$ which is also almost periodic. Recently, in [22], the authors extended the works [20] and [21] to almost automorphic case, in fact they proved the existence and uniqueness of a bounded solution on $\mathbb{R}$ which is compact almost automorphic. For more contribution on almost periodic and almost automorphic solutions of Eq.(1.1), see [13, 14, 15, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34] and references therein.

To the best of our knowledge, the existence of Stepanov-like almost automorphic mild solutions for the semilinear evolution Eq.(1.1) under more general conditions is a subject that has not been treated in the literature. Our purpose in this paper is to establish some results concerning the existence and exponential stability of Stepanov-like almost automorphic mild solutions for equations that can be modelled in the form of Eq.(1.1) under more general conditions in the framework of semigroup theory. We point out that the methods used in this paper can also be applied to deal with the existence of periodic, anti-periodic, quasi-periodic and other types of almost periodic and almost automorphic solutions for Eq.(1.1).

One of the main topological approaches to periodic, anti-periodic, quasi-periodic, almost periodic and almost automorphic problems is to formulate them as fixed point problems for a proper solution operator in a suitable space of functions defined on the interval $[0, T]$ or on the whole line $\mathbb{R}$. Here we select the second one since if we use the former, it is difficult to check that the solution operator maps a bounded subset into itself. Indeed, we define a solution operator, say, Γ on the Banach space of all Stepanov-like almost automorphic functions from $\mathbb{R}$ to X. One of our main assumptions is that the operator A generates a exponentially stable semigroup on X. Under this assumption, for any Stepanov-like almost automorphic function x, Γx does not depend on the initial value. As we will see, this fact is very helpful for estimating the bound of Γx. We note that in some literature Schauder's fixed point theorem was employed to obtain periodic type mild solutions for semilinear evolution equations. However, their proofs for the compactness of the solution operators rely heavily on some restrictive conditions. For example, in [35] the function F is assumed to satisfied some compactness condition (see H_3 in [35]) which is not required in our result. See also [36] and others where the Banach contraction mapping principle is applied to periodic type problems.

We first consider the case that F satisfies Lipschitz type conditions with respect to the second variable x. By applying the Banach contraction mapping principle we obtain the existence and uniqueness of Stepanov-like almost automorphic mild solutions. When the function F does not satisfy Lipschitz type conditions in the second arguments x, we apply Schauder's fixed point theorem to obtain the existence of Stepanov-like almost automorphic mild solutions. In this case, we

assume that the semigroup $\{T(t)\}_{t\geq 0}$ generated by A is compact. Since the functions are defined on the whole line, more technical problems have to be overcome in our proof for the compactness of the solution operator Γ. In the second part of Section 4 we show that Γ maps a closed convex bounded subset, say, Y_r of the Stepanov-like almost automorphic functions space into itself and ΓY_r is equicontinuous. Subsequently, we construct a relatively compact set of functions to approximate a subset of ΓY_r. By Schauder's fixed point theorem we deduce the existence of Stepanov-like almost automorphic solutions.

The rest of this paper is organized as follows. In Section 2 we recall some concepts and prove some preliminary results. In Section 3 we prove the existence and uniqueness of Stepanov-like almost automorphic mild solutions for the linear evolution equations. The section that follows contains four existence theorems about Stepanov-like almost automorphic mild solutions for the nonlinear evolution equations. In section 5 we further discuss the exponential stability of Stepanov-like almost automorphic mild solutions for the nonlinear evolution equations. In the last section we give two examples to illustrate our main results.

2. Preliminaries

We begin this section by giving some notations. Let $(X, \|\cdot\|)$ $(Y, \|\cdot\|_Y)$ be two Banach spaces and $A : D(A) : X \to X$ be the infinitesimal generator of a C_0-semigroup of bounded linear operators $T(t)$ defined on X. By $\sigma(A)$ and $\rho(A)$ we denote the spectrum and resolvent set of Arespectively. By $L(X)$ we denote the Banach space of all bounded linear operators on X. $BC(\mathbb{R}, X)$ will stand for the Banach space of all bounded, continuous functions from $\mathbb{R}$ to X with the uniform norm

$$\|x\|_\infty = \sup\Big\{\|x(t)\| : t \in \mathbb{R}\Big\}.$$

Now, let us recall some basic definitions and results on almost automorphic functions.

Definition 2.1. (Bochner) [3]. A continuous function $f : \mathbb{R} \to X$ is said to be almost automorphic if for every sequence of real numbers $\{s'_n\}_{n=1}^\infty$, one can extract a subsequence $\{s_n\}_{n=1}^\infty$ such that

$$g(t) = \lim_{n\to\infty} f(t + s_n),$$

is well defined in $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n) = f(t),$$

for each $t \in \mathbb{R}$.

Denote by $AA(X)$ the set of all such functions.

Definition 2.2. [3]. A continuous function $f : \mathbb{R} \times Y \to X$ is said to be almost automorphic if $f(t, x)$ is almost automorphic in $t \in \mathbb{R}$ uniformly for all $x \in K$, where K is any bounded subset of Y.

Denote by $AA(\mathbb{R} \times Y, X)$ the set of all such functions.

Remark 2.3. The function g in Definition 2.1 is measurable, but not necessarily continuous. Moreover, if g is continuous, then f is uniformly continuous (cf., e.g., Theorem 2.6 of [37]). If the convergence in Definition 2.1 is uniform in $t \in \mathbb{R}$, then f is almost periodic. A classical example of almost automorphic function (not almost periodic) is (cf. [38, 39])

$$f(t) = \sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right), \quad t \in \mathbb{R}.$$

Next, let us recall some definitions and basic results on Stepanov-like almost automorphic functions (for more details, see [7]).

Definition 2.4. The Bochner transform $f^b(t,s)$, $t \in \mathbb{R}$, $s \in [0,1]$, of a function $f : \mathbb{R} \to X$ is defined by

$$f^b(t,s) := f(t+s).$$

Definition 2.5. Let $p \in [1,\infty)$. The space $BS^p(X)$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f : \mathbb{R} \to X$ such that

$$f^b \in L^\infty(\mathbb{R}, L^p([0,1],X)).$$

This is a Banach space with the norm

$$\|f\|_{S^p} := \|f^b\|_{L^\infty(\mathbb{R},L^p)} = \sup_{t\in\mathbb{R}}\left(\int_t^{t+1} \|f(\tau)\|^p \mathrm{d}\tau\right)^{\frac{1}{p}}.$$

Definition 2.6. The space $AS^p(X)$ of Stepanov-like almost automorphic functions consists of all $f \in BS^p(X)$ such that

$$f^b \in AA(L^p([0,1],X)).$$

That is, a function $f \in L^p_{loc}(\mathbb{R},X)$ is said to be Stepanov-like almost automorphic if its Bochner transform

$$f^b : \mathbb{R} \to L^p([0,1],X),$$

is almost automorphic in the sense that for every sequence of real numbers $\{s'_n\}_{n=1}^\infty$, there exist a subsequence $\{s_n\}_{n=1}^\infty$ and a function $g \in L^p_{loc}(\mathbb{R},X)$ such that

$$\left[\int_0^1 \|f(t+s_n+s)-g(t+s)\|^p \mathrm{d}s\right]^{\frac{1}{p}} \to 0,$$

and

$$\left[\int_0^1 \|g(t-s_n+s)-f(t+s)\|^p \mathrm{d}s\right]^{\frac{1}{p}} \to 0,$$

as $n \to \infty$ for all $t \in \mathbb{R}$.

Definition 2.7. A function

$$f : \mathbb{R}\times Y \to X, \quad (t,x) \to f(t,x)$$

with $f(\cdot,x) \in L^p_{loc}(\mathbb{R},X)$ for each $x \in Y$, is said to be Stepanov-like almost automorphic in $t \in \mathbb{R}$ uniformly for $x \in Y$, if $t \to f(t,x)$ is Stepanov-like almost

automorphic for each $x \in Y$. That is, for every sequence of real numbers $\{s'_n\}_{n=1}^{\infty}$, there exist a subsequence $\{s_n\}_{n=1}^{\infty}$ and a function $g(\cdot, x) \in L^p_{loc}(\mathbb{R}, X)$ such that

$$\left[\int_0^1 \|f(t+s_n+s,x) - g(t+s,x)\|^p \mathrm{d}s\right]^{\frac{1}{p}} \to 0,$$

and

$$\left[\int_0^1 \|g(t-s_n+s,x) - f(t+s,x)\|^p \mathrm{d}s\right]^{\frac{1}{p}} \to 0,$$

as $n \to \infty$ for all $t \in \mathbb{R}$ and $x \in Y$.

Denote by $AS^p(\mathbb{R} \times Y, X)$ the set of all such functions.

Remark 2.8. It is clear that, if $x : \mathbb{R} \to X$ is an almost automorphic function, then x is a Stepanov-like almost automorphic function, that is

$$AA(X) \subset AS^p(X).$$

Now we give a lemma for Stepanov-like almost automorphic functions.

Lemma 2.9. *Let $\{x_n(t)\}_{n\in\mathbb{N}}$ be a sequence of Stepanov-like almost automorphic functions such that*

$$\int_0^1 \|x_n(t+s) - x(t+s)\|^p ds \to 0, \tag{2.1}$$

as $n \to \infty$ for each $t \in \mathbb{R}$, then $x \in AS^p(X)$.

Proof. For any $i \in \mathbb{N}$ fixed, since $x_i(t) \in AS^p(X)$, then for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exist a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a function $y_i \in L^p_{loc}(\mathbb{R}, X)$ such that

$$\left[\int_0^1 \|x_i(t+s_n+s) - y_i(t+s)\|^p \mathrm{d}s\right]^{\frac{1}{p}} \to 0, \tag{2.2}$$

and

$$\left[\int_0^1 \|y_i(t-s_n+s) - x_i(t+s)\|^p \mathrm{d}s\right]^{\frac{1}{p}} \to 0,$$

as $n \to \infty$ for all $t \in \mathbb{R}$. On the other hand, from (2.1), one can easily deduce that $\{x_n(t)\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $\|\cdot\|_{S^p}$. Observe that, for each $t \in \mathbb{R}$, the sequence y_i is also a Cauchy sequence in $BS^p(\mathbb{R}, X)$. Indeed, if we write

$$y_i(t) - y_j(t) = y_i(t) - x_i(t+s_n) + x_i(t+s_n) - x_j(t+s_n) + x_j(t+s_n) - y_j(t),$$

then for a sufficiently large n, one gets

$$\begin{aligned}
&\sup_{t\in\mathbb{R}}\left[\int_t^{t+1}\|y_i(s)-y_j(s)\|^p\mathrm{d}s\right]^{\frac{1}{p}}\\
=&\sup_{t\in\mathbb{R}}\left[\int_0^{1}\|y_i(t+s)-y_j(t+s)\|^p\mathrm{d}s\right]^{\frac{1}{p}}\\
\leq&\sup_{t\in\mathbb{R}}\Big[\int_0^{1}\Big(\|y_i(t+s)-x_i(t+s+s_n)\|+\|x_i(t+s+s_n)-x_j(t+s+s_n)\|\\
&+\|x_j(t+s+s_n)-y_j(t+s)\|\Big)^p\mathrm{d}s\Big]^{\frac{1}{p}}\\
\leq&3\sup_{t\in\mathbb{R}}\Big[\int_0^{1}\Big(\|y_i(t+s)-x_i(t+s+s_n)\|^p+\|x_i(t+s+s_n)-x_j(t+s+s_n)\|^p\\
&+\|x_j(t+s+s_n)-y_j(t+s)\|^p\Big)\mathrm{d}s\Big]^{\frac{1}{p}}.
\end{aligned}$$

From (2.2) and the fact that $\{x_n(t)\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $\|\cdot\|_{S^p}$, it follows that the sequence of y_i is a Cauchy sequence in $BS^p(\mathbb{R},X)$. Using the completeness of $BS^p(\mathbb{R},X)$, we denote by $y(t)$ the pointwise limit of $y_i(t)$. Now let us prove that $x(t)\in AS^p(X)$. Note that the inequality below holds for any index i and any $t\in\mathbb{R}$,

$$\begin{aligned}
&\left[\int_0^{1}\|x(t+s+s_n)-y(t+s)\|^p\mathrm{d}s\right]^{\frac{1}{p}}\\
\leq&\Big[\int_0^{1}\Big(\|x(t+s+s_n)-x_i(t+s+s_n)\|+\|x_i(t+s+s_n)-y_i(t+s)\|\\
&+\|y_i(t+s)-y(t+s)\|\Big)^p\mathrm{d}s\Big]^{\frac{1}{p}}\\
\leq&3\Big[\int_0^{1}\Big(\|x(t+s+s_n)-x_i(t+s+s_n)\|^p+\|x_i(t+s+s_n)-y_i(t+s)\|^p\\
&+\|y_i(t+s)-y(t+s)\|^p\Big)\mathrm{d}s\Big]^{\frac{1}{p}}.
\end{aligned}$$

So, from (2.2) and and the fact that $\{x_n(t)\}_{n\in N}$ is a Cauchy sequence with respect to $\|\cdot\|_{S^p}$, for any sufficiently small $\varepsilon>0$ there exists a sufficiently large i, such that for each $t\in\mathbb{R}$,

$$\int_0^{1}\|x_i(t+s+s_n)-y_i(t+s)\|^p\mathrm{d}s<\frac{\varepsilon^p}{3^{p+1}},$$
$$\int_0^{1}\|x(t+s+s_n)-x_i(t+s+s_n)\|^p\mathrm{d}s<\frac{\varepsilon^p}{3^{p+1}}.$$

Now for this sufficiently large i, from the fact that $y(t)$ is the pointwise limit of $y_i(t)$, there exists a sufficient N such that for any $n>N$ one has

$$\int_0^{1}\|y_i(t+s)-y(t+s)\|^p\mathrm{d}s<\frac{\varepsilon^p}{3^{p+1}}.$$

Thus

$$\left[\int_0^1 \|x(t+s+s_n)-y(t+s)\|^p \mathrm{d}s\right]^{\frac{1}{p}} < \varepsilon, \quad \text{for } n > N,$$

which implies

$$\left[\int_0^1 \|x(t+s+s_n)-y(t+s)\|^p \mathrm{d}s\right]^{\frac{1}{p}} \to 0,$$

as $n \to \infty$ pointwise on $\mathbb{R}$. One can use the same steps to prove that

$$\left[\int_0^1 \|y(t+s-s_n)-x(t+s)\|^p \mathrm{d}s\right]^{\frac{1}{p}} \to 0,$$

as $n \to \infty$ pointwise on $\mathbb{R}$. That is $x(t) \in AS^p(X)$. The proof is finished. □

Now we give other two lemmas.

Lemma 2.10. *Let*

$$f : \mathbb{R} \times X \to X, \quad (t,x) \to f(t,x)$$

be Stepanov-like almost automorphic in $t \in \mathbb{R}$ uniformly in $x \in X$, and assume that f satisfies the following Lipschitz condition

$$\|f(t,x) - f(t,y)\| \le L\|x-y\|,$$

for all $x, y \in X$ and $t \in \mathbb{R}$, where $L > 0$ is a constant which is independent of t. Then for any Stepanov-like almost automorphic function $x : \mathbb{R} \to X$, the function $F : \mathbb{R} \to X$ given by

$$F(t) = f(t, x(t))$$

is Stepanov-like almost automorphic.

Proof. Let $\{s'_m\}_{m\in\mathbb{N}}$ be an arbitrary sequence of real numbers. Since $x \in AS^p(X)$, there exist a subsequence $\{s_m\}_{m\in\mathbb{N}}$ of $\{s'_m\}_{m\in\mathbb{N}}$ and a function $\tilde{x} \in L^p_{loc}(\mathbb{R}, X)$ such that

$$\left[\int_0^1 \|x(t+s_m+s)-\tilde{x}(t+s)\|^p \mathrm{d}s\right]^{\frac{1}{p}} \to 0, \tag{2.3}$$

and

$$\left[\int_0^1 \|\tilde{x}(t-s_m+s)-x(t+s)\|^p \mathrm{d}s\right]^{\frac{1}{p}} \to 0,$$

as $m \to \infty$ pointwise on $\mathbb{R}$. On the other hand, since $(t,x) \to f(t,x)$ is Stepanov-like almost automorphic in $t \in \mathbb{R}$ uniformly in $x \in X$, one can extract a subsequence $\{s_m\}_{m\in\mathbb{N}}$ of $\{s'_m\}_{m\in\mathbb{N}}$ (for convenience, we also denote it by $\{s_m\}_{m\in\mathbb{N}}$) and a function

$$\tilde{f}(\cdot, x) \in L^p_{loc}(\mathbb{R}, X)$$

such that

$$\left[\int_0^1 \|f(t+s_m+s,x)-\tilde{f}(t+s,x)\|^p \mathrm{d}s\right]^{\frac{1}{p}} \to 0, \tag{2.4}$$

and

$$\left[\int_0^1 \|\tilde{f}(t-s_m+s,x)-f(t+s,x)\|^p \mathrm{d}s\right]^{\frac{1}{p}} \to 0,$$

as $m \to \infty$ pointwise on $\mathbb{R}$ for each $x \in X$. Now, let us consider the function $\tilde{F}:\mathbb{R}\to X$ defined by

$$\tilde{F}(t) := \tilde{f}(t,\tilde{x}(t)), \quad t \in \mathbb{R}.$$

Note that

$$\begin{aligned} &F(t+s+s_m)-\tilde{F}(t+s)\\ =&f(t+s+s_m,x(t+s+s_m))-f(t+s+s_m,\tilde{x}(t+s))\\ &+f(t+s+s_m,\tilde{x}(t+s))-\tilde{f}(t+s,\tilde{x}(t+s)). \end{aligned}$$

So, one obtains

$$\begin{aligned} &\left[\int_0^1 \|F(t+s+s_m)-\tilde{F}(t+s)\|^p \mathrm{d}s\right]^{\frac{1}{p}}\\ \le&\Big[\int_0^1 \Big(\|f(t+s+s_m,x(t+s+s_m))-f(t+s+s_m,\tilde{x}(t+s))\|\\ &+\|f(t+s+s_m,\tilde{x}(t+s))-\tilde{f}(t+s,\tilde{x}(t+s))\|\Big)^p \mathrm{d}s\Big]^{\frac{1}{p}}\\ \le&2\Big[\int_0^1 \Big(\|f(t+s+s_m,x(t+s+s_m))-f(t+s+s_m,\tilde{x}(t+s))\|^p\\ &+\|f(t+s+s_m,\tilde{x}(t+s))-\tilde{f}(t+s,\tilde{x}(t+s))\|^p\Big)\mathrm{d}s\Big]^{\frac{1}{p}}\\ \le&2\Big[\int_0^1 \Big(L^p\|x(t+s+s_m)-\tilde{x}(t+s)\|^p\\ &+\|f(t+s+s_m,\tilde{x}(t+s))-\tilde{f}(t+s,\tilde{x}(t+s))\|^p\Big)\mathrm{d}s\Big]^{\frac{1}{p}}. \end{aligned}$$

One can deduce from (2.3) and (2.4) that

$$\left[\int_0^1 \|F(t+s+s_m)-\tilde{F}(t+s)\|^p \mathrm{d}s\right]^{\frac{1}{p}} \to 0,$$

as $m \to \infty$ pointwise on $\mathbb{R}$. Similarly one can prove that

$$\left[\int_0^1 \|\tilde{F}(t+s-s_m)-F(t+s)\|^p \mathrm{d}s\right]^{\frac{1}{p}} \to 0,$$

as $m \to \infty$ pointwise on $\mathbb{R}$. That is

$$F(t) \in AS^p(\mathbb{R},X).$$

The proof is finished. □

Lemma 2.11. *Assume that the semigroup $\{T(t)\}_{t\ge 0}$ is uniformly exponentially stable, i.e. there exist two constants $M>1$ and $\delta>0$ such that*

$$\|T(t)\| \le Me^{-\delta t}, \text{ for } t \ge 0. \tag{2.5}$$

Given a function $F(t) \in AS^p(X)$. Let

$$[\Phi F](t) := \int_{-\infty}^{t} T(t-s)F(s)\,ds.$$

Then $[\Phi F](t)$ is Stepanov-like almost automorphic.

Proof. By conditions (2.5), one has

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \left\| \int_{-\infty}^{\sigma} T(\sigma - s)F(s)\mathrm{d}s \right\|^p \mathrm{d}\sigma \right)^{\frac{1}{p}} \\
=&\sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \left\| \int_{0}^{\infty} T(\tau)F(\sigma - \tau)\mathrm{d}\tau \right\|^p \mathrm{d}\sigma \right)^{\frac{1}{p}} \\
\leq& M \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \int_{0}^{\infty} e^{-p\delta\tau} \left\| F(\sigma - \tau) \right\|^p \mathrm{d}\tau\mathrm{d}\sigma \right)^{\frac{1}{p}} \\
\leq& M\|F\|_{S^p} \left(\int_0^{\infty} e^{-p\delta\tau}\mathrm{d}\tau \right)^{\frac{1}{p}} \\
\leq& M\|F\|_{S^p} (p\delta)^{-\frac{1}{p}}.
\end{aligned}$$

Thus, Φ is well defined and ΦF is bounded. On the other hand, for any $t,\ h \in \mathbb{R}$,

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \left\| [\Phi F](\sigma + h) - [\Phi F](\sigma) \right\|^p \mathrm{d}\sigma \right)^{\frac{1}{p}} \\
=&\sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \left\| \int_{-\infty}^{\sigma+h} T(\sigma + h - s)F(s)\mathrm{d}s - \int_{-\infty}^{\sigma} T(\sigma - s)F(s)\mathrm{d}s \right\|^p \mathrm{d}\sigma \right)^{\frac{1}{p}} \\
=&\sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \left\| \int_{-\infty}^{\sigma} T(\sigma - s)[F(s + h) - F(s)]\mathrm{d}s \right\|^p \mathrm{d}\sigma \right)^{\frac{1}{p}} \\
=&\sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \left\| \int_{0}^{\infty} T(\tau)[F(\sigma - \tau + h) - F(\sigma - \tau)]\mathrm{d}\tau \right\|^p \mathrm{d}\sigma \right)^{\frac{1}{p}} \\
\leq& M\|F(\sigma + h) - F(\sigma)\|_{S^p} \left(\int_0^{\infty} e^{-p\delta\tau}\mathrm{d}\tau \right)^{\frac{1}{p}} \\
=& M(p\delta)^{-\frac{1}{p}}\|F(\sigma + h) - F(\sigma)\|_{S^p},
\end{aligned}$$

which yields that ΦF is continuous. Let $\{s'_m\}_{m\in\mathbb{N}}$ be a sequence of real numbers. Since $F \in AS^p(X)$, there exist a subsequence $\{s_m\}_{m\in\mathbb{N}}$ of $\{s'_m\}_{m\in\mathbb{N}}$ and a function $\tilde{F} \in L^p_{loc}(\mathbb{R}, X)$ such that

$$\left[\int_0^1 \|F(t + s_m + s) - \tilde{F}(t + s)\|^p \mathrm{d}s \right]^{\frac{1}{p}} \to 0, \tag{2.6}$$

and

$$\left[\int_0^1 \|\tilde{F}(t - s_m + s) - F(t + s)\|^p \mathrm{d}s \right]^{\frac{1}{p}} \to 0,$$

as $m \to \infty$ pointwise on $\mathbb{R}$ for each $x \in X$. Let

$$[\Phi\tilde{F}](t) := \int_{-\infty}^{t} T(t-s)\tilde{F}(s)\mathrm{d}s.$$

Thus

$$\begin{aligned}
&\left[\int_0^1 \|[\Phi F](t+s+s_m) - [\Phi\tilde{F}](t+s)\|^p \mathrm{d}s\right]^{\frac{1}{p}} \\
&= \left[\int_0^1 \left\|\int_0^\infty T(\sigma)F(t+s+s_m-\sigma)\mathrm{d}\sigma - \int_0^\infty T(\sigma)\tilde{F}(t+s-\sigma)\mathrm{d}\sigma\right\|^p \mathrm{d}s\right]^{\frac{1}{p}} \\
&= \left[\int_0^1 \left\|\int_0^\infty T(\sigma)\Big[F(t+s+s_m-\sigma) - \tilde{F}(t+s-\sigma)\Big]\mathrm{d}\sigma\right\|^p \mathrm{d}s\right]^{\frac{1}{p}} \\
&\le \left[\int_0^1 \left(\int_0^\infty \|T(\sigma)\|\|F(t+s+s_m-\sigma) - \tilde{F}(t+s-\sigma)\|\mathrm{d}\sigma\right)^p \mathrm{d}s\right]^{\frac{1}{p}} \\
&\le \left[\int_0^1 \int_0^\infty \|T(\sigma)\|^p\|F(t+s+s_m-\sigma) - \tilde{F}(t+s-\sigma)\|^p \mathrm{d}\sigma\mathrm{d}s\right]^{\frac{1}{p}} \\
&\le \left[\int_0^1 \int_0^\infty e^{-p\delta\sigma}\|F(t+s+s_m-\sigma) - \tilde{F}(t+s-\sigma)\|^p \mathrm{d}\sigma\mathrm{d}s\right]^{\frac{1}{p}} \\
&= \left[\int_0^\infty e^{-p\delta\sigma}\int_0^1 \|F(t+s+s_m-\sigma) - \tilde{F}(t+s-\sigma)\|^p \mathrm{d}s\mathrm{d}\sigma\right]^{\frac{1}{p}}.
\end{aligned}$$

From (2.6), obviously, the last inequality goes to 0 as $m \to \infty$ pointwise on $\mathbb{R}$. Similarly one can prove that

$$\left[\int_0^1 \|[\Phi\tilde{F}](t+s-s_m) - [\Phi F](t+s)\|^p \mathrm{d}s\right]^{\frac{1}{p}} \to 0,$$

as $m \to \infty$ pointwise on $\mathbb{R}$. Thus we conclude that

$$[\Phi F] \in AS^p(X).$$

The proof is now complete. □

3. Stepanov-like almost automorphic mild solutions for linear evolution equations

To study the existence of Stepanov-like almost automorphic mild solutions for Eq.(1.1), we first consider the existence of Stepanov-like almost automorphic mild solutions for the linear evolution equations

$$x'(t) = Ax(t) + F(t),\ t \in \mathbb{R}, \tag{3.1}$$

where $A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0-semigroup of bounded linear operator on a Banach space X and $F : \mathbb{R} \to X$ is a Stepanov-like almost automorphic function.

Definition 3.1. A mild solution of Eq.(3.1) is a continuous function $x : \mathbb{R} \to X$ satisfying the corresponding integral equation

$$x(t) = T(t-s)x(s) + \int_s^t T(t-\sigma)F(\sigma)\mathrm{d}\sigma, \tag{3.2}$$

for all $t \geq \sigma$ and each $\sigma \in \mathbb{R}$.

The following are the main result for the linear evolution Eq.(3.1).

Theorem 3.2. *Assume that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ which satisfies (2.5). Then Eq.(3.1) admits a unique Stepanov-like almost automorphic mild solution.*

Proof. Let us first prove the uniqueness. Assume that $x : \mathbb{R} \to X$ is a Stepanov-like almost automorphic function and satisfies the homogeneous equation

$$x'(t) = Ax(t), \quad t \in \mathbb{R}. \tag{3.3}$$

Then

$$x(t) = T(t-s)x(s), \text{ for any } t \geq s.$$

Thus

$$\|x(t)\| \leq e^{-\delta(t-s)}K$$

with $\|x(s)\| \leq K$ for $s \in \mathbb{R}$. So

$$x(t) \to 0, \text{ as } t \to +\infty.$$

Since $x(t) \in AS^p(X)$, for any sequence of real numbers $\{s'_n\}$, there exists a subsequence $\{s_n\}$ of $\{s'_n\}$ such that for some function $\tilde{x}(t)$,

$$\left[\int_0^1 \|x(t+s_n+s) - \tilde{x}(t+s)\|^p \mathrm{d}s\right]^{\frac{1}{p}} \to 0, \tag{3.4}$$

and

$$\left[\int_0^1 \|\tilde{x}(t-s_n+s) - x(t+s)\|^p \mathrm{d}s\right]^{\frac{1}{p}} \to 0, \tag{3.5}$$

as $n \to \infty$ pointwise on $\mathbb{R}$. In particular, if $\lim\limits_{n\to\infty} s'_n = \infty$, then $\tilde{x}(t) \equiv 0$ by (3.4). Hence $x(t) \equiv 0$ by (3.5). Now, if $x_1,\ x_2 : \mathbb{R} \to X$ are Stepanov-like almost automorphic solutions of Eq.(3.1), then $x = x_1 - x_2$ is a Stepanov-like almost automorphic solution of Eq.(3.3). In view of the above, $x = x_1 - x_2 = 0$, that is, $x_1 = x_2$. Now let us investigate the existence. Consider for each $n = 1, 2, \cdots$, the integrals

$$x_n(t) = \int_{n-1}^n T(\sigma)F(t-\sigma)\mathrm{d}\sigma,$$

for each $t \in \mathbb{R}$. Firstly, by using Hölder inequality, one gets

$$\begin{aligned}
\sup_{t\in\mathbb{R}}\left(\int_t^{t+1}\|x_n(s)\|^p\mathrm{d}s\right)^{\frac{1}{p}} =& \sup_{t\in\mathbb{R}}\left(\int_t^{t+1}\left\|\int_{n-1}^{n}T(\sigma)F(s-\sigma)\mathrm{d}\sigma\right\|^p\mathrm{d}s\right)^{\frac{1}{p}}\\
\leq& \sup_{t\in\mathbb{R}}\left(\int_t^{t+1}\left(\int_{n-1}^{n}\|T(\sigma)\|\|F(s-\sigma)\|\mathrm{d}\sigma\right)^p\mathrm{d}s\right)^{\frac{1}{p}}\\
\leq& \sup_{t\in\mathbb{R}}\left(\int_t^{t+1}\int_{n-1}^{n}\|T(\sigma)\|^p\|F(s-\sigma)\|^p\mathrm{d}\sigma\mathrm{d}s\right)^{\frac{1}{p}}\\
\leq& \sup_{t\in\mathbb{R}}\left(\int_t^{t+1}\int_{n-1}^{n}e^{-p\delta\sigma}\|F(s-\sigma)\|^p\mathrm{d}\sigma\mathrm{d}s\right)^{\frac{1}{p}}\\
=& \left(\int_{n-1}^{n}e^{-p\delta\sigma}\sup_{t\in\mathbb{R}}\int_t^{t+1}\|F(s-\sigma)\|^p\mathrm{d}s\mathrm{d}\sigma\right)^{\frac{1}{p}}\\
=& \left(\int_{n-1}^{n}e^{-p\delta\sigma}\|F\|_{S^p}^p\mathrm{d}\sigma\right)^{\frac{1}{p}}\\
=& \|F\|_{S^p}\left(\int_{n-1}^{n}e^{-p\delta\sigma}\mathrm{d}\sigma\right)^{\frac{1}{p}}\\
\leq& \|F\|_{S^p}e^{-p\delta(n-1)}.
\end{aligned}$$

From

$$\|F\|_{S^p}\sum_{n=1}^{\infty}e^{-p\delta(n-1)} < \infty,$$

one can deduce from the well-known Weierstrass test that the series $\sum\limits_{n=1}^{\infty}x_n(t)$ is convergent in the sense of the norm $\|\cdot\|_{S^p}$ uniformly on $\mathbb{R}$. Now let

$$\Phi(t) := \sum_{n=1}^{\infty}x_n(t), \quad \text{for each} \quad t \in \mathbb{R}.$$

Observe that

$$\Phi(t) = \int_{-\infty}^{t}T(t-\sigma)F(\sigma)\mathrm{d}\sigma, \quad \text{for each} \quad t \in \mathbb{R}.$$

Clearly, $\Phi(t) \in C(\mathbb{R}, X)$. Now let us show that each $x_n \in AS^p(X)$. Indeed, let $\{s'_m\}_{m\in\mathbb{N}}$ be a sequence of real numbers. Since $F \in AS^p(X)$, there exist a subsequence $\{s_m\}_{m\in\mathbb{N}}$ of $\{s'_m\}_{m\in\mathbb{N}}$ and a function $\tilde{F} \in L^p_{loc}(\mathbb{R}, X)$ such that

$$\left[\int_0^1\|F(t+s_m+s)-\tilde{F}(t+s)\|^p\mathrm{d}s\right]^{\frac{1}{p}} \to 0,$$

and

$$\left[\int_0^1\|\tilde{F}(t-s_m+s)-F(t+s)\|^p\mathrm{d}s\right]^{\frac{1}{p}} \to 0,$$

as $m \to \infty$ pointwise on $\mathbb{R}$. Moreover, if we let

$$\tilde{x}_n(t) = \int_{n-1}^{n} T(\sigma)\tilde{F}(t-\sigma)\mathrm{d}\sigma,$$

one has

$$\begin{aligned}
&\left[\int_0^1 \|x_n(t+s+s_m) - \tilde{x}_n(t+s)\|^p \mathrm{d}s\right]^{\frac{1}{p}} \\
&= \left[\int_0^1 \left\|\int_{n-1}^{n} T(\sigma)F(t+s+s_m-\sigma)\mathrm{d}\sigma - \int_{n-1}^{n} T(\sigma)\tilde{F}(t+s-\sigma)\mathrm{d}\sigma\right\|^p \mathrm{d}s\right]^{\frac{1}{p}} \\
&= \left[\int_0^1 \left\|\int_{n-1}^{n} T(\sigma)\Big[F(t+s+s_m-\sigma) - \tilde{F}(t+s-\sigma)\Big]\mathrm{d}\sigma\right\|^p \mathrm{d}s\right]^{\frac{1}{p}} \\
&\le \left[\int_0^1 \left(\int_{n-1}^{n} \|T(\sigma)\|\|F(t+s+s_m-\sigma) - \tilde{F}(t+s-\sigma)\|\mathrm{d}\sigma\right)^p \mathrm{d}s\right]^{\frac{1}{p}} \\
&\le \left[\int_0^1 \int_{n-1}^{n} \|T(\sigma)\|^p \|F(t+s+s_m-\sigma) - \tilde{F}(t+s-\sigma)\|^p \mathrm{d}\sigma\mathrm{d}s\right]^{\frac{1}{p}} \\
&\le \left[\int_0^1 \int_{n-1}^{n} e^{-p\delta\sigma}\|F(t+s-\sigma)\|^p \mathrm{d}\sigma\mathrm{d}s\right]^{\frac{1}{p}} \\
&= \left[\int_{n-1}^{n} e^{-p\delta\sigma}\int_0^1 \|F(t+s+s_m-\sigma) - \tilde{F}(t+s-\sigma)\|^p \mathrm{d}s\mathrm{d}\sigma\right]^{\frac{1}{p}}.
\end{aligned}$$

Obviously, the last inequality goes to 0 as $m \to \infty$ pointwise on $\mathbb{R}$. Similarly one can prove that

$$\left[\int_0^1 \|\tilde{x}_n(t+s-s_m) - x_n(t+s)\|^p \mathrm{d}s\right]^{\frac{1}{p}} \to 0,$$

as $m \to \infty$ pointwise on $\mathbb{R}$. Thus we conclude that each $x_n \in AS^p(X)$ and consequently their uniform limit $\Phi(t) \in AS^p(X)$, by using Lemma 2.9. In view of the above, it follows that x is the only bounded Stepanov-like almost automorphic mild solution of Eq.(3.1). The proof is now complete. □

Remark 3.3. The mild solution of Eq.(3.1) with the initial value

$$x(s) = \int_{-\infty}^{s} T(s-\sigma)F(\sigma)\mathrm{d}\sigma,$$

is also a Stepanov-like almost automorphic mild solution of Eq.(3.1). In particular, with the initial value specified above, (3.2) can be changed into

$$x(t) = \int_{-\infty}^{t} T(t-\sigma)F(\sigma)\mathrm{d}\sigma.$$

Proof. It is well known that for given $s \in \mathbb{R}$ and given initial value x_s at 'time' s, the function

$$x(t) = T(t-s)x_s + \int_s^t T(t-\sigma)F(\sigma)\mathrm{d}\sigma, \quad t \geq s, \tag{3.6}$$

is the unique mild solution of Eq.(3.1) with the initial value condition $x(s) = x_s$. So to prove the existence of Stepanov-like almost automorphic mild solution, we need to find an initial value x_s such that the function given by (3.6) is Stepanov-like almost automorphic. Let $\{s'_m\}_{m\in\mathbb{N}}$ be a sequence of real numbers. Since $F \in AS^p(X)$, there exist a subsequence $\{s_m\}_{m\in\mathbb{N}}$ of $\{s'_m\}_{m\in\mathbb{N}}$ and a function $\tilde{F} \in L^p_{loc}(\mathbb{R}, X)$ such that

$$\left[\int_0^1 \|F(t+s_m+s) - \tilde{F}(t+s)\|^p \mathrm{d}s\right]^{\frac{1}{p}} \to 0,$$

and

$$\left[\int_0^1 \|\tilde{F}(t-s_m+s) - F(t+s)\|^p \mathrm{d}s\right]^{\frac{1}{p}} \to 0,$$

as $m \to \infty$ pointwise on $\mathbb{R}$. Now we consider

$$x(t) := \int_{-\infty}^t T(t-s)F(\sigma)\mathrm{d}\sigma,$$

defined as

$$\lim_{r\to-\infty} \int_r^t T(t-\sigma)F(\sigma)\mathrm{d}\sigma.$$

From Theorem 3.1 of [40], we know that

$$\int_r^t T(t-\sigma)F(\sigma)\mathrm{d}\sigma,$$

exists for each $r < t$. Moreover, if we let

$$\tilde{x}(t) = \int_{-\infty}^t T(t-\sigma)\tilde{F}(\sigma)\mathrm{d}\sigma,$$

one has

$$\begin{aligned}
&\left[\int_0^1 \|x(t+s+s_m)-\tilde{x}(t+s)\|^p \mathrm{d}s\right]^{\frac{1}{p}} \\
=&\left[\int_0^1 \left\|\int_{-\infty}^{t+s+s_m} T(t+s+s_m-\sigma)F(\sigma)\mathrm{d}\sigma - \int_{-\infty}^{t+s} T(t+s-\sigma)\tilde{F}(\sigma)\mathrm{d}\sigma\right\|^p \mathrm{d}s\right]^{\frac{1}{p}} \\
=&\left[\int_0^1 \left\|\int_{-\infty}^{t} T(t-\sigma)F(s+s_m-\sigma)\mathrm{d}\sigma - \int_{-\infty}^{t} T(t-\sigma)\tilde{F}(s-\sigma)\mathrm{d}\sigma\right\|^p \mathrm{d}s\right]^{\frac{1}{p}} \\
=&\left[\int_0^1 \left\|\int_{-\infty}^{t} T(t-\sigma)\Big[F(s+s_m-\sigma)-\tilde{F}(s-\sigma)\Big]\mathrm{d}\sigma\right\|^p \mathrm{d}s\right]^{\frac{1}{p}} \\
\leq&\left[\int_0^1 \left(\int_{-\infty}^{t} \|T(t-\sigma)\|\|F(s+s_m-\sigma)-\tilde{F}(s-\sigma)\|\mathrm{d}\sigma\right)^p \mathrm{d}s\right]^{\frac{1}{p}} \\
\leq&\left[\int_0^1 \int_{-\infty}^{t} \|T(t-\sigma)\|^p\|F(s+s_m-\sigma)-\tilde{F}(s-\sigma)\|^p \mathrm{d}\sigma\mathrm{d}s\right]^{\frac{1}{p}} \\
\leq&\left[\int_0^1 \int_{-\infty}^{t} e^{-p\delta\sigma}\|F(s+s_m-\sigma)-\tilde{F}(s-\sigma)\|^p \mathrm{d}\sigma\mathrm{d}s\right]^{\frac{1}{p}} \\
=&\left[\int_{-\infty}^{t} e^{-p\delta\sigma}\int_0^1 \|F(s+s_m-\sigma)-\tilde{F}(s-\sigma)\|^p \mathrm{d}s\mathrm{d}\sigma\right]^{\frac{1}{p}}.
\end{aligned}$$

Obviously, the last inequality goes to 0 as $m\to\infty$ pointwise on $\mathbb{R}$. Similarly one can prove that

$$\left[\int_0^1 \|\tilde{x}(t+s-s_m)-x(t+s)\|^p \mathrm{d}s\right]^{\frac{1}{p}} \to 0,$$

as $m\to\infty$ pointwise on $\mathbb{R}$. This indicates that x is Stepanov-like almost automorphic. Note that

$$x(s)=\int_{-\infty}^{s} T(s-\sigma)F(\sigma)\mathrm{d}\sigma.$$

If $t\geq s$, then

$$\begin{aligned}
\int_s^t T(t-\sigma)F(\sigma)\mathrm{d}\sigma =& \int_{-\infty}^{t} T(t-\sigma)F(\sigma)\mathrm{d}\sigma - \int_{-\infty}^{s} T(t-\sigma)F(\sigma)\mathrm{d}\sigma \\
=&x(t)-T(t-s)x(s),
\end{aligned}$$

i.e.,

$$x(t)=T(t-s)x(s)+\int_s^t T(t-\sigma)F(\sigma)\mathrm{d}\sigma.$$

If we choose initial value $x_s=x(s)$, then the function $x(t)$ given by (3.6) is Stepanov-like almost automorphic. In fact, denote

$$\tilde{x}(t)=T(t-s)x(s)+\int_s^t T(t-\sigma)\tilde{F}(\sigma)\mathrm{d}\sigma.$$

Then, for each $t \in \mathbb{R}$, one has

$$
\begin{aligned}
&\Big[\int_0^1 \|x(t+s+s_m)-\tilde{x}(t+s)\|^p \mathrm{d}s\Big]^{\frac{1}{p}} \\
=&\Big[\int_0^1 \Big\|T(t-s+s_m)x(s)-T(t-s)x(s)+\int_{-\infty}^t T(t-\sigma)F(s+s_m-\sigma)\mathrm{d}\sigma \\
&-\int_{-\infty}^t T(t-\sigma)\tilde{F}(s-\sigma)\mathrm{d}\sigma\Big\|^p \mathrm{d}s\Big]^{\frac{1}{p}} \\
=&\Big[\int_0^1 \Big[\Big\|T(t-s+s_m)x(s)-T(t-s)x(s)\Big\| \\
&+\Big\|\int_{-\infty}^t T(t-\sigma)\Big[F(s+s_m-\sigma)-\tilde{F}(s-\sigma)\Big]\mathrm{d}\sigma\Big\|\Big]^p \mathrm{d}s\Big]^{\frac{1}{p}} \\
\leq&2\Big[\int_0^1 \Big[\Big\|T(t-s+s_m)x(s)-T(t-s)x(s)\Big\|^p \\
&+\Big\|\int_{-\infty}^t T(t-\sigma)\Big[F(s+s_m-\sigma)-\tilde{F}(s-\sigma)\Big]\mathrm{d}\sigma\Big\|^p\Big]\mathrm{d}s\Big]^{\frac{1}{p}} \\
\leq&2\Big[\int_0^1 \Big[\Big\|T(t-s+s_m)x(s)-T(t-s)x(s)\Big\|^p \\
&+\Big(\int_{-\infty}^t \|T(t-\sigma)\|\|F(s+s_m-\sigma)-\tilde{F}(s-\sigma)\|\mathrm{d}\sigma\Big)^p \mathrm{d}s\Big)\Big]^{\frac{1}{p}} \\
\leq&2\Big[\int_0^1 \Big[\Big\|T(t-s+s_m)x(s)-T(t-s)x(s)\Big\|^p \\
&+\int_{-\infty}^t \|T(t-\sigma)\|^p\|F(s+s_m-\sigma)-\tilde{F}(s-\sigma)\|^p \mathrm{d}\sigma\mathrm{d}s\Big]^{\frac{1}{p}} \\
\leq&2\Big[\int_0^1 \Big[\Big\|T(t-s+s_m)x(s)-T(t-s)x(s)\Big\|^p \\
&+\int_{-\infty}^t e^{-p\delta\sigma}\|F(s+s_m-\sigma)-\tilde{F}(s-\sigma)\|^p \mathrm{d}\sigma\mathrm{d}s\Big]^{\frac{1}{p}}.
\end{aligned}
$$

Obviously, the last inequality goes to 0 as $m \to \infty$ pointwise on $\mathbb{R}$. Similarly one can prove that

$$\Big[\int_0^1 \|\tilde{x}(t+s-s_m)-x(t+s)\|^p \mathrm{d}s\Big]^{\frac{1}{p}} \to 0,$$

as $m \to \infty$ pointwise on $\mathbb{R}$. Up to now, the existence is proved. We finally prove the uniqueness of the Stepanov-like almost automorphic mild solution of Eq.(3.1). Assume that $x(t)$ and $y(t)$ are both Stepanov-like almost automorphic mild solutions of Eq.(3.1) with different initial value $x(s)$ and $y(s)$ at 'time' s. That is, for $t \geq s$,

$$x(t) = T(t-s)x(s) + \int_s^t T(t-\sigma)F(\sigma)\mathrm{d}\sigma,$$

$$y(t) = T(t-s)y(s) + \int_s^t T(t-\sigma)F(\sigma)\mathrm{d}\sigma,$$

and $x(s) \neq y(s)$. Let

$$z(t) = x(t) - y(t).$$

Then $z(t)$ satisfies the equation

$$z'(t) = Az(t)dt, \quad t \geq s,$$

with initial condition

$$z(s) = x(s) - y(s).$$

Hence

$$z(t) = T(t-s)z(s)$$

and

$$\|z(t)\| \leq Me^{-(t-s)}\|z(s)\|, \quad \text{for all } t \geq s.$$

So

$$z(t) \to 0, \text{ as } t \to +\infty.$$

Since $z(t) \in AS^p(X)$, for any sequence of real numbers $\{s'_n\}$, there exists a subsequence $\{s_n\}$ of $\{s'_n\}$ such that for some function $\tilde{z}(t)$,

$$\left[\int_0^1 \|z(t+s_n+s) - \tilde{z}(t+s)\|^p \mathrm{d}s\right]^{\frac{1}{p}} \to 0, \tag{3.7}$$

and

$$\left[\int_0^1 \|\tilde{z}(t-s_n+s) - z(t+s)\|^p \mathrm{d}s\right]^{\frac{1}{p}} \to 0, \tag{3.8}$$

for each $t \in \mathbb{R}$. In particular, if $\lim\limits_{n\to\infty} s'_n = \infty$, then $\tilde{z}(t) \equiv 0$ by (3.7). Hence $z(t) \equiv 0$ by (3.8), so we must have $x(s) = y(s)$, a contradiction. The proof is complete. □

4. Stepanov-like almost automorphic mild solutions for nonlinear evolution equations

Now we investigate the Stepanov-like almost automorphic mild solutions for the nonlinear evolution Eq.(1.1).

Definition 4.1. A mild solution of Eq.(1.1) is a continuous function $x : \mathbb{R} \to X$ satisfying the corresponding integral equations

$$x(t) = T(t-s)x(s) + \int_s^t T(t-\sigma)F(\sigma, x(\sigma))\mathrm{d}\sigma,$$

for all $t \geq \sigma$ and each $\sigma \in \mathbb{R}$.

We first consider the case the function F satisfies Lipschitz type conditions with respect to the second variable x, the following are the main results.

Theorem 4.2. *Assume that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ which satisfies (2.5). Let $F:\mathbb{R}\times X\to X$ satisfy*

$$F(t,x)\in AS^p(\mathbb{R}\times X,X)$$

and the following Lipschitz type condition

$$\|F(t,x)-F(t,y)\|\leq L(t)\|x-y\|,\quad \text{for all}\ \ x,\ y\in X,\quad t\in\mathbb{R},\tag{4.1}$$

where $L(t)\in L^p(\mathbb{R})$ is bounded. Then Eq.(1.1) admits a unique Stepanov-like almost automorphic mild solution.

Proof. Define operator Γ on $AS^p(X)$ by

$$\Gamma x(t)=\int_{-\infty}^{t}T(t-\sigma)F(\sigma,x(\sigma))\mathrm{d}\sigma.$$

Since $L(t)$ is bounded, from Lemma 2.9, it follows that

$$F(\cdot)=f(\cdot,x(\cdot))\in AS^p(X).$$

From the proof of Lemma 2.11, one can easily see that Γx is well-defined and continuous. Then by using the proof of Theorem 3.2 with the above Lemma 2.11, one has that $\Gamma x\in AS^p(X)$ whenever $x\in AS^p(X)$. Thus Γ maps $AS^p(X)$ into itself. We now prove that $x\in AS^p(X)$ is a mild solution of Eq.(1.1) if and only if x is a fixed point of Γ. It is easy to show that if $x=\Gamma x$, then x is a mild solution of Eq.(1.1). Let $x\in AS^p(X)$ be a mild solution of Eq.(1.1). Then for all $t>s$,

$$x(t)=T(t-s)x(s)+\int_{s}^{t}T(t-\sigma)F(\sigma,x(\sigma))\mathrm{d}\sigma.\tag{4.2}$$

Noticing the estimates (2.5), letting $s\to-\infty$ in (4.2) one obtains that

$$x(t)=\int_{-\infty}^{t}T(t-\sigma)F(\sigma,x(\sigma))\mathrm{d}\sigma.$$

Hence,

$$x(t)=[\Gamma x](t)\ (\forall t\in\mathbb{R}),$$

i.e., x is a fixed point of Γ. It suffices now to show that the operator Γ has a unique fixed point in $AS^p(X)$. For this, let x, y be in $AS^p(X)$ and define

$$C:=\sup_{t\in\mathbb{R}}\|T(t)\|,$$

one has

$$\begin{aligned}\|\Gamma x(t)-\Gamma y(t)\|_{S^p}=&\sup_{\tau\in\mathbb{R}}\left(\int_{\tau}^{\tau+1}\Big\|\int_{-\infty}^{t}T(t-\sigma)\Big[F(\sigma,x(\sigma))-F(\sigma,y(\sigma))\Big]\mathrm{d}\sigma\Big\|^p\mathrm{d}t\right)^{\frac{1}{p}}\\ \leq&\sup_{\tau\in\mathbb{R}}\left(\int_{\tau}^{\tau+1}\int_{-\infty}^{t}L^p(\sigma)\|T(t-\sigma)\|^p\|x(\sigma)-y(\sigma)\|^p\mathrm{d}\sigma\mathrm{d}t\right)^{\frac{1}{p}}\\ \leq&C\|L\|_p\|x-y\|_{S^p}.\end{aligned}$$

In general we get

$$\begin{aligned}\|[\Gamma^n x](t)-[\Gamma^n y](t)\|_{S^p} \leq & \frac{C^n}{(n-1)!}\left(\int_{-\infty}^{t} L^p(\sigma)\left(\int_{-\infty}^{\sigma} L^p(\tau)\mathrm{d}\tau\right)^{n-1}\mathrm{d}\sigma\right)^{\frac{1}{p}}\|x-y\|_{S^p}\\ \leq & \frac{C^n}{n!}\left(\left(\int_{-\infty}^{t} L^p(\sigma)\mathrm{d}\sigma\right)^{\frac{1}{p}}\right)^{n}\|x-y\|_{S^p}\\ \leq & \frac{(C\|L\|_p)^n}{n!}\|x-y\|_{S^p}.\end{aligned}$$

Hence, since

$$\frac{(C\|L\|_p)^n}{n!}<1$$

for n sufficiently large, by the contraction principle Γ has a unique fixed point $x\in AS^p(X)$. □

Note that conditions of type (4.1) have been previously considered in the literature for almost automorphic functions [41]. Our motivation comes from their use in the study of pseudo-almost periodic solutions of semilinear Cauchy problems [42]. Now we consider a more general case of equations introducing a new class of functions L which do not necessarily belong to $L^p(\mathbb{R})$. We have the following result.

Theorem 4.3. *Assume that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ which satisfies (2.5). Let $F:\mathbb{R}\times X\to X$ satisfy*

$$F(t,x)\in AS^p(\mathbb{R}\times X,X)$$

and the Lipschitz type condition (4.1) where the integral $\int_{-\infty}^{t}L(\sigma)d\sigma$ exists for all $t\in\mathbb{R}$ and $L(t)$ is bounded. Then Eq.(1.1) admits a unique Stepanov-like almost automorphic mild solution.

Proof. Define a new norm

$$|||x|||:=\sup_{t\in\mathbb{R}}\left\{v(t)\|x(t)\|_{S^p}\right\},$$

where

$$v(t):=\left[e^{-k\int_{-\infty}^{t}L(\sigma)\mathrm{d}\sigma}\right]^{\frac{1}{p}}$$

and k is a fixed positive constant greater than

$$C:=\sup_{t\in\mathbb{R}}\|T(t)\|.$$

Let x, y be in $AS^p(X)$, then one has

$$\begin{aligned}
&v(t)\|\Gamma x(t)-\Gamma y(t)\|_{S^p}\\
=&v(t)\sup_{\tau\in\mathbb{R}}\left(\int_\tau^{\tau+1}\Big\|\int_{-\infty}^t T(t-\sigma)\Big[F(\sigma,x(\sigma))-F(\sigma,y(\sigma))\Big]\mathrm{d}\sigma\Big\|^p\mathrm{d}t\right)^{\frac{1}{p}}\\
\leq&C\sup_{\tau\in\mathbb{R}}\left(\int_\tau^{\tau+1}\int_{-\infty}^t v^p(t)L^p(\sigma)\|x(\sigma)-y(\sigma)\|^p\mathrm{d}\sigma\mathrm{d}t\right)^{\frac{1}{p}}\\
=&C\sup_{\tau\in\mathbb{R}}\left(\int_\tau^{\tau+1}\int_{-\infty}^t v^p(t)v^p(\sigma)L^p(\sigma)(v^p(\sigma))^{-1}\|x(\sigma)-y(\sigma)\|^p\mathrm{d}\sigma\mathrm{d}t\right)^{\frac{1}{p}}\\
\leq&C|||x-y|||\sup_{\tau\in\mathbb{R}}\left(\int_\tau^{\tau+1}\int_{-\infty}^t v^p(t)(v^p(\sigma))^{-1}L^p(\sigma)\mathrm{d}\sigma\mathrm{d}t\right)^{\frac{1}{p}}\\
=&\frac{C}{k}|||x-y|||\sup_{\tau\in\mathbb{R}}\left(\int_\tau^{\tau+1}\int_{-\infty}^t ke^{k\int_t^\sigma L(\tau)\mathrm{d}\tau}L(\sigma)\mathrm{d}\sigma\mathrm{d}t\right)^{\frac{1}{p}}\\
=&\frac{C}{k}|||x-y|||\sup_{\tau\in\mathbb{R}}\left(\int_\tau^{\tau+1}\int_{-\infty}^t \frac{\mathrm{d}}{\mathrm{d}\sigma}\Big(e^{k\int_t^\sigma L(\tau)\mathrm{d}\tau}\Big)\mathrm{d}\sigma\mathrm{d}t\right)^{\frac{1}{p}}\\
=&\frac{C}{k}\sup_{\tau\in\mathbb{R}}\left(\int_\tau^{\tau+1}\Big(1-e^{-k\int_{-\infty}^t L(\tau)\mathrm{d}\tau}\Big)\mathrm{d}t\right)^{\frac{1}{p}}|||x-y|||\\
\leq&\frac{C}{k}|||x-y|||.
\end{aligned}$$

Hence, since $\frac{C}{k}<1$, Γ has a unique fixed point $x\in AS^p(X)$. □

Note that the above result does not include the cases where L is a constant.

Theorem 4.4. *Assume that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ which satisfies (2.5). Let $F:\mathbb{R}\times X\to X$ satisfy*

$$F(t,x)\in AS^p(\mathbb{R}\times X,X)$$

and the following Lipschitz condition

$$\|F(t,x)-F(t,y)\|\leq L\|x-y\|,\quad \textit{for all}\ \ x,\ y\in X,\quad t\in\mathbb{R}.$$

Then Eq.(1.1) admits a unique Stepanov-like almost automorphic mild solution whenever

$$CML(p\delta)^{-\frac{1}{p}}<1.$$

Proof. For $x,\ y \in AS^p(X)$, one has

$$\begin{aligned}
&\|\Gamma x(t) - \Gamma y(t)\|_{S^p}\\
=&\sup_{s\in\mathbb{R}}\left(\int_s^{s+1}\Big\|\int_{-\infty}^{t} T(t-\sigma)\Big[F(\sigma,x(\sigma)) - F(\sigma,y(\sigma))\Big]\mathrm{d}\sigma\Big\|^p\mathrm{d}t\right)^{\frac{1}{p}}\\
=&\sup_{s\in\mathbb{R}}\left(\int_s^{s+1}\Big\|\int_{0}^{\infty} T(\tau)\Big[F(t-\tau,x(t-\tau)) - F(t-\tau,y(t-\tau))\Big]\mathrm{d}\tau\Big\|^p\mathrm{d}t\right)^{\frac{1}{p}}\\
\le& L\sup_{s\in\mathbb{R}}\left(\int_s^{s+1}\int_0^{\infty}\|T(\tau)\|^p\|x(t-\tau)) - y(t-\tau)\|^p\mathrm{d}\tau\mathrm{d}t\right)^{\frac{1}{p}}\\
\le& CML\|x-y\|_{S^p}\left(\int_0^{\infty} e^{-p\delta\tau}\mathrm{d}\tau\right)^{\frac{1}{p}}\\
=& CML(p\delta)^{-\frac{1}{p}}\|x-y\|_{S^p}.
\end{aligned}$$

This proves that Γ is a strict contraction, so it follows from the Banach contraction mapping principle that Γ admits a unique fixed point $x \in AS^p(X)$, which is the unique Stepanov-like almost automorphic mild solution of Eq.(1.1). □

Next, we consider the case that the function F does not satisfy Lipschitz type conditions with respect to the second variable x. For any positive number r, set

$$\bar{B}_r := \Big\{x \in X : \|x\| \le r\Big\},\quad Y_r := \Big\{\varphi \in AS^p(\mathbb{R},X) : \|\varphi\|_{S^p} \le r\Big\}.$$

We make the following assumptions.

(H_1) A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\ge 0}$ satisfying (2.5).

(H_2) The function $F : \mathbb{R}\times X \to X$ is a Caratheodory map, i.e., for every $x \in X$, $F(\cdot,x)$ is measurable and for any $t \in \mathbb{R}$, $F(t,\cdot)$ is continuous. Moreover, there exists a function $\eta \in L(\mathbb{R},\mathbb{R}^+)$ such that

$$\|F(t,x)\| \le \eta(t),\ \text{a.e. } t \in \mathbb{R}, \forall\ x \in \bar{B}_r.$$

(H_3) $M\|\eta\|_{S^p}(p\delta)^{-\frac{1}{p}} < r$.

(H_4) T is a compact C_0-semigroup, i.e., for any $t > 0$, the operator $T(t)$ is compact.

It follows from (H_4) and Theorem 2.3.2 of [43] that $T(t)$ is continuous in the uniform operator topology for all $t > 0$, i.e.,

$$\lim_{\tau\to 0}\|T(t+\tau - T(t))\| \to 0,\ \forall t \ge 0. \tag{4.3}$$

Now we are in position to prove another theorem.

Theorem 4.5. *Suppose that there exists a constant $r > 0$ such that the hypotheses (H_1)-(H_4) are satisfied. Then Eq.(1.1) has at least one Stepanov-like almost automorphic mild solution in Y_r.*

Proof. We divide the proof into four steps.

Step 1. Define a mapping Γ on $AS^p(X)$ by

$$\Gamma x(t) = \int_{-\infty}^{t} T(t-\sigma)F(\sigma, x(\sigma))\mathrm{d}\sigma.$$

It follows from the proof of Theorem 4.2 that Γ is well defined and maps $AS^p(X)$ into itself. Moreover $x \in AS^p(X)$ is a mild solution of Eq.(1.1) if and only if x is a fixed point of Γ.

Step 2. Show that Γ maps Y_r into itself. For any $x \in Y_r$,

$$\begin{aligned}
\|\Gamma x(t)\|_{S^p} =& \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \left\| \int_{-\infty}^{\sigma} T(\sigma - s)F(s, x(s))\mathrm{d}s \right\|^p \mathrm{d}\sigma \right)^{\frac{1}{p}} \\
=& \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \left\| \int_{0}^{\infty} T(\tau)F(\sigma-\tau, x(\sigma-\tau))\mathrm{d}\tau \right\|^p \mathrm{d}\sigma \right)^{\frac{1}{p}} \\
\leq& M \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \int_{0}^{\infty} e^{-p\delta\tau} \left\| \eta(\sigma-\tau) \right\|^p \mathrm{d}\tau\mathrm{d}\sigma \right)^{\frac{1}{p}} \\
\leq& M\|\eta\|_{S^p} \left(\int_0^{\infty} e^{-p\delta\tau}\mathrm{d}\tau \right)^{\frac{1}{p}} \\
\leq& M\|\eta\|_{S^p}(p\delta)^{-\frac{1}{p}}.
\end{aligned}$$

By (H_3) one has $\Gamma x \in Y_r$.

Step 3. Show that Γ is continuous in Y_r. Letting $x,\ y \in Y_r$, one has

$$\begin{aligned}
&\|[\Gamma x](t) - [\Gamma y](t)\|_{S^p} \\
=& \sup_{\tau\in\mathbb{R}} \left(\int_\tau^{\tau+1} \left\| \int_{-\infty}^{t} T(t-\sigma)[F(\sigma, x(\sigma)) - F(\sigma, y(\sigma))]\mathrm{d}\sigma \right\|^p \mathrm{d}t \right)^{\frac{1}{p}} \\
\leq& M \sup_{\tau\in\mathbb{R}} \left(\int_\tau^{\tau+1} \int_{-\infty}^{t} e^{-p\delta(t-\sigma)} \left\| F(\sigma, x(\sigma)) - F(\sigma, y(\sigma)) \right\|^p \mathrm{d}\sigma\mathrm{d}t \right)^{\frac{1}{p}} \\
\leq& M(p\delta)^{-\frac{1}{p}} \left\| F(t, x(t)) - F(t, y(t)) \right\|_{S^p}.
\end{aligned}$$

Noticing the condition (H_2), one can deduces that Γ is continuous.

Step 4. Prove that ΓY_r is compact. Let $\{x_k : k \geq 1\} \subset Y_r$. We need to show that $\{\Gamma x_k : k \geq 1\}$ is relatively compact in $BS^p(\mathbb{R}, X)$. To this end, we first prove that $\{\Gamma x_k : k \geq 1\}$ is equicontinuous. Let

$$\varphi_k = \int_{-\infty}^{t} T(t-\sigma)F(\sigma, x_k(\sigma))\mathrm{d}\sigma, \quad t \in \mathbb{R}.$$

For any $t \in \mathbb{R}$, $h \in (0,1)$,

$$\begin{aligned}
\Big\|[\varphi_k(t+h)-\varphi_k(t)\Big\|_{S^p} =& \sup_{\tau\in\mathbb{R}}\Big[\int_\tau^{\tau+1}\Big\|\int_{-\infty}^{t+h} T(t+h-\sigma)F(\sigma,x_k(\sigma))\mathrm{d}\sigma \\
&-\int_{-\infty}^{t} T(t-\sigma)F(\sigma,x_k(\sigma))\mathrm{d}\sigma\Big\|^p\mathrm{d}t\Big]^{\frac{1}{p}} \\
\le& \sup_{\tau\in\mathbb{R}}\Big[\int_\tau^{\tau+1}\Big(\Big\|\int_t^{t+h} T(t+h-\sigma)F(\sigma,x_k(\sigma))\mathrm{d}\sigma\Big\| \\
&+\Big\|\int_{t-N}^{t}[T(t+h-\sigma)-T(t-\sigma)]F(\sigma,x_k(\sigma))\mathrm{d}\sigma\Big\| \\
&+\Big\|\int_{-\infty}^{t-N}[T(t+h-\sigma)-T(t-\sigma)]F(\sigma,x_k(\sigma))\mathrm{d}\sigma\Big\|\Big)^p\mathrm{d}t\Big]^{\frac{1}{p}} \\
\le& 3\sup_{\tau\in\mathbb{R}}\Big[\int_\tau^{\tau+1}\Big\|\int_t^{t+h} T(t+h-\sigma)F(\sigma,x_k(\sigma))\mathrm{d}\sigma\Big\|^p \\
&+\Big\|\int_{t-N}^{t}[T(t+h-\sigma)-T(t-\sigma)]F(\sigma,x_k(\sigma))\mathrm{d}\sigma\Big\|^p \\
&+\Big\|\int_{-\infty}^{t-N}[T(t+h-\sigma)-T(t-\sigma)]F(\sigma,x_k(\sigma))\mathrm{d}\sigma\Big\|^p\mathrm{d}t\Big]^{\frac{1}{p}} \\
=& I_1+I_2+I_3.
\end{aligned}$$

For each k and any $\varepsilon>0$, by some direct estimation, one can find some constant $\theta\in(0,1)$, independent of k and t, such that

$$\Big\|\int_t^{t+h} T(t+h-\sigma)F(\sigma,x_k(\sigma))\mathrm{d}\sigma\Big\|^p < \frac{\varepsilon^p}{3^{p+1}},$$

provided that $h<\theta$. On the other hand

$$\begin{aligned}
&\Big\|\int_{-\infty}^{t-N}[T(t+h-\sigma)-T(t-\sigma)]F(\sigma,x_k(\sigma))\mathrm{d}\sigma\Big\|^p \\
\le& \int_{-\infty}^{t-N}\|[T(t+h-\sigma)-T(t-\sigma)]F(\sigma,x_k(\sigma))\|^p\mathrm{d}\sigma \\
\le& M^p\int_{-\infty}^{t-N}(e^{-\delta(t+h-s)}+e^{-\delta(t-s)})^p\eta^p(s)\mathrm{d}s \\
\le& M^p\|\eta\|^p(e^{-\delta(N+h)}+e^{-\delta N})^p.
\end{aligned}$$

Therefore, for sufficiently large N, independent of k and t, such that

$$M^p\|\eta\|^p(e^{-\delta(N+h)}+e^{-\delta N})^p < \frac{\varepsilon^p}{3^{p+1}}.$$

Thus

$$\Big\|\int_{-\infty}^{t-N}[T(t+h-\sigma)-T(t-\sigma)]F(\sigma,x_k(\sigma))\mathrm{d}\sigma\Big\|^p < \frac{\varepsilon^p}{3^{p+1}}, \tag{4.4}$$

for sufficiently large N independent of k and t. Now, fix N such that (4.4) holds. Then

$$\begin{aligned}
&\left\|\int_{t-N}^{t}[T(t+h-\sigma)-T(t-\sigma)]F(\sigma,x_k(\sigma))\mathrm{d}\sigma\right\|^p\\
\leq&\int_{t-N}^{t}\|T(t+h-\sigma)-T(t-\sigma)\|^p\eta^p(\sigma)\mathrm{d}\sigma\\
=&\int_{t-N}^{t}\|(T(\tau+h)-T(\tau))T(t-\tau-\sigma)\|^p\eta^p(\sigma)\mathrm{d}\sigma\\
\leq&M^p\int_{t-N}^{t}\|T(\tau+h)-T(\tau)\|^p e^{-p\delta(t-\tau-\sigma)}\eta^p(\sigma)\mathrm{d}\sigma\\
\leq&M^p\|T(\tau+h)-T(\tau)\|^p\int_{t-N}^{t}e^{-p\delta(t-\tau-\sigma)}\eta(\sigma)\mathrm{d}\sigma.
\end{aligned}$$

Noticing (4.3), one can deduce that there exists a constant $\theta_1\in(0,\theta)$, independent of k and t, such that

$$M^p\|T(\tau+h)-T(\tau)\|^p\int_{t-N}^{t}e^{-p\delta(t-\tau-\sigma)}\eta(\sigma)\mathrm{d}\sigma<\frac{\varepsilon^p}{3^{p+1}},$$

provided that $h<\theta_1$. This implies that

$$\left\|\int_{t-N}^{t}[T(t+h-\sigma)-T(t-\sigma)]F(\sigma,x_k(\sigma))\mathrm{d}\sigma\right\|^p<\frac{\varepsilon^p}{3^{p+1}},$$

provided that $h<\theta_1$. Therefore, for every k,

$$\|\varphi_k(t+h)-\varphi_k(t)\|_{S^p}<\varepsilon,\quad\forall t\in\mathbb{R},\quad h\in(0,\theta_1).$$

This shows that $\{\Gamma x_k:k\geq 1\}$ is equicontinuous. In the following we prove that the family of functions $\{\Gamma x_k:k\geq 1\}$ has a Cauchy subsequence in $BS^p(\mathbb{R},X)$. In fact, for each $k\geq 1$, one has

$$\begin{aligned}
[\Gamma x_k](t)=&\int_{-\infty}^{t}T(t-\sigma)F(\sigma,x_k(\sigma))\mathrm{d}\sigma\\
=&\int_{-\infty}^{t-\frac{1}{n}}T(t-\sigma)F(\sigma,x_k(\sigma))\mathrm{d}\sigma+\int_{t-\frac{1}{n}}^{t}T(t-\sigma)F(\sigma,x_k(\sigma))\mathrm{d}\sigma\qquad(4.5)\\
=&S(\frac{1}{n})[\Gamma x_k](t-\frac{1}{n})+\int_{t-\frac{1}{n}}^{t}T(t-\sigma)F(\sigma,x_k(\sigma))\mathrm{d}\sigma.
\end{aligned}$$

It follows from (H_2) that there exists a constants $C>0$ such that

$$\left\|\int_{t-\frac{1}{n}}^{t}T(t-\sigma)F(\sigma,x_k(\sigma))\mathrm{d}\sigma\right\|_{S^p}\leq C\int_{t-\frac{1}{n}}^{t}\eta(\sigma)\mathrm{d}\sigma.$$

Set

$$\xi_n=\sup\left\{C\int_{t-\frac{1}{n}}^{t}\eta(\sigma)\mathrm{d}\sigma:t\in\mathbb{R}\right\}.$$

Since $\eta \in L(\mathbb{R}, \mathbb{R}^+)$, one has $\xi_n < \infty$ and

$$\xi_n \to 0, \quad \text{as} \quad n \to \infty. \tag{4.6}$$

Consider the family of functions $\{S(1)[\Gamma x_k](\cdot - 1) : k \geq 1\}$. Clearly, it is a subset of $BS^p(\mathbb{R}, X)$. As is shown above, $\{\Gamma x_k : k \geq 1\}$ is equicontinuous, and so is $\{S(1)[\Gamma x_k](\cdot - 1) : k \geq 1\}$. On the other hand, in Step 2 we proved that $\{\Gamma x_k : k \geq 1\} \subset Y_r$. This implies that

$$\|\Gamma x_k(t-1)\|_{S^p} \leq r, \quad t \in \mathbb{R}, \quad k \geq 1.$$

Since $S(1)$ is a compact operator, for any $t \in \mathbb{R}$, the set $\{S(1)[\Gamma x_k](\cdot - 1) : k \geq 1\}$ is relatively compact in X. Now it follows from Arzela-Ascoli's Theorem that $\{S(1)[\Gamma x_k](\cdot - 1) : k \geq 1\}$ is relatively compact. Therefore, there exists a subsequence of $\{\Gamma x_k : k \geq 1\}$ denoted by $\{\Gamma x_k^1 : k \geq 1\}$ such that $\{S(1)[\Gamma x_k^1](\cdot - 1) : k \geq 1\}$ is a Cauchy sequence in $BS^p(\mathbb{R}, X)$.Similarly, we can select a subsequence of $\{\Gamma x_k^1 : k \geq 1\}$ denoted by $\{\Gamma x_k^2 : k \geq 1\}$ such that $\{S(\frac{1}{2})[\Gamma x_k^2](\cdot - \frac{1}{2}) : k \geq 1\}$ is a Cauchy sequence in $BS^p(\mathbb{R}, X)$. Repeating the above approach and using a diagonal argument, we get a subsequence of $\{\Gamma x_k : k \geq 1\}$, say, $\{\Gamma x_k^n : k \geq 1\}$, such that for every n, $\{S(\frac{1}{n})[\Gamma x_k^n](\cdot - \frac{1}{n}) : k \geq 1\}$ is a Cauchy sequence in $BS^p(\mathbb{R}, X)$. This, together with (4.5) and (4.6), implies that $\{\Gamma x_k^n : k \geq 1\}$ is a Cauchy sequence in $BS^p(\mathbb{R}, X)$, i.e., $\{\Gamma x_k : k \geq 1\}$ is relatively compact in $BS^p(\mathbb{R}, X)$. Now by Schauder's fixed point theorem we immediately deduce that the mapping Γ has a fixed point, i.e., Eq.(1.1) has a Stepanov-like almost automorphic mild solution. □

As an immediate consequence of Theorem 4.5, one has

Corollary 4.6. *Assume that the hypotheses (H_1) and (H_4) hold. Suppose that the function F satisfies the Caratheodory condition. Moreover, there exist a function $\beta \in L(\mathbb{R}, \mathbb{R}^+)$ and a nondecreasing function $\Gamma : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\|F(t,x)\| \leq \beta(t)\Gamma(\|x\|),$$

a.e. $t \in \mathbb{R}$, $\forall x \in X$,

$$\Omega = \lim_{r \to \infty} \frac{\Gamma(r)}{r} < \frac{(p\delta)^{\frac{1}{p}}}{M\|\beta\|_{S^p}}.$$

Then Eq.(1.1) haS at least one mild solution in $AS^p(\mathbb{R}, X)$.

5. Stability of Stepanov-like almost automorphic mild solutions for the nonlinear evolution equations

With the established theory of the existence of Stepanov-like almost automorphic mild solutions for the nonlinear evolution Eq.(1.1), we are in a position to investigate their stability.

Theorem 5.1. *Assume that all the conditions of Theorem 4.4 hold, and that*

$$\delta^2 p^2 > 8M^p L^p.$$

Then the Stepanov-like almost automorphic mild solution $x^*(t)$ of Eq.(1.1) is globally p-th exponentially stable.

Proof. Let $x(t)$ be any solution of Eq.(1.1) with initial value $x(0)$ with an interval of existence, denoted by $[0, T')$. Then, one has that, for any $t \in [0, T')$,

$$\begin{aligned}
&\|x(t) - x^*(t)\|^p \\
=&\Big\|T(t)[x(0) - x^*(0)] + \int_0^t T(t-\sigma)\Big[F(\sigma, x(\sigma)) - F(\sigma, x^*(\sigma))\Big]\mathrm{d}\sigma\Big\|^p \\
\leq& 2\Big[\|T(t)[x(0) - x^*(0)]\|^p + \Big\|\int_0^t T(t-\sigma)\Big[F(\sigma, x(\sigma)) - F(\sigma, x^*(\sigma))\Big]\mathrm{d}\sigma\Big\|^p\Big] \\
\leq& 2M^p e^{-p\delta t}\|x(0) - x^*(0)]\|^p + 2\int_0^t \|T(t-\sigma)\|^{\frac{p}{2}}\mathrm{d}\sigma \int_0^t \|T(\tau-\sigma)\|^{\frac{p}{2}} \\
&\|F(\sigma, x(\sigma)) - F(\sigma, x^*(\sigma))\|^p\mathrm{d}\sigma \\
\leq& 2M^p e^{-p\delta t}\|x(0) - x^*(0)]\|^p + 2\int_0^t M^{\frac{p}{2}} e^{-\frac{\delta p(t-\sigma)}{2}}\mathrm{d}\sigma \int_0^t M^{\frac{p}{2}} e^{-\frac{\delta p(t-\sigma)}{2}} L^p \\
&\|x(\sigma) - x^*(\sigma)\|^p\mathrm{d}\sigma \\
\leq& 2M^p e^{-p\delta t}\|x(0) - x^*(0)]\|^p + \frac{4M^pL^p}{\delta p}\int_0^t e^{-\frac{\delta p(t-\sigma)}{2}}\|x(\sigma) - x^*(\sigma)\|^p\mathrm{d}\sigma.
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_0^T e^{\varepsilon t}\|x(t) - x^*(t)\|^p\mathrm{d}t \\
\leq& 2M^p\left[\|x(0) - x^*(0)\|^p\int_0^T e^{\varepsilon t - p\delta t}\mathrm{d}t + \frac{2L^p}{\delta p}\int_0^T e^{\varepsilon t}\mathrm{d}t\int_0^t e^{-\frac{\delta p(t-\sigma)}{2}}\|x(\sigma) - x^*(\sigma)\|^p\mathrm{d}\sigma\right] \\
\leq& \frac{2M^p}{p\delta - \varepsilon}\|x(0) - x^*(0)]\|^p + \frac{4M^pL^p}{\delta p}\int_0^T \|x(\sigma) - x^*(\sigma)\|^p\mathrm{d}\sigma\int_\sigma^T e^{\varepsilon t}e^{-\frac{\delta p(t-\sigma)}{2}}\mathrm{d}t \\
\leq& \frac{2M^p}{p\delta - \varepsilon}\|x(0) - x^*(0)]\|^p + \frac{4M^pL^p}{\delta p(\frac{\delta p}{2} - \varepsilon)}\int_0^T e^{\varepsilon\sigma}\|x(\sigma) - x^*(\sigma)\|^p\mathrm{d}\sigma.
\end{aligned}$$

Since $\delta^2p^2 > 8M^pL^p$ as assumed above, there exists a number $\varepsilon \in (0, \delta)$ such that

$$\delta p(\frac{\delta p}{2} - \varepsilon) > 4M^pL^p.$$

With this setting and using the estimation obtained above, one obtains that, for any $T \in [0, T')$,

$$\begin{aligned}
\int_0^T e^{\varepsilon t}\|x(t) - x^*(t)\|^p\mathrm{d}t \leq& \frac{2M^p}{p\delta - \varepsilon}\|x(0) - x^*(0)]\|^p \\
&+ \frac{4M^pL^p}{\delta p(\frac{\delta p}{2} - \varepsilon)}\int_0^T e^{\varepsilon\sigma}\|x(\sigma) - x^*(\sigma)\|^p\mathrm{d}\sigma,
\end{aligned}$$

which implies

$$\int_0^T e^{\varepsilon t}\|x(t) - x^*(t)\|^p\mathrm{d}t \leq \frac{\delta p(\frac{\delta p}{2} - \varepsilon)}{\delta p(\frac{\delta p}{2} - \varepsilon) - 4M^pL^p}\frac{2M^p}{p\delta - \varepsilon}\|x(0) - x^*(0)]\|^p,$$

for any $T \in [0, T')$. Obviously, the last estimation is independent of T as well as T'. Therefore, the maximal interval of the existence of the Stepanov-like almost automorphic mild solutions for Eq.(1.1) can be extended to the positive infinity under the conditions assumed above. In particular, one has

$$e^{\varepsilon t}\|x(t) - x^*(t)\|^p \to 0,$$

as $t \to \infty$. This eventually implies that the Stepanov-like almost automorphic mild solution $x^*(t)$ of Eq.(1.1) is globally p-th exponentially stable. □

6. Applications

In this section we give two examples to illustrate our abstract results. Consider the the following partial differential equation

$$\begin{aligned} &\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + F(t,u(t,x)), \quad t \in \mathbb{R}, \quad x \in [0,\pi], \\ &u(t,0) = u(t,\pi) = 0, \quad t \in \mathbb{R}, \end{aligned} \tag{6.1}$$

where $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function. Take $X = L^2([0,\pi])$ and define the operator A by

$$A\varphi := \varphi'', \quad \varphi \in D(A),$$

where

$$D(A) := \left\{\varphi \in L^2[0,\pi] : \varphi'' \in L^2[0,\pi], \varphi(0) = \varphi(\pi)\right\} \subset L^2[0,\pi].$$

It is known that A generates a compact C_0-semigroup $T = \{T(t)\}_{t\geq 0}$ on X satisfying

$$\|T(t)\| \leq e^{-t}, \text{ for } t \geq 0.$$

Eq.(6.1) can be formulated by the inhomogeneous problem (1.1), where $u(t) = u(t,\cdot)$.

Example 6.1. Let us consider the nonlinearity

$$F(t,x)(s) = \beta b(t)\sin(x(s)),$$

for all $x \in X$ and $s \in [0,\pi]$, $t \in \mathbb{R}$, where $b(t)$ is a Stepanov-like almost automorphic function. Thus one has

$$F(t,x) \in AS^p(\mathbb{R} \times X, X)$$

and

$$\begin{aligned} \|F(t,x) - F(t,y)\|_2^2 \leq &\int_0^\pi \beta^2|b(t)|^2|\sin(x(s)) - \sin(y(s))|^2 \mathrm{d}s \\ \leq &\beta^2|b(t)|^2\|x(s) - y(s)\|_2^2. \end{aligned}$$

In consequence, the semilinear evolution Eq.(6.1) has unique Stepanov-like almost automorphic mild solutions if either $b \in L^p(\mathbb{R})$ (Theorem 4.2) or

$$\int_{-\infty}^t b(\sigma)\mathrm{d}\sigma$$

exists for all $t \in \mathbb{R}$ (Theorem 4.3). If we assume that $b \in L(\mathbb{R})$ and

$$C|\beta||b|(p)^{-\frac{1}{p}} < 1,$$

then the same conclusion holds by Theorem 4.4, moreover, if

$$p^2 > 8|\beta|^p|b|^p,$$

then the unique Stepanov-like almost automorphic mild solution of Eq.(6.1) is globally p-th exponentially stable by Theorem 5.1.

Example 6.2. Suppose that the function F satisfies the Carathéodory condition. Moreover, assume that there exist a function $\alpha \in L(\mathbb{R}, \mathbb{R}^+)$ and a nondecreasing function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|F(t,u)| \leq \alpha(t)\gamma(|u|), \tag{6.2}$$

a.e. $t \in \mathbb{R}$, $\forall u \in \mathbb{R}$,

$$\omega = \lim_{r\to\infty} \frac{\gamma(r)}{r} = 0. \tag{6.3}$$

Then for any $u(\cdot) \in L^2([0,\pi])$, by (6.3) one has

$$\gamma(|u(\cdot)|) \in L^2([0,\pi]).$$

This, together with (6.2), implies that

$$F(t,u(\cdot)) \in L^2([0,\pi]).$$

Now, define a nondecreasing function $\Gamma : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\Gamma(r) = \sup\Big\{\|\gamma(|u(\cdot)|_{L^2([0,\pi])} : u(\cdot) \in L^2([0,\pi]), \|u(\cdot)\|_{L^2([0,\pi])} = r\Big\}.$$

By (6.3) one can deduce that for any $\delta > 0$, there exists $M > 0$ such that

$$\gamma(r) < \delta r, \quad \forall\ r > M. \tag{6.4}$$

On the other hand, noticing that Γ is nondecreasing, one has

$$\gamma(r) \leq \gamma(M), \quad \forall\ 0 \leq r \leq M. \tag{6.5}$$

By (6.4) and (6.5) one has the following estimate

$$\|\gamma(|u(\cdot)|)\|_{L^2([0,\pi])} \leq \delta(\|u(\cdot)\|_{L^2([0,\pi])} + \pi\gamma(M), \quad \forall\ u(\cdot) \in L^2([0,\pi])$$

which implies that

$$\Gamma(r) \leq \delta r + \pi\gamma(M).$$

This shows that the function Γ is well defined. Clearly, by the definition of Γ and (6.2) one has

$$\|F(t,u(\cdot))\|_{L^2([0,\pi])} \leq \alpha(t)\Gamma(|u(\cdot)|)$$

for all $u(\cdot) \in L^2([0,\pi])$ and a.e. $t \in \mathbb{R}$. On the other hand, for any $\varepsilon > 0$, let $\delta = \frac{1}{2}\varepsilon$. Then

$$\frac{\Gamma(r)}{r} < \varepsilon,$$

whenever $r > \pi\gamma(M)$. Therefore,

$$\Omega = \lim_{r\to\infty} \frac{\Gamma(r)}{r} = 0.$$

In view of the above, according to Corollary 4.6, we arrive at the following result.

Proposition 6.3. *Let the above mentioned conditions for the system (6.1) be satisfied. Then (6.1) admits at least one Stepanov-like almost automorphic mild solution.*

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