Gulf Journal of Mathematics
Vol 17, Issue 1 (2024) 29-43

ISSN: 2309-4966
https://doi.org/10.56947/gjom.v17i1.1978

OPTIMIZING THE FIRST EIGENVALUE OF NONLINEAR QUANTUM GRAPHS

MOHAMMED AHRAMI[1*] AND ZAKARIA EL ALLALI[2]

Abstract. The main objective of the present work is to study the problem of minimizing or maximizing the first eigenvalue of nonlinear Schrödinger operators on an appropriately specified subset. In the case of an interval, we find that the minimizing and maximizing potentials can be immediately extended to the nonlinear Schrödinger operators. In the case of metric graphs, we show that the maximizing potential on a finite compact metric graph G coincides with the maximizing potential on the loop graph.

1. Introduction

Nonlinear quantum graphs are metric graphs, meaning that the edges of the graphs are endowed with a metric structure, and equipped with nonlinear Schrödinger operators defined as

$$H(p,q)u(x) = -\Delta_p u(x) + q(x)|u(x)|^{p-2}u(x) = \lambda|u(x)|^{p-2}u(x) \ ,$$

where $1 < p < \infty$, and a real absolutely summable potential $q \in L^P(G)$. This operator is pivotal due to its broad applicability across various fields including fluid dynamics, non-Newtonian fluid flows [3, 22], and glaciology [6, 20, 23]. This operator is particularly significant in the modeling of generalized reaction–diffusion problems [2, 17], non-Newtonian filtration [16], and turbulent flows [9].

Nonlinear quantum graphs are of interest in physics as they are simplified models of the dynamics in elongated structures. The primary focus of our paper is to explore the optimization—both minimization and maximization of the first eigenvalue of these operators and and to describe the form of the optimal estimate within the framework of quantum graphs. This particular aspect of quantum mechanics has been garnering increasing attention due to its potential implications in theoretical physics and applied mathematics. The nonlinear nature of the operator introduces complexities that challenge existing theories, which predominantly address linear operators.

Our work builds upon foundational studies such as those by Kurasov and Serio [19], who proved that the maximizing of the first eigenvalue for linear Schrödinger

Date: Received: May 5, 2024; Accepted: May 25, 2024.

* Corresponding author.

2020 *Mathematics Subject Classification.* 34B45, 34L15, 35J05, 81Q35.

Key words and phrases. Nonlinear Schrödinger operators, First eigenvalue, quantum graphs, δ-coupling conditions.

operators on finite compact metric graphs with δ-coupling vertex conditions coincide with the maximizing potential on the loop graphs. In [8] for the nonlinear Schrödinger operator and when $q = 0$, Del Pezzo and Rossi proved the novel lower bound for the first eigenvalue that

$$\lambda_1(G, q) \geqslant c(p) \left(\frac{1}{L}\right)^p$$

where $c(p)$ is a constant that depends only on p, and L is the length of the graph G.

Optimizing the first eigenvalue of the nonlinear Schrödinger operator on quantum graphs is significant for several reasons. Firstly, it aids in understanding the fundamental quantum state or energy level of the quantum graph. This has practical implications for designing and developing more efficient systems where wave propagation or quantum mechanics play a crucial role, such as in semiconductor physics, nanotechnology, and materials science. Secondly, the results could advance the mathematical understanding of nonlinear differential operators, thereby broadening the scope of theoretical physics and applied mathematics.

The remaining part of the paper proceeds as follows. In §2, notation and background results related to the first eigenvalue of nonlinear Schrödinger operators on an arbitrary finite metric graphs G are presented. In §3, we determine the explicit form of the maximizing and minimizing potentials of the nonlinear Schrödinger operators on an interval. In §4, we show that for an arbitrary finite metric graphs G the maximizing potential exists and coincides with the maximizing potential for the loop graphs. In §5, we study the bounds of the first eigenvalue for the nonlinear Schrödinger operators on metric graphs G.

2. Preliminaries and Basics

2.1. **Metric graphs.** We start with some basic facts about metric graphs, for background, we refer to [4].

Let G be a compact connected metric graphs and $\mathcal{E}$ a finite set of edges e, where each edge $e \in \mathcal{E}$ is identified with a compact interval $[0, |e|]$ of length $|e| > 0$, and $\mathcal{V}$ the set of vertex v. We assume in this paper that G is connected with finite total length, which we denote $L := \sum_{e\in\mathcal{E}} |e|$.

We introduce two important functional spaces on G, the Lebesgue space $L^p(G)$ and the Sobolev space $W^{1,p}(G)$:

Definition 2.1. We say that $u \in L^p(G)$ if $u_e \in L^p(0, l_e)$ for all $e \in \mathcal{E}$, where u is a function on G and u_e ia a collection function defined on $(0, l_e)$ for all $e \in \mathcal{E}$, and

$$\|u\|^p_{L^p(G)} = \sum_{e\in\mathcal{E}} \|u_e\|^p_{L^p(0,l_e)} < \infty.$$

Definition 2.2. The Sobolev space $W^{1,p}(G)$ consists of all continuous functions on G that belong to $W^{1,p}(e)$ for each edge $e \in \mathcal{E}$ and such that

$$\|u\|^p_{W^{1,p}(G)} = \sum_{e\in\mathcal{E}} \|u_e\|^p_{L^p(0,l_e)} + \|u'_e\|^p_{L^p(0,l_e)} < \infty.$$

We consider on each edge of the graph G the nonlinear Schrödinger operators H defined on an appropriate domain of definition in $L^p(G)$, whereby for $u \in W^{1,p}(G)$

$$H(p,q)u(x) = -\Delta_p u(x) + q(x)|u(x)|^{p-2}u(x) = \lambda|u(x)|^{p-2}u(x).$$

At each vertex $v \in \mathcal{V}$, we could impose either Kirchhoff conditions

$$\sum_{x_i \in v} \partial u(x_i) = 0$$

or δ-coupling vertex conditions

$$\sum_{x_i \in v} \partial u(x_i) = \alpha_v u(v),$$

where $\partial u(x_i)$ is the normal derivative of u on the edge e_i pointing into v.

In this paper we consider an important subclass of graphs, the class of flower graph, we recall that a flower graph $\mathcal{F}$ is defined as a graph with only one vertex and all edges being loops starting and ending at this vertex.

Definition 2.3. [21, 12] The Ljusternik-Schnirelmann theory prove the existence of a sequence $(\lambda_n, u_n)_{n\geqslant 0}$ for the nonlinear Schrödinger operators with δ-coupling conditions
The first eigenvalue is the minimum of the Rayleigh quotient

$$\lambda_1(G,q) = \min_{u\in\wp} R(u,q).$$

with $\wp = \left(\bigoplus_{n=1}^{N} W^{1,p}(e_n) \bigcap C(G)\right)$, and the Rayleigh quotient $R(u,q)$ is given by

$$R(u,q) = \frac{\|u'\|^p + \displaystyle\int_G q(x)|u(x)|^p dx + \sum_{v\in V} \alpha_v |u(v)|^p}{\|u\|^p}.$$

Proposition 2.4. *Consider a finite compact connected metric graph G. Subsequently, the ground-state eigenfunction of H can be selected to be exclusively positive. Furthermore, the associated eigenvalue is simple.*

Proof. See [18]. □

Remark 2.5. The discreteness of the spectrum of linear Schrödinger operators is a well-established fact. However, it has not been proven that the spectrum of nonlinear Schrödinger operators is also discrete.

Lemma 2.6. *Consider a finite compact connected metric graph G. Let u_1 denote the eigenfunction associated with the first eigenvalue $\lambda_1(G,q)$ of nonlinear Schrödinger operators subject to Kirchhoff boundary conditions. Then $u_1 \in L^\infty(G)$.*

Proof. By Proposition 2.4, we have $u_1 \geqslant 0$. We write

$$u_1(x) = u_1(x_{\min}) + \int_{x_{\min}}^{x} u_1' dz.$$

Using the Hölder inequality, we get

$$\|u_1(x)\|_{L^\infty(G)} \leqslant u_1(x_{\min}) + \left(\int_G (\lambda_1 - q(x))|u_1|^p dz\right)^{\frac{1}{p}} \times L^{(1-\frac{1}{p})}.$$

Consequently, the inequality stated above implies that

$$\|u_1(x)\|_{L^\infty(G)} \leqslant \left(\frac{1}{L} + \max\left(\lambda_1 - q(x)\right)^{\frac{1}{p}} \times L^{(1-\frac{1}{p})}\right) \|u_1\|_{L^p(G)}.$$

Thus $u_1 \in L^\infty(G)$. This completes the proof.

□

3. Characterization of optimizers on an interval

In this section, we look at the explicit form of the maximizing and minimizing potentials of the first eigenvalue for the nonlinear Schrödinger operators on an interval with Dirichlet boundary conditions and we will see that the minimizer and maximizer potentials can be extended immediately to the nonlinear Schrödinger operators.

Throughout this section, we consider the eigenvalues problem (3.1)

$$\begin{cases} H(p,q)u(x) = -\Delta_p u(x) + q(x)|u(x)|^{p-2}u(x) = \lambda|u(x)|^{p-2}u(x) \ x \in [0,\widehat{\pi}], \\ u(0) = u(\widehat{\pi}) = 0, \end{cases} \tag{3.1}$$

with $1 < p < \infty$ and $q(x)$ is the potential-energy which is satisfied a various assumptions and $\widehat{\pi} = \displaystyle\int_0^1 \frac{2dt}{(1-t^p)^{\frac{1}{p}}} = \frac{2\pi}{p\sin\left(\frac{\pi}{p}\right)}$ is the first zero of $\sin_p(x)$, here $\sin_p(x)$ is called the generalized sine function (See [10, 13]).

As usual, we denote by $\displaystyle\int_0^{\widehat{\pi}} q(x)dx = \mathcal{M}$ the total strength of the potential.

We put

$$E = \{q \in L^\infty([0,\widehat{\pi}]) : 0 \leqslant q(x) \leqslant N \text{ and } \int_0^{\widehat{\pi}} q(x)dx = \mathcal{M}\}.$$

Lemma 3.1. *Let $q \in E$, and suppose that $q_{\max} - q_{\min} \leqslant \frac{p}{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$ then $\lambda_1(q) \geqslant q(x)$ on $(0,\widehat{\pi})$.*

Proof. Let $u_1(x)$ the first eigenfunction associated to the eigenvalue $\lambda_1(q)$ of problem (3.1).

We have

$$\begin{aligned} \lambda_1(q) &= \frac{\displaystyle\int_0^{\widehat{\pi}} |u_1'(x)|^p dx + \int_0^{\widehat{\pi}} q(x)|u_1(x)|^p dx}{\displaystyle\int_0^{\widehat{\pi}} |u_1(x)|^p dx} \\ &\geqslant \lambda_1(0) + q_{min} \\ &\geqslant q_{\max}. \end{aligned}$$

Then $\lambda_1(q) \geqslant q(x)$. □

Remark 3.2. [13] The eigenvalues λ of the problem (3.1) with Dirichlet boundary conditions with $q = 0$ on an interval $[0, L]$ are of the form

$$\lambda_n(0) = \left(\frac{n\widehat{\pi}}{L}\right)^p \left(\frac{p}{p'}\right),$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.
In particular the first eigenvalue is

$$\lambda_1(0) = \left(\frac{\widehat{\pi}}{L}\right)^p \left(\frac{p}{p'}\right).$$

Next, we will generally follow the strategy of [7].

Theorem 3.3. *The maximizing potential $\widehat{q}$ of $\lambda_1(q)$ must be a step function.*

Proof. Let

$$\widehat{q}(x) = \begin{cases} N \text{ if } 0 < x < \zeta, \\ 0 \text{ elsewhere,} \end{cases}$$

with $\zeta = \frac{M}{2N}$.
Suppose that $q_1(x)$ is the maximizing potential of $\lambda_1(q)$, since $0 \leqslant q_1(x) \leqslant N$ then

$$q_1(x) = \begin{cases} \leqslant \widehat{q}(x) \text{ if } 0 < x < \zeta, \\ \geqslant \widehat{q}(x) \text{ elsewhere.} \end{cases}$$

Note that $|u_1(x)|$ may be taken as a positive and symmetric decreasing (see [7]), since $|u_1(x)|^p$ is also positive and symmetric decreasing. Therefore by Beesak's theorem [5]

$$\int_0^{\widehat{\pi}} q_1(x)u_1(x)^p dx \leqslant \int_0^{\widehat{\pi}} \widehat{q}(x)u_1(x)^p dx.$$

It follows that

$$R(u, q_1) \leqslant R(u, \widehat{q}).$$

Therefore, we obtain

$$\lambda_1(q_1) \leqslant \lambda_1(\widehat{q}).$$

Since $q_1(x)$ is a maximizing potential of $\lambda_1(q)$, we conclude that $q_1 = \widehat{q}$. This ends the proof. □

Theorem 3.4. *The minimizing potential $q^\star$ of $\lambda_1(q)$ must be a step function.*

Proof. Let

$$q^\star(x) = \begin{cases} 0 \text{ if } 0 < x < \eta, \\ N \text{ elsewhere,} \end{cases}$$

with $\eta = \frac{\widehat{\pi}N - M}{2N}$.
Suppose that $q_0(x)$ is the minimizing potential of of $\lambda_1(q)$, since $0 \leqslant q_0(x) \leqslant N$ then

$$q_0(x) = \begin{cases} \geqslant q^\star(x) \text{ if } 0 < x < \eta, \\ \leqslant q^\star(x) \text{ elsewhere.} \end{cases}$$

Note that $|u_1(x)|$ may be taken as a positive and symmetric decreasing (see [7]), since $|u_1(x)|^p$ is also positive and symmetric decreasing. Therefore, by Beesack's theorem [5] we get

$$\int_0^{\widehat{\pi}} q_0(x)u_1(x)^p dx \geqslant \int_0^{\widehat{\pi}} q^\star(x)u_1(x)^p dx.$$

It follows that

$$R(u, q_0) \geqslant R(u, q^\star),$$

and consequently

$$\lambda_1(q_0) \geqslant \lambda_1(q^\star).$$

Since $q_0(x)$ is a minimizing potential of $\lambda_1(q)$ then $q_0 = q^\star$. Thus the minimizing potential must be a step function. □

Theorem 3.5. *If $\lambda > N$, the eigenvalues $\lambda_1(\widehat{q})$ corresponding to the maximizing potential $\widehat{q}$ of $\lambda_1(q)$ is given by the first real roots of the following transcendental equation*

$$(\lambda - N)^{\frac{1}{p}} \tan_p\left(\left(\frac{\lambda}{p-1}\right)^{\frac{1}{p}}(\widehat{\pi} - \zeta)\right) = -\lambda^{\frac{1}{p}} \tan_p\left(\left(\frac{\lambda - N}{p-1}\right)^{\frac{1}{p}}\zeta\right).$$

Proof. The solutions of the problem (3.1) with the maximizing potential $\widehat{q}$ are of the form

$$u(x) = \begin{cases} \alpha \sin_p((\frac{\lambda-N}{p-1})^{\frac{1}{p}}x) \text{ if } 0 < x < \zeta, \\ \alpha \sin_p((\frac{\lambda}{p-1})^{\frac{1}{p}}(\widehat{\pi} - x)) \text{ elsewhere.} \end{cases}$$

The function $u(x)$ is of class C^1 at ζ. This allows us to write

$$\left.\frac{u'}{u}\right|_{x=\zeta^-} = \left.\frac{u'}{u}\right|_{x=\zeta^+}. \tag{3.2}$$

Invonking the relation (3.2), we infer that

$$\frac{(\lambda - N)^{\frac{1}{p}} \cos_p((\frac{\lambda-N}{p-1})^{\frac{1}{p}}\zeta)}{\sin_p((\frac{\lambda-N}{p-1})^{\frac{1}{p}}\zeta)} = -\frac{\lambda^{\frac{1}{p}} \cos_p((\frac{\lambda}{p-1})^{\frac{1}{p}}(\widehat{\pi} - \zeta))}{\sin_p((\frac{\lambda}{p-1})^{\frac{1}{p}}(\widehat{\pi} - \zeta))}.$$

Hence

$$(\lambda - N)^{\frac{1}{p}} \cot_p\left(\left(\frac{\lambda - N}{p-1}\right)^{\frac{1}{p}}\zeta\right) = -\lambda^{\frac{1}{p}} \cot_p\left(\left(\frac{\lambda}{p-1}\right)^{\frac{1}{p}}(\widehat{\pi} - \zeta)\right),$$

and therefore

$$(\lambda - N)^{\frac{1}{p}} \tan_p\left(\left(\frac{\lambda}{p-1}\right)^{\frac{1}{p}}(\widehat{\pi} - \zeta)\right) = -\lambda^{\frac{1}{p}} \tan_p\left(\left(\frac{\lambda - N}{p-1}\right)^{\frac{1}{p}}\zeta\right).$$

□

Remark 3.6. If $\lambda > N$, the eigenvalues $\lambda_1(q^\star)$ corresponding to the minimizing potential $q^\star$ of $\lambda_1(q)$ is given by the first real roots of the following transcendental equation

$$(\lambda - N)^{\frac{1}{p}} \tan_p\left(\left(\frac{\lambda}{p-1}\right)^{\frac{1}{p}} \eta\right) = -\lambda^{\frac{1}{p}} \tan_p\left(\left(\frac{\lambda - N}{p-1}\right)^{\frac{1}{p}} (\widehat{\pi} - \eta)\right).$$

4. Characterization of Optimizers on quantum graphs

Our intent in this section is to find the explicit form of the maximizing potential of problem (4.1) for arbitrary compact finite metric graphs G, closely following the strategy of [19].

Throughout this section, for $1 < p < \infty$, we consider the eigenvalues problem for nonlinear Schrödinger operators with δ-coupling conditions

$$\begin{cases} H(p,q)u(x) = \lambda |u(x)|^{p-2}u(x) \text{ on } G \\ \displaystyle\sum_{x_i \in v} |\partial u(x_i)|^{p-2}\partial u(x_i) = \alpha_v |u(v)|^{p-2}u(v) \ v \in V(G), \end{cases} \tag{4.1}$$

where $\partial u(x_i) = (-1)^{i+1}u'(x_i)$ denote normal derivatives at the edge end points, and the function α_v parametrises the δ-couplings at the vertices, and note that the value $u(v)$ is well defined, since u is supposed to be continuous on G.

As usual, we denote by $q = q_+ - q_-$ with $q_+(x) = \frac{1}{2}(|q|+q)$ and $q_-(x) = \frac{1}{2}(|q|-q)$ the positive and negative part of q. We assume that the total strength of the potential is fixed $\displaystyle\int_G q(x)dx = \mathcal{M}$ and the total strength of the singular interaction is fixed $\displaystyle\sum_{v\in V} \alpha_v = \mathcal{N}$.

We define the total interaction by

$$I = \int_G q(x)dx + \sum_{i=1}^{n} \alpha_i,$$

and the total positive interaction by

$$I_+ = \int_G q_+(x)dx + \sum_{\alpha_i > 0} \alpha_i,$$

and the total negative interaction by

$$I_- = \int_G q_-(x)dx - \sum_{\alpha_i < 0} \alpha_i.$$

Proposition 4.1. *Let $\widehat{q}$ be a non-negative potential such that*

$$\text{supp } \widehat{q} \subseteq F_{\widehat{q}} = \{x \in G : u_1^{\widehat{q}}(x) = \max_{y\in G} \ u_1^{\widehat{q}}(y)\} \text{ and } \int_G \widehat{q}(x)dx = \mathcal{M},$$

then the potential $\widehat{q}$ is optimal in the sense that we have

$$\lambda_1(\widehat{q}) \geq \lambda_1(q)$$

for any non-negative potential q.

Proof. Let $u_1(x)$ the first normalized eigenfunction i.e: $\int_G |u_1(x)|^p dx = 1$ associated to the first eigenvalue $\lambda_1(q)$ of problem (4.1). We have

$$\begin{aligned} R(u,\widehat{q}) &= \|u_1'\|^p + \left(\int_G \widehat{q}(x)dx\right) \max_{y\in G} |u_1(y)|^p + \sum_{v\in V} h(v)|u_1(v)|^p \\ &\geqslant \|u_1'\|^p + \int_G q(x)|u_1(x)|^p dx + \sum_{v\in V} h(v)|u_1(v)|^p \\ &\geqslant R(u,q). \end{aligned}$$

Thus

$$\lambda_1(\widehat{q}) \geq \lambda_1(q)$$

□

A similar result holds for non-positive potential:

Proposition 4.2. *Let $\widehat{q}$ be a non-positive potential such that*

$$supp\ \widehat{q} \subseteq G_{\widehat{q}} = \{x \in G : u_1^{\widehat{q}}(x) = \min_{y\in G}\ u_1^{\widehat{q}}(y)\}\ and\ \int_G \widehat{q}(x)dx = \mathcal{M},$$

then the potential $\widehat{q}$ is optimal in the sense that we have

$$\lambda_1(\widehat{q}) \geq \lambda_1(q)$$

for any non-positive potential q.

On refined the same arguments as in section (3) and following [19], the maximizing potential $\widehat{q}$ for nonlinear Schrödinger operators on loop graphs $G = (-\frac{l}{2}, \frac{l}{2})$ is of the form

$$\widehat{q}(x) = \begin{cases} k^p\ on\ |x| < \frac{l}{2} - \eta, \\ 0\ elsewhere, \end{cases}$$

and the corresponding eigenfunction $u_1^{\widehat{q}}$ is

$$u_1^{\widehat{q}}(x) = \begin{cases} 1\ on\ |x| < \frac{l}{2} - \eta, \\ \cos_p(k(|x| - \frac{l}{2} + \eta))\ elsewhere, \end{cases}$$

where $k = \lambda^{\frac{1}{p}}$ and η are real constants.

Theorem 4.3. *Parameters η and k are the smallest positive solutions of the system*

$$\begin{cases} k^p(l - 2\eta) = \mathcal{M}, \\ 2k^{p-1}\left(\tan_p(k\eta)\right)^{(p-1)^2} = \mathcal{N}. \end{cases}$$

Proof. The total strength of the maximizing potential $\widehat{q}$ is fixed. Then

$$\int_{\frac{-l}{2}}^{\frac{l}{2}} \widehat{q}dx = \int_{\eta-\frac{l}{2}}^{\frac{l}{2}-\eta} k^p dx = \mathcal{M}$$

Hence

$$k^p(l-2\eta)=\mathcal{M}.$$

The eigenfunction $u_1^{\widehat{q}}$ satisfies δ-vertex conditions

$$\sum_{x=\frac{\pm l}{2}}|\partial u(x)|^{p-2}\partial u(x)=\mathcal{N}|u(v)|^{p-2}u(v).$$

Then

$$2k^{p-1}\left(\cos_p(k\eta)\right)^{(2-p)(p-1)}\left(\sin_p(k\eta)\right)^{(p-1)^2}=\mathcal{N}\left(\cos_p(k\eta)\right)^{p-1}.$$

Therefore

$$2k^{p-1}\left(\cos_p(k\eta)\right)^{(2-p)(p-1)-(p-1)}\left(\sin_p(k\eta)\right)^{(p-1)^2}=\mathcal{N}.$$

It follows that

$$2k^{p-1}\left(\tan_p(k\eta)\right)^{(p-1)^2}=\mathcal{N}.$$

Thus, parameters η and k satisfy

$$\begin{cases} k^p(l-2\eta)=\mathcal{M}, \\ 2k^{p-1}\left(\tan_p(k\eta)\right)^{(p-1)^2}=\mathcal{N}. \end{cases}$$

□

Remark 4.4. (1) Linear Case: when $(p=2)$, the system simplifies to:

$$\begin{cases} k^2(l-2\eta)=\mathcal{M}, \\ 2k\tan(k\eta)=\mathcal{N} \end{cases}$$

which is the same result proved in [19].
(2) If $\mathcal{M}, \mathcal{N}$ are opposite signs, then parameters η and k are the smallest positive solutions of the system

$$\begin{cases} -k^p(l-2\eta)=\mathcal{M}, \\ -2k^{p-1}\left(\tanh_p(k\eta)\right)^{(p-1)^2}=\mathcal{N}. \end{cases}$$

Theorem 4.5. *Let G be a compact connected metric graph. Then the maximizing potential for problem (4.1) on G coincide with the maximizing potential on loop graphs.*

Proof. We essentially follow the proof given in [19]. Let $\mathcal{F}$ be a flower graphs. According to [19], we parametrize the edges in the graph as $e_n=(\frac{-l_n}{2}\frac{l_n}{2})$. Additionally, we make the assumption that $\int_{e_n}q(x)dx=\mathcal{M}_n$ and

$$\partial u_1^{\widehat{q}}\left(\frac{-l_n}{2}\right)^{p-1}+\partial u_1^{\widehat{q}}\left(\frac{l_n}{2}\right)^{p-1}=\mathcal{N}_n.$$

The maximizing potential and the ground state eigenfunction restricted to e_n coincide with those for the loop graphs

$$\widehat{q}_\eta(x)=\begin{cases} k^p \text{ on } |x|<\frac{l_n}{2}-\eta, \\ 0 \text{ elsewhere}, \end{cases}$$

and

$$u_{1,\eta}^{\widehat{q}}(x) = \begin{cases} 1 \text{ on } |x| < \frac{l_n}{2} - \eta, \\ \cos_p(k(|x| - \frac{l_n}{2} + \eta)) \text{ elsewhere}, \end{cases}$$

and on some short edges, the potential may be identically equal to zero. Therefore parameter η and k of problem (4.1) satisfy

$$k^p \sum_{l_n \geqslant 2\eta} (l_n - 2\eta) = \mathcal{M}$$

and

$$2\left(\sum_{l_n \geqslant 2\eta} k^{p-1} \left(\tan_p(k\eta)\right)^{(p-1)^2} + \sum_{l_n < 2\eta} k^{p-1} \left(\tan_p\left(k\frac{l_n}{2}\right)\right)^{(p-1)^2}\right) = \mathcal{N}.$$

Let $\sup \lambda_1(G, q)$ the maximizing potential on G and $\sup \lambda_1(\mathcal{F}, q)$ the maximizing potential on $\mathcal{F}$. It is easy to demonstrate that

$$\sup \lambda_1(G, q) \leqslant \sup \lambda_1(\mathcal{F}, q),$$

where $\mathcal{F}$ is a flower graph cutting into G along the ground state eigenfunction u_1.
Let $\widehat{q}$ the maximizing potential on flower graphs $\mathcal{F}$ cutting into G along the ground state eigenfunction $u_1^{\widehat{q}}$. Then

$$\lambda_1(G, \widehat{q}) = \lambda_1(\mathcal{F}, \widehat{q}).$$

Thus

$$\begin{aligned} \sup \lambda_1(\mathcal{F}, q) &= \lambda_1(\mathcal{F}, \widehat{q}) \\ &= \lambda_1(G, \widehat{q}) \\ &\leqslant \sup \lambda_1(G, q). \end{aligned}$$

Hence, we obtain

$$\sup \lambda_1(\mathcal{F}, q) = \sup \lambda_1(G, q).$$

Therefore, we conclude that the maximizing potential on G coincide with the maximizing potential on loop graphs. □

5. Bounds on the first eigenvalue

In this section, we will study the bound of the first eigenvalue $\lambda_1(G, q)$ of nonlinear Schrödinger operators on metric graphs G.

Theorem 5.1. *For each $p \in (2, +\infty)$ and $q = 0$, let G be a connected finite compact metric graph and let G' be a metric graph obtained from G by adding an edge between the vertices v_1 and v_2 such that $u_1(v_1) = u_1(v_2)$ with u_1 is an eigenfunction associated to the first excited eigenvalue of nonlinear Schrödinger operators with Kirchhoff boundary conditions at the nodes. Then the following inequality holds*

$$\lambda_1(G, 0) \geqslant \lambda_1(G', 0).$$

Proof. Consider the following problem

$$\begin{cases} H(p,0)u(x) = -\Delta_p u(x) = \lambda |u(x)|^{p-2}u(x) \text{ on } G, \\ \sum\limits_{x_i \in v} |\partial u(x_i)|^{p-2}\partial u(x_i) = 0, \end{cases} \tag{5.1}$$

and we define the function ψ by

$$\psi(x) = \begin{cases} (u_1(x))^{\frac{p}{2}} \text{ on } G, \\ (u_1(v_1))^{\frac{p}{2}} \text{ on } G' \setminus G. \end{cases} \tag{5.2}$$

Let

$$\eta(x) = \psi(x) + c,$$

it's clear that $\eta \in L^2(G')$ because $u_1 \in L^p(G)$ and c is chosen such that

$$< \eta(x), 1 >_{L^2(G')} = 0.$$

Then

$$\begin{aligned} < \eta(x), 1 >_{L^2(G')} &= \int_{G'} \eta(x)dx \\ &= \int_{G'} (\psi(x) + c)dx \\ &= \int_{G} (u_1(x))^{\frac{p}{2}}\, dx + (u_1(v_1))^{\frac{p}{2}}\, l + cL'. \end{aligned}$$

In other words, for each $p > 2$ we have $\frac{p}{2} < p - 1$. It follows that

$$0 \leqslant \int_G u_1(x)^{\frac{p}{2}} dx \leqslant \int_G u_1(x)^{p-1} dx.$$

Thus

$$\int_G u_1(x)^{\frac{p}{2}} dx = 0.$$

Therefore

$$c = -\frac{u_1(v_1)^{\frac{p}{2}} l}{L'},$$

where l and L' are the length of the added edge and the total length of G'. The first eigenvalue for problem (5.1) is given by

$$\lambda_1(G,0) = \min_{u \in X_p(G)} \frac{\|u'\|^p_{L^p(G)}}{\|u\|^p_{L^p(G)}} = \frac{\|u_1'\|^p_{L^p(G)}}{\|u_1\|^p_{L^p(G)}}$$

with

$$X_p(G) = \{u \in W^{1,p}(G) \bigcap C(G) : \int_G |u(x)|^{p-2}u(x)dx = 0 \ , \ u \neq 0\}.$$

Now, we will get an estimate for $\lambda_1(G',0)$ using Rayleigh quotient

$$\lambda_1(G',0) = \min_{u \in X_p(G')} \frac{\|u'\|^p_{L^p(G')}}{\|u\|^p_{L^p(G')}} \leqslant \frac{\|f'\|^p_{L^p(G')}}{\|f\|^p_{L^p(G')}},$$

where $f(x) = \psi(x)^{\frac{2}{p}}$ and $f \in L^p(G')$. We have

$$\begin{aligned}\|f\|^p_{L^p(G')} &= \|(\psi)^{\frac{2}{p}}\|^p_{L^p(G')} \\ &= \|(u_1^{\frac{p}{2}})^{\frac{2}{p}}\|^p_{L^p(G)} + \|(u_1(v_1)^{\frac{p}{2}})^{\frac{2}{p}}\|^p_{L^p(G'\setminus G)} \\ &= \|u_1\|^p_{L^p(G)} + |u_1(v_1)|^p l \\ &\geqslant \|u_1\|^p_{L^p(G)}\end{aligned}$$

and

$$\|f'\|^p_{L^p(G')} = \|(\psi^{\frac{2}{p}})'\|^p_{L^p(G')} = \|u_1'\|^p_{L^p(G)}.$$

Therefore

$$\frac{\|u_1'\|^p_{L^p(G')}}{\|u_1\|^p_{L^p(G')}} \geqslant \frac{\|f'\|^p_{L^p(G)}}{\|f\|^p_{L^p(G)}}.$$

Thus

$$\lambda_1(G, 0) \geqslant \lambda_1(G', 0).$$

□

Lemma 5.2. *Let G be a finite compact metric graph, then*

$$\lambda_1(G, q) \geqslant \lambda_1(G, 0)$$

or $\lambda_1(G, 0)$ is the first eigenvalue of nonlinear Schrödinger operators for $q = 0$ and satisfying the δ-coupling conditions

$$\begin{cases} \sum\limits_{x_{\min}} |u'(x_{\min})|^{p-2}u'(x_{\min}) = I_+|u(x_{\min})|^{p-2}u(x_{\min}) \\ \sum\limits_{x_{\max}} |u'(x_{\max})|^{p-2}u'(x_{\max}) = -I_-|u(x_{\max})|^{p-2}u(x_{\max}). \end{cases} \tag{5.3}$$

Proof. Let $u_1(x)$ the first eigenfunction corresponding to the first eigenvalue $\lambda_1(G, q)$, and let $u_1(x_{\max})$ and $u_1(x_{\min})$ the maximum and minimum of $u_1(x)$. Let $H(p, 0)$ the Laplacian operators satisfying the δ-coupling conditions

$$\begin{cases} \sum\limits_{x_{\min}} |u'(x_{\min})|^{p-2}u'(x_{\min}) = I_+|u(x_{\min})|^{p-2}u(x_{\min}) \\ \sum\limits_{x_{\max}} |u'(x_{\max})|^{p-2}u'(x_{\max}) = -I_-|u(x_{\max})|^{p-2}u(x_{\max}). \end{cases} \tag{5.4}$$

The first eigenvalue of problem (4.1) is given by

$$\lambda_1(G, q) = \min_{u\in\wp} R(u, q)$$

with

$$R(u,q)=\frac{\|u_1'\|^p+\displaystyle\int_G q(x)|u_1(x)|^p dx+\sum_{v\in V}\alpha_v|u_1(v)|^p}{\|u_1\|^p}.$$

Then

$$R(u,q)\geqslant\frac{\|u_1'\|^p+I_+|u_1(x_{\min})|^p-I_-|u_1(x_{\max})|^p}{\|u_1\|^p}.$$

Thus, we conclude that

$$\lambda_1(G,q)\geqslant\lambda_1(G,0).$$

□

We will employ the symmetrization technique, drawing inspiration from the papers [11, 15], we have:

Theorem 5.3. *Let G be a finite compact metric graph, then*

$$\lambda_1(G,q)\geqslant\lambda_1(G,0)$$

or $\lambda_1(G,0)$ is the first eigenvalue of nonlinear Schrödinger operators for $q=0$ and defined on the single interval $[0,L]$ satisfying the following Robin boundary conditions

$$\begin{cases}|u'(0)|^{p-2}u'(0)=I_+|u(0)|^{p-2}u(0)\\ |u'(L)|^{p-2}u'(L)=I_-|u(L)|^{p-2}u(L).\end{cases}\tag{5.5}$$

Proof. The function $u^\star$ on the interval $[0,L]$ is the unique decreasing continuous function such that

$$\begin{cases}u^\star(0)=u_1(x_{\min}),\\ u^\star(L)=u_1(x_{\max}),\end{cases}\tag{5.6}$$

and $m(t)=measure\{x\in G:u_1(x)>t\}=measure\{s\in[0,L]:u_1^\star(s)>t\}$. The function $u_1^\star$ constructed in this way satisfies

$$\int_G|u_1(x)|^p dx=\int_0^L|u^\star(x)|^p dx$$

By the co-area formula we have

$$\int_G|u_1'(x)|^p dx\geqslant\int_0^L|(u^\star)'(x)|^p dx.$$

Then

$$\frac{\|u_1'\|^p+I_+|u_1(x_{\min})|^p-I_-|u_1(x_{\max})|^p}{\|u_1\|^p}\geqslant\frac{\|(u^\star)'\|^p+I_+|u^\star(0)|^p-I_-|u^\star(L)|^p}{\|u^\star\|^p}.$$

Thus

$$\lambda_1(G,q)\geqslant\lambda_1(G,0).$$

□

Acknowledgement. We would like to thank Evans M. Harrell II for useful conversations and helpful comments.
The authors would like to thank the referee for the valuable comments and suggestions.

References

1. Ahrami, M., El Allali, Z., Harrell II, E.M., & Kennedy, J.B. (2024). Optimizing the fundamental eigenvalue gap of quantum graphs. https://doi.org/10.48550/arXiv.2401.04344
2. Aris, R. (1975). The Mathematical Theory of Diffusion and Reaction in Permeable Catalyst. The theory of the steady state (Vol. 1). Oxford University Press, USA. https://doi.org/10.1007/BF02459545
3. Astrita, G., & Marrucci, G. (1994) Principles of Non-Newtonian Fuid Mechanics. McGraw-Hill, NewYork. https://DOI:10.1115/1.3423693
4. Berkolaiko, G. & Kuchment, P. (2013). Introduction to Quantum Graphs. Mathematical Surveys and Monographs, vol. 186. American Mathematical Society, Providence. https://bookstore.ams.org/surv-186
5. Beesack, P.R. (1957). A note on integral inequality. Proc.Amer.Math.soc, 875-879. https://doi.org/10.2307/2035005
6. Bognár,G., & Rontò, M. (2005). Numerical-analytic investigation of the radially symmetric solutions for some nonlinear PDEs. Comput. Math. Appl. 50, 983–991. https://doi.org/10.1016/j.camwa.2005.08.002
7. Chern, H.H., & Shen, C.L. (1994). On the maximum and minimum of some functional for the eigenvalue problem of Sturm-Liouville type. J. Diff. Eqns, 107, 68-79. https://doi.org/10.1006/jdeq.1994.1003
8. Del Pezzo, L.M., & Rossi J.D. (2016). The first eigenvalue of the p-Laplacian on quantum graph. Analysis Mathematical Physics, 6, 365-391. https://doi.org/10.1007/s13324-016-0123-y
9. Diaz, J.I., & DE Thelin, F. (1994). On a nonlinear parabolic problem arising in some models related to turbulent flows. SIAM J. Math. Anal. 25 (4), 1085–1111. https://doi.org/10.1137/S0036141091217731
10. Drábek, P., Krejčí,P., & Takáč, P. (1999). Nonlinear Differential Equations. Chapman and Hal/CRC.
11. Friedlander L., (2005). Extremal properties of eigenvalues for a metric graph Annales de l'Institut Fourier,55 no 1, 199-211. https://doi:10.5802/aif.2095.
12. Fučík, S., Nečas, J., Souček, J., & Souček, V. (1973). Spectral analysis of nonlinear operators. Lecture Notes in Mathematics. 346. Springer-Verlag, Berlin-New York.
13. Lang, j.,& Edmunds,D. (2016). Eigenvalues, embeddings and generalised trigonometric functions. Lecture Notes in Mathematics. Springer, Heidelberg, 2011. xii+220 pp.
14. Lindqvist, P. (1995). Some remarkable sine and cosine functions. Ricerche Mat. 44(2), 269-290. https://api.semanticscholar.org/CorpusID:116154577
15. Karreskog, T., Kurasov, P. & Kupersmidt, T. (2016). Schrödinger operators: Symmetrization and Eulerian cycles. Proceedings of American Mathematical Society Volume 144, Number 3, 1197–1207.
16. Kalashnikov, A.S. (1978). On a nonlinear equation appearing in the theory of non-stationary filtration. Tr. Sem. Petrovskogo 5 (1978) 60–68, (in Russian).
17. Keller, B., & Cohen, D.S. (1967). *Some positone problems suggested by nonlinear heat generation. J. Math. Mech. 16, 1361–1376.* https://www.jstor.org/stable/45277157
18. *Kurasov P., (2019). On the ground state for Quantum Graphs. Letters in Mathematical Physics 109, 2491-2512.* https://doi.org/10.1007/s11005-019-01192-w
19. *Kurasov, P., & Serio A. (2019). Optimal Potentials for Quantum Graphs. Annales Henri Poincaré volume 20, 1517–1542.* https://doi.org/10.1007/s00023-019-00783-6

20. *Glowinski, R., & Rappaz, J. (2003). Approximation of a nonlinear elliptic problem arising in a non-Newtonian fluid flow model in glaciology. ESAIM Math. Model. Numer. Anal. 37 (1), 175–186.* https://doi.org/10.1051/m2an:2003012
21. *Lê A. (2006). Eigenvalue problems for the p-Laplacian. k Nonlinear Analysis: Theory, Methods and Applications, 64(5), 1057–1099.* http//doi:10.1016/j.na.2005.05.056
22. *Martinson, L.K, & Pavlov, K.B. (1971). Unsteady shear flows of a conducting fluid with a rheological power law. Magnetohydrodynamics (USSR)(Engl. Transl.);(United States), 7(2).* https://www.osti.gov/biblio/7360371
23. *Picasso, M., Rappaz, J., Reist, A., Funk, M., & Blatter, H. (2004). Numerical simulation of the motion of a two dimensional glacier. J. Numer. Methods Engrg. 60, 995–1009.* https://doi.org/10.1002/nme.997

[1] Modeling and Scientific Computing Team, Multidisciplinary Faculty of Nador, Mohammed I University, Morocco.
Email address: m.ahrami@ump.ac.ma

[2] Modeling and Scientific Computing Team, Multidisciplinary Faculty of Nador, Mohammed I University, Morocco.
Email address: z.elallali@ump.ma