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IDENTITY ON $\theta-$CENTRALIZERS IN SEMIPRIME RINGS

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Abstract. Let m and n be fixed integers with $m \geq 1$, $n \geq 1$ and let R be a $(m+n+2)!-$ torsion free semiprime ring with the identity element suppose there exists an additive mapping $T : R \to R$ such that $T(x^{m+n+1}) = \theta(x^m)T(x)\theta(x^n)$ holds for all $x \in R$ where θ is an automorphisms of R. In this case T is a centralizer.

1. Introduction and preliminaries

Through this paper, R will represent an associative ring with center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ will stand for the commutator $xy - yx$. A ring R is called $n-$torsion free, where $n \geq 1$ is an integer, if whenever $nx = 0$, with $x \in R$, then $x = 0$. Recall that a ring R is prime if for any $a, b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$ and is semiprime if for any $a \in R$, $aRa = (0)$ implies $a = 0$.

Following B. Zalar [9], an additive mapping $T : R \to R$ is called a left (right) centralizer of R if $T(xy) = T(x)y$ ($T(xy) = xT(y)$) holds for all $x, y \in R$. If T is both left as well right centralizer, it is called a centralizer. An additive mapping $T : R \to R$ is called a left (right) Jordan centralizer in case $T(x^2) = T(x)x$ ($T(x^2) = xT(x)$) holds for all $x \in R$. Following ideas from [2], Zalar [9] proved that any left (right) Jordan Centralizer on $2-$torsion free semiprime ring is a left (right) centralizer. In [7], Vukman has proved that in case $T : R \to R$ is an additive mapping, where R is an $2-$torsion free semiprime ring, which satisfies the identity

$$2T(x^2) = T(x)x + xT(x), \text{ for all } x \in R,$$

then T is a centralizer.

E. Albas [1] introduced the definition of $\theta-$centralizer of R, i.e an additive mapping $T : R \to R$ is called a left (right) $\theta-$Centralizer of R if $T(xy) = T(x)\theta(y)$ ($T(xy) = \theta(x)T(y)$) holds for all $x, y \in R$ where θ is an endomorphism of R. If T is left and right $\theta-$Centralizer, it is natural to call T a $\theta-$Centralizer. Clearly, every centralizer is a special case of a $\theta-$Centralizer with $\theta = 1_R$, the identity map on R. An additive mapping $D : R \to R$ is called Jordan triple derivation, in case $D(xyx) = D(x)yx + xD(y)x + xyD(x)$ holds for all pairs $x, y \in R$. It is easy to prove that any Jordan derivation on an arbitrary ring is a Jordan triple derivation, [4]. Bresar [3] has proved that any Jordan triple derivation on

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a 2−trosion free Semiprime ring is a derivation. This result inspired Vukman [8] to prove that if $T : R \to R$ is an additive mapping, where R is a 2−trosion free semiprime ring. Suppose that

$$T(xyx) = xT(y)x, \tag{1.1}$$

holds for all $x \in R$. In this case T is a centralizer. For $x = y$ identity (1) reduces to

$$T(x^3) = xT(x)x, \qquad x \in R. \tag{1.2}$$

In [5], Daif and Tammam generalized the above result to $\theta-$centralizer for semiprime rings.

The question arises whether the identity (2) on a 2−torsion free semiprime ring implies that T is a Centralizer. Vukman [8] proved that the answer to this question is affirmative in case R has the identity element. The identity (2) leads to the following conjecture:

Let R be a semiprime ring with suitable torsion restrictions. Suppose there exists an additive mapping $T : R \to R$, such that $T(x^{m+n+1}) = x^m T(x)x^n$ holds for all $x \in R$, where $m \geq 1$, $n \geq 1$ are some integers. In this case T is a centralizer.

Ulbul [6] proved that the answer to this question is affirmative in case R has the identity element. She proved the following:

Theorem 1.1. ([6]). *Let $m \geq 1$, $n \geq 1$ be fixed integers and let R be a $(m+n+1)!-$ torsion free semiprime ring with the identity element. Suppose there exists an additive mapping $T : R \to R$, such that $T(x^{m+n+1}) = x^m T(x)x^n$ holds for all $x \in R$. In this case T is a Centralizer.*

In this paper we intend to generalize Theorem 1 to the case of $\theta-$centralizer where θ is an automorphism of R.

2. Main results

Theorem 2.1. *Let $m \geq 1$, $n \geq 1$ be a fixed integers and let R be a $(m+n+1)!-$torsion free semiprime with the identity element. Let θ be an epimorphism of R. Suppose these exist an additive mapping $T : R \to R$, such that $T(x^{m+n+1}) = \theta(x^m)T(x)\theta(x^n)$ for all $x \in R$. In this case T is a $\theta-$Centralizer.*

Proof. We have the relation

$$T(x^{m+n+1}) = \theta(x^m)T(x)\theta(x^n), \qquad x \in R \tag{2.1}$$

Replacing in the above relation $x + e$ for x, where e denotes the identity element. We obtain

$$\sum_{i=0}^{m+n+1} \binom{m+n+1}{i} T(x^{m+n+1-i}) =$$

$$(\sum_{i=0}^{m} \binom{n}{i} \theta(x^{m-i}))(T(x)+a)(\sum_{i=0}^{n} \binom{n}{i} \theta(x^{n-i})), \qquad x \in R, \tag{2.2}$$

Where $a = T(e)$ using (3) and rearranging the equation (4) in sense of collecting together terms involving equal number of factors of e we obtain:

$$\begin{pmatrix} m+n+1 \\ 1 \end{pmatrix} T(x^{m+n}e) - \begin{pmatrix} m \\ 0 \end{pmatrix}\begin{pmatrix} n \\ 1 \end{pmatrix}\theta(x^m)T(x)\theta(x^{n-1})e-$$

$$\begin{pmatrix} m \\ 0 \end{pmatrix}\begin{pmatrix} n \\ 0 \end{pmatrix}\theta(x^m)a\theta(x^n) - \begin{pmatrix} m \\ 1 \end{pmatrix}\begin{pmatrix} n \\ 0 \end{pmatrix}\theta(x^{m-1})T(x)\theta(x^n)e+$$

$$\begin{pmatrix} m+n+1 \\ 2 \end{pmatrix} T(x^{m+n-1}e^2) - \begin{pmatrix} m \\ 0 \end{pmatrix}\begin{pmatrix} n \\ 2 \end{pmatrix}\theta(x^m)T(x)\theta(x^{n-1})e^2-$$

$$\begin{pmatrix} m \\ 0 \end{pmatrix}\begin{pmatrix} n \\ 1 \end{pmatrix}\theta(x^m)a\theta(x^{n-1})e - \begin{pmatrix} m \\ 1 \end{pmatrix}\begin{pmatrix} n \\ 1 \end{pmatrix}\theta(x^{m-1})T(x)\theta(x^{n-1})e^2+$$

$$\begin{pmatrix} m \\ 1 \end{pmatrix}\begin{pmatrix} n \\ 0 \end{pmatrix}\theta(x^{m-1})a\theta(x^n)e - \begin{pmatrix} m \\ 2 \end{pmatrix}\begin{pmatrix} n \\ 0 \end{pmatrix}\theta(x^{m-2})T(x)\theta(x^n)e^2 + \cdots+$$

$$\begin{pmatrix} m+n+1 \\ m+n \end{pmatrix} T(xe^{m+n}) - \begin{pmatrix} m \\ m-1 \end{pmatrix}\begin{pmatrix} n \\ n \end{pmatrix}\theta(x)ae^{m+n-1}-$$

$$\begin{pmatrix} m \\ m \end{pmatrix}\begin{pmatrix} n \\ n \end{pmatrix} T(x)e^{m+n} - \begin{pmatrix} m \\ m \end{pmatrix}\begin{pmatrix} n \\ n-1 \end{pmatrix} a\theta(x)e^{m+n-1} = 0, \qquad x \in R.$$

Or shortly,

$$\sum_{i=1}^{m+n} f_i(\theta(x), e) = 0, \qquad x \in R, \tag{2.3}$$

where $f_i(\theta(x), e) = 0$ stands for the expression i of terms involving factors of e.

Replacing e by $2e, 3e, \cdots, (m+n)e$ in turn in the equation (5), and expressing the resulting system of $m+n$ homogeneous equation.

So, we get

$$\begin{aligned} f_1(\theta(x), e) + f_2(\theta(x), e) + \cdots + f_{m+n}(\theta(x), e) &= 0 \\ 2f_1(\theta(x), e) + 2^2 f_2(\theta(x), e) + \cdots + 2^{m+n} f_{m+n}(\theta(x), e) &= 0 \\ &\vdots \\ (m+n)f_1(\theta(x), e) + (m+n)^2 f_2(\theta(x), e) + \cdots + (m+n)^{m+n} f_{m+n}(\theta(x), e) &= 0 \end{aligned}$$

we see that the coefficient matrix of the system is a Vander Monole matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{m+n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ (m+n) & (m+n)^2 & \cdots & (m+n)^{m+n} \end{pmatrix}$$

Since the determinant of the matrix is different from zero, it follows that the

system has only a trivial solution. In particular,

$$f_{m+n-1}(\theta(x),e)=\left(\begin{array}{c} m+n+1 \\ m+n-1 \end{array}\right)T(x^2)-\left(\begin{array}{c} m \\ m-2 \end{array}\right)\theta(x^2)a$$
$$-\left(\begin{array}{c} n \\ n-2 \end{array}\right)a\theta(x^2)\left(\begin{array}{c} m \\ m-1 \end{array}\right)\theta(x)T(x)-\left(\begin{array}{c} n \\ n-1 \end{array}\right)T(x)\theta(x)$$
$$-\left(\begin{array}{c} m \\ m-1 \end{array}\right)\left(\begin{array}{c} n \\ n-1 \end{array}\right)\theta(x)a\theta(x)=0, \qquad x\in R,$$

and,

$$f_{m+n}(x,e)=\left(\begin{array}{c} m+n+1 \\ m+n \end{array}\right)T(x)-\left(\begin{array}{c} m \\ m-1 \end{array}\right)\theta(x)a-$$
$$\left(\begin{array}{c} n \\ n-1 \end{array}\right)a\theta(x)-\left(\begin{array}{c} m \\ m \end{array}\right)\left(\begin{array}{c} n \\ n \end{array}\right)T(x)=0, \qquad x\in R.$$

Since R is a $(m+n+2)!-$torsion free ring, the above equation reduce to

$$(m+n+1)(m+n)T(x^2)=m(m-1)\theta(x^2)a+n(n-1)a\theta(x^2)+$$
$$2mn\theta(x)a\theta(x)+2m\theta(x)T(x)+2nT(x)\theta(x), \qquad x\in R, \tag{2.4}$$

and

$$(m+n)T(x)=m\theta(x)a+na\theta(x), \qquad x\in R, \tag{2.5}$$

respectively.

According to (7) one obtain the relation

$$(m+n)T(x^2)=m\theta(x^2)a+na\theta(x^2), \qquad x\in R.$$

Using the above connection one can replace the expression $(m+n)T(x^2)$ with $m\theta(x^2)a+na\theta(x^2)$ in the relation (6). Thus, we have

$$(m+n+1)(m\theta(x^2)a+na\theta(x^2))=$$
$$m(m-1)\theta(x^2)a+n(n-1)a\theta(x^2)+$$
$$2mn\theta(x)a\theta(x)+2m\theta(x)T(x)+2nT(x)\theta(x).$$

From the above relation we obtain

$$(2m+mn)\theta(x^2)a+(2n+mn)a\theta(x^2)-2mn\theta(x)a\theta(x)-2m\theta(x)T(x)-$$
$$2nT(x)\theta(x)=0, \qquad x\in R. \tag{2.6}$$

Rearranging the above relation gives

$$2m(\theta(x^2)a-\theta(x)T(x))+2n(a\theta(x^2)-T(x)\theta(x))+mn(\theta(x^2)a+a\theta(x^2)-$$
$$2\theta(x)a\theta(x))=0, \qquad x\in R. \tag{2.7}$$

Left and right multiplication of the relation (7) by $\theta(x)$ gives:

$$(m+n)\theta(x)T(x)=m\theta(x^2)a+n\theta(x)a\theta(x), \qquad x\in R, \tag{2.8}$$

and

$$(m+n)T(x)\theta(x)=m\theta(x)a\theta(x)+na\theta(x^2), \qquad x\in R, \tag{2.9}$$

respectively.

Multiplication of the relation (9) by $(m+n)e$ gives

$$\begin{aligned}&2m((m+n)\theta(x^2)a-(m+n)\theta(x)T(x))+\\&2n((m+n)a\theta(x^2)-(m+n)T(x)\theta(x))+\\&mn(m+n)(\theta(x^2)a+a\theta(x^2)-2\theta(x)a\theta(x))=0, \qquad x\in R. \quad (2.10)\end{aligned}$$

Using (10) and (11) in the relation (12) one obtain

$$\begin{aligned}0 &= 2m((m+n)\theta(x^2)a-m\theta(x^2)a-n\theta(x)a\theta(x))+\\&\quad 2n((m+n)a\theta(x^2)-m\theta(x)a\theta(x)-na\theta(x^2))+\\&\quad mn(m+n)(\theta(x^2)a+a\theta(x^2)-2\theta(x)a\theta(x))\\&= (2m(m+n)-2m^2+mn(m+n))\theta(x^2)a+\\&\quad (2n(m+n)-2n^2+mn(m+n))a\theta(x^2)-\\&\quad (4mn+2mn(m+n))\theta(x)a\theta(x)\\&= mn(m+n+2)\theta(x^2)a+\\&\quad mn(m+n+2)a\theta(x^2)-2mn(m+n+2)\theta(x)a\theta(x), \qquad x\in R.\end{aligned}$$

Since R is a $(m+n+2)!-$torsion free ring we obtain

$$\theta(x^2)a+a\theta(x^2)-2\theta(x)a\theta(x)=0, \qquad x\in R. \quad (2.11)$$

The above relation can be written in the from

$$[[a,\theta(x)],\theta(x)]=0, \qquad x\in R. \quad (2.12)$$

The rest of the proof goes through in the same way as in the end of the proof of Theorem 2 in [7], but we proceed for the sake of completeness. Putting $x+y$ for x in the above relation we obtain

$$[[a,\theta(x)],\theta(y)]+[[a,\theta(y)],\theta(x)]=0, \qquad x,y\in R. \quad (2.13)$$

Putting xy for y in relation (15) we obtain because of (14) and (15):

$$\begin{aligned}0 &= [[a,\theta(x)],\theta(xy)]+[[a,\theta(xy)],\theta(x)]=\\&\quad [[a,\theta(x)],\theta(x)]\theta(y)+\theta(x)[[a,\theta(x)],\theta(y)]+\\&\quad [[a,\theta(x)]\theta(y)+\theta(x)[a,\theta(y)],\theta(x)]=\end{aligned}$$

$$\begin{aligned}&\theta(x)[[a,\theta(x)],\theta(y)]+[[a,\theta(x)],\theta(x)]\theta(y)+\\&[a,\theta(x)][\theta(y),\theta(x)]+\theta(x)[[a,\theta(y)],\theta(x)]=\\&[a,\theta(x)][\theta(y),\theta(x)], \qquad x,y\in R. \quad (2.14)\end{aligned}$$

Thus we have

$$[a,\theta(x)][\theta(y),\theta(x)]=0, \qquad x,y\in R.$$

"Since θ is an epimorphism, then there exist $z\in R$ such that $\theta(z)=a$ and replacing y by yz we get

$$[a,\theta(x)][\theta(y)\theta(z),\theta(x)]=[a,\theta(x)][\theta(y)a,\theta(x)]=0$$

That is

$$[a,\theta(x)]\theta(y)[a,\theta(x)]"$$

$$[a, \theta(x)]\theta(y)[a, \theta(x)] = 0.$$

Since R is semiprime ring so

$$[a, \theta(x)] = 0, \qquad x \in R.$$

This reduces the relation (7) to

$$T(x) = a\theta(x), \qquad x \in R.$$

The proof of the theorem is completed. □

In particular we get the following as a corollary

Corollary 2.2 (Theorem 2.2, [5]). *Let R be a $2-$torsion free semiprime ring with an identity, and let θ be a nonzero epimorphism of R. $T : R \to R$ be an additive mapping such that $T(x^3) = \theta(x)T(x)\theta(x)$ for all $x \in R$. Then T is a $\theta-$Centralizer.*

Corollary 2.3 (Theorem 2.1, [5]). *Let R be a $2-$torsion-free semiprime ring, and let $T : R \to R$ be a additive mapping such that $T(xyx) = \theta(x)T(y)\theta(x)$ holds for all pairs $x, y \in R$ where θ is a nonzero epimorphism of R with $\theta(Z(R)) = Z(R)$. Then T is a $\theta-$Centralizer.*

References

1. E. Albas, On τ-Centralizers of Semiprime rings, Siberian Math. I. 48(2)(2007), 191-196.
2. M. Bresar, Jordan derivations of Semiprime rings, Proc. Amer. Math. Soc. 104(1988), 1003-1006.
3. M. Bresar, Jordan Mappings of Semiprime rings, J. Algebra. 127(1984), 218-228.
4. M. Bresar and J. Vukman, Jordan derivations on Prime rings, Bull. Austral. Math. Soc. 3(1988), 321-322.
5. M. N. Daif, M. S. Tammam, and C.Haetinger, On θ-Centralizers of semiprime rings, Aligarh Bulletin of Mathematics, Vol. 30, n(1)(2001), 1-9.
6. I. Kosi-Ulbl, A remark on Centralizers in semiprime rings, Glasnik Mathematics, Vol. 39 (59)(2004), 21-26.
7. J. Vukman, An identity related to centralizers in semprime rings, Comment. Math. Univ. Carolina. 40(1999), 447-456.
8. J. Vukman, Centralizers of semiprime rings, Comment. Math. Univ. Carolina, 42 (2001), 237-245.
9. B. Zalar, On Centralizers of semiprime rings, Comment. Math. Univ. Carolina. 32(1991), 609-614.

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