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COVERING COVER PEBBLING NUMBER FOR SQUARE OF A PATH

A. LOURDUSAMY[1], AND T.MATHIVANAN[2]

Abstract. Let G be a connected graph. Let p be the number of pebbles distributed on the vertices of G. A pebbling move is defined by removing two pebbles from one vertex and put a pebble on an adjacent vertex. The covering cover pebbling number, $\sigma(G)$, is the least p such that after a sequence of pebbling moves, the set of vertices should form a covering for G from every configuration of p pebbles on the vertices of G. In this paper, we determine the covering cover pebbling number for square of a path.

1. Introduction

Pebbling, one of the latest evolutions in graph theory proposed by Lakarias and Saks, has been the topic of vast investigation with significant observations. Having Chung [1] as the forerunner to familiarize pebbling into writings, many other authors too have developed this topic. Hulbert published a survey of graph pebbling [3]. Given a connected graph G, distribute certain number of pebbles on its vertices in some configuration. Precisely, a configuration on a graph G, is a function from $V(G)$ to $N \cup \{0\}$ representing a placement of pebbles on G. The size of the configuration is the total number of pebbles placed on the vertices. A pebbling move is the removal of two pebbles from one vertex and the addition of one pebble to an adjacent vertex. In (regular) pebbling, the target is selected and the aim is to move a pebble to the target vertex. The minimum number of pebbles, such that regardless of their initial placement and regardless of the target vertex, we can pebble that target vertex is called the pebbling number of G, denoted by $f(G)$. In cover pebbling, the aim is to cover all the vertices with at least one pebble, when the pebbling process ends. The minimum number of pebbles required such that regardless of their initial placement on G, there is a sequence of pebbling moves, at the end of which, every vertex has at least one pebble on it, is called the cover pebbling number of G. In [2], Crull et al. determine the cover pebbling number for complete graphs, paths, and trees. In [4], Hulbert and Munyan, determine the cover pebbling number of the d-cube, and in [5], we find the cover pebbling number for square of a path. A set $K \subseteq V(G)$ is a covering, if every edge of G has at least one end in K. The covering cover pebbling number of G, denoted by $\sigma(G)$, is the smallest number of pebbles, such

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that, however the pebbles are initially placed on the vertices of the graph, after a sequence of pebbling moves, the set of vertices with pebbles forms a covering of G. In [7], Lourdusamy and Punitha Tharani determine the covering cover pebbling number for complete graphs, paths, wheel graphs, complete r-partite graphs and binary trees. For more results on covering cover pebbling number, please refer to [6, 7, 8, 9, 10, 11]. In the next section, we determine the covering cover pebbling number of square of a path, that is $\sigma(P_n^2)$, for $n \geq 3$.

2. The covering cover pebbling number for square of a path

Definition 2.1. [12] Let $G = (V(G), E(G))$ be a connected graph. The p-th power of G, denoted by G^p, is the graph obtained from G by adding the edge uv to G whenever $2 \leq d(u,v) \leq p$ in G, that is, $G^p = (V(G), E(G) \cup \{uv : 2 \leq d(u,v) \leq p \text{ in } G\})$. If $p = 1$, we define $G^1 = G$.

We know that if p is large enough, that is, $p \geq n-1$, then $G^p = K_n$.

Notation 2.2. The vertices of P_n^2 will be labeled $v_1, v_2, \cdots v_{n-1}, v_n$. Let p_i be the number of pebbles on v_i, and $s(i,j) = p_i + p_{i+1} + \cdots + p_j$.

It is pretty obvious that to pebble a vertex v_k of P_k^2 from the vertex v_1, one needs at most $2^{\lfloor \frac{k}{2} \rfloor}$ pebbles.

It is easy to see that, $\sigma(P_3^2) = 3$, since $P_3^2 \cong K_3$[7].

Theorem 2.3. *The covering cover pebbling number of P_4^2 is $\sigma(P_4^2) = 4$.*

Proof. Let $p_1 = 3$ and $p_i = 0$ for all $i \neq 1$. Then we cannot cover at least one of the edges of P_4^2. Thus, $\sigma(P_4^2) \geq 4$.

Now consider the distribution of four pebbles on the vertices of P_4^2. If P_4^2 contains three or four vertices with at least one pebble or both p_2 and p_3 are greater than or equal to one, then we are done. Thus, $\sigma(P_4^2) \leq 4$.
Therefore, $\sigma(P_4^2) = 4$. □

Theorem 2.4. *The covering cover pebbling number of P_5^2 is $\sigma(P_5^2) = 7$.*

Proof. Let $p_1 = 6$ and $p_i = 0$ for all $i \neq 1$. Then we cannot cover at least one of the edges of P_5^2. Thus, $\sigma(P_5^2) \geq 7$.

Now consider the distribution of seven pebbles on the vertices of P_5^2. If P_5^2 contains more than three vertices with at least one pebble or if $p_2 \geq 1$, $p_3 \geq 1$, and $p_4 \geq 1$ or $p_1 \geq 1$, $p_3 \geq 1$, and $p_4 \geq 1$ or $p_2 \geq 1$, $p_3 \geq 1$, and $p_5 \geq 1$, then we are done. Thus, $\sigma(P_5^2) \leq 7$.
Therefore, $\sigma(P_5^2) = 7$. □

Theorem 2.5. *The covering cover pebbling number of P_6^2 is $\sigma(P_6^2) = 11$.*

Proof. Let $p_1 = 10$ and $p_i = 0$ for all $i \neq 1$. Then we cannot cover at least one of the edges of P_6^2. Thus, $\sigma(P_6^2) \geq 11$.

Now consider the distribution of eleven pebbles on the vertices of P_6^2. It is obvious that if $s(5,6) \geq 3$, then all edges incident with v_5, v_6 can be covered by pebbled vertices. If moreover $s(1,4) \geq 4 = \sigma(C_4^2)$, we are done. Our goal is to show that we can always pebble such a configuration.

If $s(1,2) \leq 2$ and $s(5,6) \leq 2$, we have $s(3,4) \geq 7$. Then WLOG, $p_4 \geq 4$. We can then pebble both v_5 and v_6 and still have $s(1,4) \geq 5$ as desired.
If $3 \leq s(5,6) \leq 7$, we have $s(1,4) \geq 4$ and we are done.
If $8 \leq s(5,6) \leq 9$, then $s(1,4) \geq 2$. We remove 4 pebbles from $\{v_5, v_6\}$ and place 2 on v_4. Then we have $s(1,4) = 4$, $s(5,6) \geq 4$ and we are done.
If $s(5,6) = 10$, then $s(1,4) = 1$. We remove 6 pebbles from $\{v_5, v_6\}$ and place 3 on v_4, obtaining $s(1,4) = 4$, $s(5,6) = 4$.
If $s(5,6) = 11$, then $s(1,4) = 0$. We remove 8 pebbles from $\{v_5, v_6\}$ and place 4 on v_4, obtaining $s(1,4) = 4$, $s(5,6) = 3$.
We covered all cases, and hence $\sigma(P_6^2) \leq 11$.
Therefore, $\sigma(P_6^2) = 11$. □

Observation 2.6. $\sigma(P_{n-3}^2) < 2^{\lfloor \frac{n-2}{2} \rfloor} + 2^{\lfloor \frac{n-1}{2} \rfloor}$ and $\sigma(P_n^2) < 2\left(2^{\lfloor \frac{n-2}{2} \rfloor} + 2^{\lfloor \frac{n-1}{2} \rfloor}\right)$

Proof. The proof can easily be done by induction on n. □

Theorem 2.7. *For $n \geq 6$, let $n \equiv \alpha (mod\ 3)$. Then the covering cover pebbling number for P_n^2 is $\sigma(P_n^2) = \sum_{i \in I}\left(2^{\lfloor \frac{i}{2} \rfloor} + 2^{\lfloor \frac{i+1}{2} \rfloor}\right) + \begin{cases} 1 & \text{if } \alpha = 2 \\ 0 & \text{if } \alpha = 0 \text{ or } 1 \end{cases}$, where $I = \{\alpha + 1, \alpha + 4, \alpha + 7, \cdots, n-5, n-2\}$.*

Proof. To cover the incident edges of v_n, we need at least $2^{\lfloor \frac{n-2}{2} \rfloor} + 2^{\lfloor \frac{n-1}{2} \rfloor}$ pebbles from v_1. In a similar way, we need at least $2^{\lfloor \frac{\alpha+1}{2} \rfloor} + 2^{\lfloor \frac{\alpha+2}{2} \rfloor} + 2^{\lfloor \frac{\alpha+4}{2} \rfloor} + 2^{\lfloor \frac{\alpha+5}{2} \rfloor} + \cdots + 2^{\lfloor \frac{n-5}{2} \rfloor} + 2^{\lfloor \frac{n-4}{2} \rfloor}$ pebbles from v_1 to cover the remaining edges of P_n^2 except possibly the edge v_1v_2. Thus we need $\sum_{i \in I}\left(2^{\lfloor \frac{i}{2} \rfloor} + 2^{\lfloor \frac{i+1}{2} \rfloor}\right)$ pebbles from v_1 to cover the edges of P_n^2 except possibly the edge v_1v_2 where $I = \{\alpha + 1, \alpha + 4, \alpha + 7, ..., n-5, n-2\}$. Note that the edge v_1v_2 is not covered if $\alpha = 2$ and we are already done if $\alpha = 0$ or 1, where $n \equiv \alpha (mod\ 3)$. Thus we need at least $\sum_{i \in I}\left(2^{\lfloor \frac{i}{2} \rfloor} + 2^{\lfloor \frac{i+1}{2} \rfloor}\right) + \begin{cases} 1 & \text{if } \alpha = 2 \\ 0 & \text{if } \alpha = 0 \text{ or } 1 \end{cases}$ pebbles from v_1 to cover the edges of P_n^2, where $I = \{\alpha + 1, \alpha + 4, \alpha + 7, \cdots, n-5, n-2\}$.
Let us use the induction on n to prove the upper bound for the covering cover pebbling number of P_n^2. Now consider the distribution of $\sigma(P_n^2)$ pebbles on the vertices of P_n^2. We know that $\sigma(P_n^2) = \sigma(P_{n-3}^2) + 2^{\lfloor \frac{n-2}{2} \rfloor} + 2^{\lfloor \frac{n-1}{2} \rfloor}$.

If $4 \leq s(n-2, n) \leq 2^{\lfloor \frac{n-2}{2} \rfloor} + 2^{\lfloor \frac{n-1}{2} \rfloor}$, then we can pebble both v_{n-2} and v_{n-1} while at the same time $s(1, n-3) \geq \sigma(P_{n-3}^2)$ and we are done. By symmetry, the same is true for $s(1,3)$.

If the above is not true for either $s(n-2, n)$ or $s(1,3)$, then either we have both $s(n-2, n) \leq 3$ and $s(1,3) \leq 3$, or WLOG $s(n-2, n) \leq 3$ and $s(1,3) > 2^{\lfloor \frac{n-2}{2} \rfloor} + 2^{\lfloor \frac{n-1}{2} \rfloor}$.

We look at vertices v_{n-2}, v_{n-1}, v_n. In either case we have $s(n-2, n) \leq 3$. If v_{n-2} and v_{n-1} are pebbled, then all edges incident with v_{n-2}, v_{n-1}, v_n are covered by pebbled vertices, and since $s(1, n-3) \geq \sigma(P_{n-3}^2) + 2^{\lfloor \frac{n-2}{2} \rfloor} + 2^{\lfloor \frac{n-1}{2} \rfloor} - 3 > \sigma(P_{n-3}^2)$, we are done.

If there are two or three pebbles on $\{v_{n-2}, v_{n-1}, v_n\}$, we can always pebble at least v_{n-1}. Then we need at most $2^{\lfloor \frac{n-2}{2} \rfloor}$ pebbles to pebble v_{n-2}. We still have $s(1, n-3) \geq \sigma(P_{n-3}^2) + 2^{\lfloor \frac{n-1}{2} \rfloor} - 3 > \sigma(P_{n-3}^2)$ and are done.

If there is one pebble on either v_{n-2} or v_{n-1}, then we need at most $2^{\lfloor \frac{n-1}{2} \rfloor}$ pebbles to pebble the non-pebbled vertex v_{n-2} or v_{n-1}. Similarly as above, we still have $s(1, n-3) \geq \sigma(P_{n-3}^2) + 2^{\lfloor \frac{n-2}{2} \rfloor} - 1 > \sigma(P_{n-3}^2)$ and are done.

If there is one pebble on v_n, then we need at most $2^{\lfloor \frac{n-2}{2} \rfloor}$ pebbles to pebble the non-pebbled vertex v_{n-2}. So all edges incident with v_{n-2} and v_n are now covered with pebbled vertices. We still need to make sure that $v_{n-3}v_{n-1}$ is properly covered by pebbling v_{n-3}. This can be done by using at most $2^{\lfloor \frac{n-3}{2} \rfloor}$ pebbles. Similarly as above, we still have $s(1, n-3) \geq \sigma(P_{n-3}^2) + 2^{\lfloor \frac{n-1}{2} \rfloor} - 2^{\lfloor \frac{n-3}{2} \rfloor} - 1 > \sigma(P_{n-3}^2)$ and are done.

Finally, if we have $s(n-2, n) = 0$, we want to pebble v_{n-2} and v_{n-1}. This is achieved by pebbling v_{n-2} using at most $2^{\lfloor \frac{n-2}{2} \rfloor}$ pebbles, and v_{n-1} using at most $2^{\lfloor \frac{n-1}{2} \rfloor}$ pebbles. Then there are at least $\sigma(P_{n-3}^2)$ pebbles on $v_1, v_2, \cdots v_{n-3}$ and the whole P_n^2 is pebbled as desired.

Thus $\sigma(P_n^2) \leq \sum_{i \in I} \left(2^{\lfloor \frac{i}{2} \rfloor} + 2^{\lfloor \frac{i+1}{2} \rfloor}\right) + \begin{cases} 1 & \text{if } \alpha = 2 \\ 0 & \text{if } \alpha = 0 \text{ or } 1 \end{cases}$.

Therefore, $\sigma(P_n^2) = \sum_{i \in I} \left(2^{\lfloor \frac{i}{2} \rfloor} + 2^{\lfloor \frac{i+1}{2} \rfloor}\right) + \begin{cases} 1 & \text{if } \alpha = 2 \\ 0 & \text{if } \alpha = 0 \text{ or } 1 \end{cases}$, where $I = \{\alpha + 1, \alpha + 4, \alpha + 7, \cdots, n-5, n-2\}$. □

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[1] Department of Mathematics,, St.Xavier's College (Autonomous), Palayamkottai-627 002, India
E-mail address: lourdugnanam@hotmail.com

[2] Research Scholar, Department of Mathematics,, St.Xavier's College (Autonomous), Palayamkottai-627 002, India
E-mail address: tahit_van_man@yahoo.com