

STABILIZATION OF LINEAR KDV EQUATION WITH BOUNDARY TIME-DELAY FEEDBACK AND INTERNAL SATURATION

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ABSTRACT. This research studies the stabilization of the linear KdV equation with time-delay on boundary feedback in the presence of a saturated source term. Under certain hypotheses, the proof of well-posedness is established. The result of exponential stability is demonstrated using an appropriate Lyapunov functional.

1. INTRODUCTION

In this research, we focus on the study of the linear KdV equation with time-delay on boundary feedback in the presence of a source term.

$$\begin{cases} u_t(x, t) + u_x(x, t) + u_{xxx}(x, t) = f(x, t), & t > 0, x \in [0, L]; \\ u(0, t) = u(L, t) = 0, & t > 0 \\ u_x(L, t) = \alpha u_x(0, t) + \beta u_x(0, t - h), & t > 0; \\ u(x, 0) = u_0(x), & x \in [0, L]; \\ u_x(0, t) = z_0(t), & t \in [-h, 0]. \end{cases} \quad (1.1)$$

where u symbolizes the state, f represents the source term, $L > 0$ signifies the length of the spatial domain, $h > 0$ denotes the delay, and α and β are real fixed values that satisfy certain conditions to be specified later. The KdV equation serves as a mathematical representation of wave behavior on shallow water surfaces and stands out as a classic example of a model that can be precisely solved. Various methods have been employed to investigate the mathematical properties of this equation (see, for example, [13, Pages 151-184], [5, Pages 38-50], [4, 26]). Additionally, extensive research has been conducted on its controllability and stabilizability characteristics, as detailed in [5, 27].

Both theoretical and practical interest has been sparked by problems related to the stability of systems subject to time delays. Recently, several papers have been published on the stability of partial differential equations with delay terms (see, for example, [6, 1, 8, 10, 21, 7, 11]). In [6], the authors demonstrate that a

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slight delay in the feedback mechanism can lead to the destabilization of a system. However, the presence of delay phenomena can sometimes contribute to the improvement of the system. Indeed, in [1], the authors showed that a delay can enhance system performance

During the last few years, some works have emerged on the study of the KdV equation with time-delay on the boundary or internal control. Indeed, in the literature, we find some papers that study this problem ([2, 32, 21]). In [2], the following nonlinear KdV equation with time delay on the boundary feedback

$$\begin{cases} u_t(x, t) + u_x(x, t) + u_{xxx}(x, t) + u(x, t)u_x(x, t) = 0, & t > 0, x \in [0, L]; \\ u(0, t) = u(L, t) = 0, & t > 0 \\ u_x(L, t) = \alpha u_x(0, t) + \beta u_x(0, t - h), & t > 0; \\ u(x, 0) = u_0(x), & x \in [0, L]; \\ u_x(0, t) = z_0(t), & t \in [-h, 0]. \end{cases} \quad (1.2)$$

has been studied. The authors of [2] introduced two distinct methods to demonstrate the exponential stability outcomes of the system (1.2). The first approach involves using a Lyapunov function alongside an estimation of the decay rate, under the assumption that the length L of the spatial domain satisfies $L < \pi\sqrt{3}$. Secondly, by employing an observability inequality approach without providing an estimation of the decay rate, and for any length such that $L \notin \mathcal{N} = \left\{ 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}, k, l \in \mathbb{N}^* \right\}$, the authors also demonstrate the exponential stability of the system (1.2).

The majority of practical systems are subject to constraints. These constraints can emerge due to various factors such as physical considerations or technological restrictions. There are various kinds of constraints, among these we can specify bounded control, the saturation constraint and polyhedral constraints on the control. Currently, it is generally understood that ignoring these kind of nonlinearities can result unfavorable and potentially disastrous outcomes for the control system.

The saturation constraint belong to the large class of constraints known as cone-bounded nonlinearity (see [14, 24, 3, 31, 34, 9]). In recent decades, the saturation constraint has received a lot of interest (see e.g [29, 28, 18, 16, 15, 31, 12, 33]). For the control system, handling with the saturation constraint is unavoidable. Actually, limitations on input signal amplitude, whether due to physical or practical considerations, can lead to adverse outcomes for the control system.

In the literature, there are several papers that study the Korteweg-de Vries equation with input saturation (see e.g [15, 17, 30, 20]). In [17], the global stabilization of Korteweg-de Vries equation with saturated distributed feedback has been studied. Thanks to the Banach fixed-point theorem, the well-posedness is proved. To prove the asymptotic stability, the authors have worked on two cases. In the first case, when the control acts on all the domain saturated, they used a sector condition and Lyapunov theory for infinite-dimensional systems. For the

second case, where the control acts only on a part of the saturated domain, they prove the asymptotic stability of the closed-loop system using an argument by contradiction. In [18], the question of asymptotic stability of a linear Korteweg-de Vries equation with the following saturated distributed control

$$f(x, t) = \text{sat}(au(x, t)),$$

where a is a positive constant, has been investigated. Thanks to nonlinear semi-group theory, the well-posedness is proved. They use a sector condition and a Lyapunov function to prove the asymptotic stability of the closed-loop system. In this research, the source term f is given by

$$f(x, t) = -\text{sat}(a(x, t)) \quad (1.3)$$

where $\tilde{a} = a(x) \in L^\infty[0, L]$, satisfying

$$\begin{cases} a = a(x) \geq a_0 > 0 & \text{on } \omega \subseteq [0, L], \\ \omega & \text{is a nonempty open subset of } [0, L], \end{cases} \quad (1.4)$$

and the function $\text{sat}(\cdot)$ is the saturation function is defined as follows

$$\text{sat}(s) = \begin{cases} s, & \text{if } \|s\|_{L^2} \leq 1; \\ \frac{s}{\|s\|_{L^2}}, & \text{if } \|s\|_{L^2} \geq 1. \end{cases} \quad (1.5)$$

The aim of this research is to study the stability results of KdV equation (1.1) in presence of saturated source term. As a first step, the well-posedness is proved. In the second step, using an appropriate Lyapunov functional, we prove the exponential stabilizability results of system (1.1).

The article is structured as follows: In Section 2, the problem is outlined. The well-posedness of system (1.1) in the presence of a saturated source term is addressed in Section 2.1. Section 3 is dedicated to exponential stability results. Finally, conclusions are provided in Section 4.

Notation 1. u_t, u_x and u_μ stands for the partial derivative of function u with respect to t, x and μ respectively. Given $L > 0$, $\|\cdot\|_{L^2[0, L]}$ (resp $\langle \cdot, \cdot \rangle$), denotes the norm (resp the inner product) in $L^2[0, L]$. $H^1[0, L]$ denotes the set of functions $u \in L^2[0, L]$ such that $u_x \in L^2[0, L]$. $H^2[0, L]$ denotes the set of functions $u \in L^2[0, L]$ such that $u_x, u_{xx} \in L^2[0, L]$. $H^3[0, L]$ denotes the set of functions $u \in L^2[0, L]$ such that $u_x, u_{xx}, u_{xxx} \in L^2[0, L]$.

2. PROBLEM STATMENT

The aim of this research is to study the following KdV equation with time-delay on the boundary feedback in presence of saturated source term

$$\begin{cases} u_t(x, t) + u_x(x, t) + u_{xxx}(x, t) = -\text{sat}(a(x)u(x, t)), & t > 0, x \in [0, L]; \\ u(0, t) = u(L, t) = 0, & t > 0; \\ u_x(L, t) = \alpha u_x(0, t) + \beta u_x(0, t - h), & t > 0; \\ u(x, 0) = u_0(x), & x \in [0, L]; \\ u_x(0, t) = z_0(t), & t \in [-h, 0]. \end{cases} \quad (2.1)$$

where u symbolizes the state, f represents the source term, $L > 0$ signifies the length of the spatial domain, $h > 0$ denotes the delay, and α and β are real fixed values. We assume that $a(\cdot) \in L^\infty([0, L])$ is a nonnegative function, satisfying (1.4).

Remark 2.1. The hypothesis $a = a(x) \in L^\infty([0, L])$ satisfying (1.4) is a classical assumption used to investigate the stabilization of the KdV equation (see ([17, 30, 23]).

Moreover, we define the matrix M_1 by

$$M_1 = \begin{pmatrix} \alpha^2 - 1 + |\beta| & \alpha\beta \\ \alpha\beta & \beta^2 - |\beta| \end{pmatrix} \quad (2.2)$$

where α, β are real constant. We will suppose that α and β fulfill the subsequent inequality

$$|\alpha| + |\beta| \leq 1. \quad (2.3)$$

According to [2], if (2.3) is satisfied, then the matrix M_1 is negative definite.

We introduce the Hilbert space $H = L^2[0, L] \times L^2[0, 1]$, equipped with usual inner product

$$\left\langle \begin{pmatrix} u \\ z \end{pmatrix}, \begin{pmatrix} u_1 \\ z_1 \end{pmatrix} \right\rangle_H = \int_0^L uu_1 dx + \int_0^1 zz_1 d\mu, \quad (2.4)$$

for all $(u, z), (u_1, z_1) \in H$ and its norm

$$\left\| \begin{pmatrix} u \\ z \end{pmatrix} \right\|_H^2 = \int_0^L u^2 dx + \int_0^1 z^2 d\mu$$

Now we introduce a variable $z(\cdot, \cdot)$ that takes into account the delay term $h > 0$, as in ([2, 32, 19]). The variable z is define by $z(\mu, t) = u_x(0, t - h\mu)$ for $\mu \in [0, 1]$ and $t > 0$. Then $z(\cdot, \cdot)$ satisfies the subsequent system

$$\begin{cases} hz_t(\mu, t) + z_\mu(\mu, t) = 0, & t > 0, \mu \in [0, 1]; \\ z(0, t) = u_x(0, t), & t > 0; \\ z(\mu, 0) = z_0(-h\mu), & \mu \in [0, 1]. \end{cases} \quad (2.5)$$

Let $Y = \begin{pmatrix} u \\ z \end{pmatrix}$, then the system (2.1) and (2.5) can be rewritten as the subsequent first-order system

$$\begin{cases} Y_t = AY(t) + \begin{pmatrix} -sat(a(x)u) \\ 0 \end{pmatrix}, & t > 0, \\ Y(0) = \begin{pmatrix} u_0, & z_0(-h\cdot) \end{pmatrix}^T. \end{cases} \quad (2.6)$$

where the operator A is defined by

$$\begin{aligned} D(A) &= \{(u, z) \in H^3([0, L]) \times H^1([0, L]), u(0) = u(L) = 0 \\ &\quad z(0) = u_x(0), u_x(L) = \alpha u_x(0, t) + \beta u_x(0, t - h)\} \\ A \begin{pmatrix} u \\ z \end{pmatrix} &= \begin{pmatrix} -u_x - u_{xxx} \\ \frac{-1}{h} z_\mu \end{pmatrix} \text{ for all } \begin{pmatrix} u \\ z \end{pmatrix} \in D(A). \end{aligned} \quad (2.7)$$

We also equipped the Hilbert space H with the following inner product

$$\left\langle \begin{pmatrix} u \\ z \end{pmatrix}, \begin{pmatrix} u_1 \\ z_1 \end{pmatrix} \right\rangle = \int_0^L uu_1 dx + |\beta|h \int_0^1 zz_1 d\mu,$$

The new inner product denoted $\langle \cdot, \cdot \rangle$ is clearly equivalent to the usual inner product $\langle \cdot, \cdot \rangle_H$ given by (2.4).

Furthermore, let $\mathcal{H}$ be the Hilbert space of the initial and boundary data defined as follows: $\mathcal{H} = L^2[0, L] \times L^2[-h, 0]$. The Hilbert space $\mathcal{H}$ is equipped with the norm $\| \cdot \|_{\mathcal{H}}$, defined for all $(u, z) \in \mathcal{H}$ by

$$\|(u, z)\|_{\mathcal{H}}^2 = \int_0^L u^2 dx + \int_{-h}^0 z^2(s) ds. \quad (2.8)$$

In order to study the existence and uniqueness of (2.6), we recall the definition of a mild solution.

Let's consider the abstract system within a Hilbert space

$$\begin{cases} \dot{u}(t) = \mathcal{A}u(t) + f(t), & t > 0, \\ u(0) = u_0, \end{cases} \quad (2.9)$$

where $\mathcal{A}$ is an infinitesimal generator of linear C_0 -semigroup $(T(t))_{t \geq 0}$ defined on its domain $D(\mathcal{A}) \subseteq H$, where H is a Hilbert space and $f \in L_{loc}^1([0, T], H)$.

Definition 2.2. [22, Definition 2.3] Let $\mathcal{A}$ be the infinitesimal generator of a C_0 -semigroup $(T(t))_{t \geq 0}$. Let $u_0 \in H$ and $f \in L^1(0, T, H)$. Then the function $u \in \mathcal{C}([0, T], H)$ given by

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds \quad 0 \leq t \leq T, \quad (2.10)$$

is the unique mild solution of (2.9) on $[0, T]$.

We recall in the subsequent result that the saturation function is Lipschitzian in $L^2(0, L)$.

Lemma 2.3. [29, Theorem 5.1] For all $(u, v) \in L^2(0, L)$, we have

$$\|sat(u) - sat(v)\|_{L^2(0, L)} \leq 3\|u - v\|_{L^2(0, L)}$$

The following Propositions will be needed.

Proposition 2.4. [17, Proposition 3.4] Assume that a satisfies 1.4. If $u \in L^2(0, T; H^1(0, L))$, then $sat(au) \in L^1(0, T; L^2(0, L))$ and the map $\psi : u \in L^2(0, T; H^1(0, L)) \mapsto sat(au) \in L^1(0, T; L^2(0, L))$ is continuous.

Proposition 2.5. [25, Proposition 4.1] Let $u \in L^2(0, T; H^1(0, L))$. Then $uu_x \in L^1(0, T; L^2(0, L))$ and the map

$$\phi : u \in L^2(0, T; H^1(0, L)) \mapsto uu_x \in L^1(0, T; L^2(0, L))$$

is continuous. Moreover, there exists $K_1 > 0$ such that, for any $u, \tilde{u} \in L^2(0, T; H^1(0, L))$, we have

$$\begin{aligned} \int_0^T \|uu_x - \tilde{u}\tilde{u}_x\|_{L^2(0, L)} &\leq K_1 \|u - \tilde{u}\|_{L^2(0, T; H^1(0, L))} \\ &\quad \times (\|u\|_{L^2(0, T; H^1(0, L))} + \|\tilde{u}\|_{L^2(0, T; H^1(0, L))}) \end{aligned}$$

2.1. Well-posedness of nonlinear system. The aim of this section is to prove the local well-posedness result of nonlinear system (2.1).

Before moving to the study the well posedness of solutions for the nonlinear system (2.1), we recall that the authors of [2] have previously demonstrated the well posedness of solution for the following linear system

$$\begin{cases} u_t(x, t) + u_x(x, t) + u_{xxx}(x, t) = f(x, t), & t > 0, \quad x \in (0, L); \\ u(0, t) = u(L, t) = 0, & t > 0; \\ u_x(L, t) = \alpha u_x(0, t) + \beta u_x(0, t - h), & t > 0; \\ u(x, 0) = u_0(x), & x \in (0, L); \\ u_x(0, t) = z_0(t), & t \in [-h, 0]. \end{cases} \quad (2.11)$$

where f is the source term. To prove the well-posedness of (2.11), the authors assume that the source term $f \in L^1(0, T, L^2(0, L))$.

Now, let us introduce the space $\mathcal{B} = \mathcal{C}([0, T], L^2(0, L)) \cap L^2(0, T, H^1(0, L))$ with $T > 0$. We equipped the space $\mathcal{B}$ with the following norm

$$\|u\|_{\mathcal{B}} = \|u\|_{\mathcal{C}([0, T], L^2(0, L))} + \|u\|_{L^2(0, T, H^1(0, L))}.$$

Let us state the main result of this section.

Theorem 2.6. *Let $T > 0, L > 0$ and suppose that (2.3) hold. We also assume that $a \in L^\infty(0, L)$ satisfying (1.4). Then there exists $r > 0$ and $K > 0$ such that for every $(u_0, z_0) \in H$ satisfying $\|(u_0, z_0)\|_H \leq r$, there exists a unique $u \in \mathcal{B}$ of system (2.1) satisfying $\|u\|_{\mathcal{B}} \leq K\|(u_0, z_0)\|_H$.*

Proof. Let $(u_0, z_0) \in H$ such that $\|(u_0, z_0)\|_H \leq r$ for $r > 0$ chosen small later. Let us take $u \in \mathcal{B}$ and we consider the following map

$$\begin{aligned} \chi : \mathcal{B} &\rightarrow \mathcal{B} \\ u &\mapsto \chi(u) = \tilde{u} \end{aligned}$$

where $\tilde{u}$ is the solution of the following system

$$\begin{cases} \tilde{u}_t(x, t) + \tilde{u}_x(x, t) + \tilde{u}_{xxx}(x, t) \\ = -sat(au(x, t)), & t > 0, x \in (0, L); \\ \tilde{u}(0, t) = \tilde{u}(L, t) = 0, & t > 0; \\ \tilde{u}_x(L, t) = \alpha \tilde{u}_x(0, t) + \beta \tilde{u}_x(0, t - h), & t > 0; \\ \tilde{u}(x, 0) = \tilde{u}_0(x), & x \in (0, L); \\ \tilde{u}_x(0, t) = z_0(t), & t \in [-h, 0]. \end{cases} \quad (2.12)$$

Therefore, $u \in \mathcal{B}$ is a solution of (2.1) if and only if u is a fixed point of χ . Let

$$f(x, t) = -sat(au(x, t)).$$

From Proposition 2.4, if $u \in L^2(0, T, H^1(0, L))$, hence, $sat(au(x, t)) \in L^1(0, T, L^2(0, L))$. Thus $f(x, t) \in L^1(0, T, L^2(0, L))$. Consequently, from [2, Proposition 2], if (2.3) is satisfied, then there exists $C > 0$ such that

$$\begin{aligned} \|(u, z)\|_{\mathcal{C}([0, T], H)}^2 &\leq C (\|(u_0, z_0(-h \cdot))\|_H \\ &\quad + \|sat(au)\|_{L^1(0, T, L^2(0, L))}). \end{aligned} \quad (2.13)$$

and

$$\begin{aligned} \|u_x\|_{L^2(0,T,L^2(0,L))}^2 &\leq C \left(\|(u_0, z_0(-h\cdot))\|_H \right. \\ &\quad \left. + \|sat(au)\|_{L^1(0,T,L^2(0,L))} \right). \end{aligned} \quad (2.14)$$

Hence from (2.13), (2.14), and Lemma 2.3, we obtain

$$\begin{aligned} \|\chi(u)\|_B &= \|\tilde{u}\|_B \\ &\leq C \left(\|(u_0, z_0(-h\cdot))\|_H + \int_0^T \|sat(au)\|_{L^2(0,L)} dt \right) \\ &\leq C \left(\|(u_0, z_0(-h\cdot))\|_H + \int_0^T \|sat(au)\|_{L^2(0,L)} dt \right) \\ &\leq C \left(\|(u_0, z_0(-h\cdot))\|_H + 3 \int_0^T \|au\|_{L^2(0,L)} dt \right) \\ &\leq C \left(\|(u_0, z_0(-h\cdot))\|_H + 3a_1 \int_0^T \|u\|_{L^2(0,L)} dt \right) \\ &\leq C \left(\|(u_0, z_0(-h\cdot))\|_H + 3a_1 \sqrt{T} \sqrt{L} \|u\|_{L^1(0,T,L^2(0,L))}^2 \right) \\ &\leq C \left(\|(u_0, z_0(-h\cdot))\|_H + 3a_1 \sqrt{T} \sqrt{L} \|u\|_B^2 \right) \\ &\leq C 3a_1 \sqrt{T} \sqrt{L} \left(\|(u_0, z_0(-h\cdot))\|_H + \|u\|_B^2 \right) \end{aligned}$$

Therefore,

$$\|\chi(u)\|_B \leq K \left(\|(u_0, z_0(-h\cdot))\|_H + \|u\|_B^2 \right).$$

where $K = 3a_1 C \sqrt{T} \sqrt{L}$. Following the previous argument, we have

$$\begin{aligned} &\|\chi(u_1) - \chi(u_2)\|_B \\ &\leq C \left(\int_0^T \| -sat(au_1) + sat(au_1) \|_{L^2(0,L)} dt \right) \\ &\leq C \left(\int_0^T \|sat(au_1) - sat(au_1)\|_{L^2(0,L)} dt \right) \\ &\leq C \left(3a_1 \sqrt{T} \sqrt{L} \|u_1 - u_2\|_{L^1(0,T,L^2(0,L))} \right) \\ &\leq C \left(3a_1 \sqrt{T} \sqrt{L} \|u_1 - u_2\|_B \right) \\ &\leq C \left(3a_1 \sqrt{T} \sqrt{L} \|u_1 - u_2\|_B \right) + C (T(\|u_1\|_B + \|u_2\|_B) \|u_1 - u_2\|_B) \\ &\leq C \left(2TR + 3a_1 \sqrt{T} \sqrt{L} \right) \|u_1 - u_2\|_B \end{aligned}$$

We restricted χ to the closed ball $\{u \in \mathcal{B}; \|u\|_B \leq R\}$, where $R > 0$. Thus,

$$\|\chi(u)\|_B \leq K(r + R^2),$$

and

$$\|\chi(u_1) - \chi(u_2)\|_B \leq C \left(2TR + 3a_1 \sqrt{T} \sqrt{L} \right) \|u_1 - u_2\|_B$$

Therefore, let r and R so that $r < \frac{R}{2K}$ and $R < \frac{1}{2K}$, we get,

$$\|\chi(u)\|_B \leq R.$$

Moreover, taking T small enough such that $C \left(2TR + 3a_1\sqrt{T}\sqrt{L} \right) < 1$, we obtain,

$$\|\chi(u_1) - \chi(u_2)\|_{\mathcal{B}} < \|u_1 - u_2\|_{\mathcal{B}}.$$

Thus, the Banach fixed-point theorem can be applied and we deduce that the map χ has a unique fixed-point. Consequently, the nonlinear system (2.1) has a unique solution $u \in \mathcal{B}$. Furthermore, since $\text{sat}(a(x)u) \in L^1(0, T, L^2[0, L])$, hence if $\begin{pmatrix} u_0 \\ z_0(-h\cdot) \end{pmatrix} \in D(A)$, then the solution of (2.1) is a regular solution according to [2, Proposition 2] $\square$

3. EXPONENTIAL SATBILITY

The aim of this section is to prove the exponential stability of system (2.1). Before stating the principal result of this section, let us consider the subsequent energy

$$E(t) = \frac{1}{2} \int_0^L u^2(x, t) dx + \frac{|\beta|}{2} h \int_0^1 u_x^2(0, t - h\mu) d\mu. \quad (3.1)$$

The following Lemma proves that energy (3.1) does not increase.

Lemma 3.1. *Assume that (2.3) holds and $a = a(x) \in L^\infty[0, L]$ satisfying (1.4). Let $(u_0, z_0(-h\cdot)) \in D(A)$ and $u \in L^2(0, T, H^1[0, L])$. Then for any regular solution of (2.1), the energy (3.1) satisfies the following inequality*

$$\begin{aligned} \frac{d}{dt} E(t) &\leq \begin{pmatrix} u_x(0, t) \\ u_x(0, t - h) \end{pmatrix}^T \left(\frac{1}{2} M_1 \right) \begin{pmatrix} u_x(0, t) \\ u_x(0, t - h) \end{pmatrix} \\ &\leq 0. \end{aligned} \quad (3.2)$$

Proof. Let us consider a regular solution of (2.1). By definition $z(\mu, t) = u_x(0, t - h\mu)$, hence we rewrite the energy (3.1) as follows

$$E(t) = \frac{1}{2} \int_0^L u^2(x, t) dx + \frac{|\beta|}{2} h \int_0^1 z^2(\mu, t) d\mu.$$

Differentiating $E(\cdot)$, we get

$$\begin{aligned} \frac{d}{dt} E(t) &= \int_0^L u u_t dx + |\beta| h \int_0^1 z z_t d\mu \\ &= - \int_0^L u u_x dx - \int_0^L u u_{xxx} dx - \int_0^L \text{sat}(au) u dx \\ &\quad - |\beta| \int_0^1 z z_\mu d\mu. \end{aligned} \quad (3.3)$$

Following some integrations by parts, we have

$$- \int_0^L u u_x dx = 0, \quad (3.4)$$

$$\begin{aligned}
-\int_0^L uu_{xxx}dx &= \frac{1}{2}u_x^2(L, t) - \frac{1}{2}u_x^2(0, t) \\
&= \frac{1}{2}(\alpha u_x(0, t) + \beta z(1, t))^2 - \frac{1}{2}u_x^2(0, t)
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
-|\beta| \int_0^1 zz_\mu d\mu &= -\frac{|\beta|}{2} [z^2(\mu, t)]_0^1 \\
&= -\frac{|\beta|}{2} [z^2(1, t) - z^2(0, t)] \\
&= \frac{|\beta|}{2} u_x^2(0, t) - \frac{|\beta|}{2} z^2(1, t).
\end{aligned} \tag{3.6}$$

Using (3.3), (3.4), (3.5) and (3.6), we have

$$\begin{aligned}
\frac{d}{dt}E(t) &= \frac{1}{2}(\alpha u_x(0, t) + \beta z(1, t))^2 - \frac{1}{2}u_x^2(0, t) - \int_0^L sat(au)udx \\
&+ \frac{|\beta|}{2}u_x^2(0, t) - \frac{|\beta|}{2}z^2(1, t) \\
&= \frac{1}{2}\alpha^2 u_x^2(0, t) + \alpha\beta u_x(0, t)z(1, t) + \frac{1}{2}\beta^2 z^2(1, t) - \frac{1}{2}u_x^2(0, t) \\
&- \int_0^L sat(au)udx + \frac{|\beta|}{2}u_x^2(0, t) - \frac{|\beta|}{2}z^2(1, t) \\
&= \frac{1}{2}\left(\alpha^2 - 1 + \frac{|\beta|}{2}\right)u_x^2(0, t) + \alpha\beta u_x(0, t)z(1, t) \\
&+ \frac{1}{2}((\beta^2 - |\beta|))z^2(1, t) - \int_0^L sat(au)udx
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{d}{dt}E(t) &+ \begin{pmatrix} u_x(0, t) \\ z(1, t) \end{pmatrix}^T \left(-\frac{1}{2}M_1\right) \begin{pmatrix} u_x(0, t) \\ z(1, t) \end{pmatrix} \\
&= \frac{1}{2}\left(\alpha^2 - 1 + \frac{|\beta|}{2}\right)u_x^2(0, t) + \alpha\beta u_x(0, t)z(1, t) \\
&+ \frac{1}{2}(\beta^2 - |\beta|)z^2(1, t) - \int_0^L sat(au)udx \\
&+ \begin{pmatrix} u_x(0, t) \\ z(1, t) \end{pmatrix}^T \left(-\frac{1}{2}M_1\right) \begin{pmatrix} u_x(0, t) \\ z(1, t) \end{pmatrix} \\
&= -\int_0^L sat(au)udx \\
&\leq 0.
\end{aligned}$$

Because $\int_0^L sat(au)udx \geq 0$, indeed, if $\|au\|_{L^2} \leq 1$, then

$$sat(au)u = au^2 \geq 0.$$

If $\|au\|_{L^2} \geq 1$,

$$\text{sat}(au)u = \frac{au}{\|au\|_{L^2}}u = \frac{au^2}{\|au\|_{L^2}} \geq 0.$$

where $a = a(x)$ represents a non-negative function. which concludes the proof. $\square$

Consider the Lyapunov function given by

$$V(t) = E(t) + \lambda V_1(t) + \gamma V_2(t), \quad (3.7)$$

where

$$\begin{aligned} E(t) &= \frac{1}{2} \int_0^L u^2(x, t) dx + \frac{|\beta|}{2} h \int_0^1 u_x^2(0, t - h\mu) d\mu, \\ V_1(t) &= \int_0^L x u^2(x, t) dx, \end{aligned} \quad (3.8)$$

and

$$V_2(t) = h \int_0^1 (1 - \mu) u_x^2(0, t - h\mu) d\mu, \quad (3.9)$$

The subsequent lemmas play a significant role to prove exponential stability of the system (2.1).

Lemma 3.2. *We suppose that $a = a(x) \in L^\infty[0, L]$ satisfies (1.4), $(u_0, z_0(-h \cdot)) \in D(A)$ and $u \in L^2(0, T, H^1[0, L])$, then for any regular solution of (2.1), the following equation is satisfied*

$$\begin{aligned} \frac{d}{dt} V_1(t) &\leq L(\alpha^2 u_x^2(0, t) + 2\alpha\beta u_x(0, t)u_x(0, t - h) + \beta^2 u_x^2(0, t - h)) \\ &\quad + \int_0^L u^2 dx - 3 \int_0^L u_x^2 dx - 2 \int_0^L x \text{sat}(au) u dx. \end{aligned} \quad (3.10)$$

Proof. Let us consider a regular solution, then Differentiate $V_1(\cdot)$, we get

$$\begin{aligned} \frac{d}{dt} V_1(t) &= 2 \int_0^L x u_t u dx \\ &= -2 \int_0^L x u u_x dx - 2 \int_0^L x u u_{xxx} dx - 2 \int_0^L x \text{sat}(au) u dx. \end{aligned}$$

Following certain integrations by parts, we obtain

$$-2 \int_0^L x u u_x dx = \int_0^L u^2 dx;$$

and

$$\begin{aligned} -2 \int_0^L x u u_{xxx} dx &= L u^2(L, t) - 3 \int_0^L u_x^2 dx \\ &= L(\alpha u_x(0, t) + \beta u_x(0, t - h))^2 - 3 \int_0^L u_x^2 dx. \end{aligned}$$

Using the last equations, we get

$$\begin{aligned}
\frac{d}{dt}V_1(t) &= \int_0^L u^2 dx + L(\alpha u_x(0, t) + \beta u_x(0, t-h))^2 \\
&- 3 \int_0^L u_x^2 dx - 2 \int_0^L x \text{sat}(au) u dx \\
&= L(\alpha^2 u_x^2(0, t) + 2\alpha\beta u_x(0, t)u_x(0, t-h) + \beta^2 u_x^2(0, t-h)) \\
&+ \int_0^L u^2 dx - 3 \int_0^L u_x^2 dx - 2 \int_0^L x \text{sat}(au) u dx
\end{aligned}$$

□

Lemma 3.3. Suppose that $(u_0, z_0(-h \cdot)) \in D(A)$ and $u \in L^2(0, T, H^1[0, L])$, then for any regular solution of (2.1), the subsequent equation is satisfied

$$\frac{d}{dt}V_2(t) = - \int_0^1 u_x^2(0, t-h\mu) d\mu + u_x^2(0, t). \quad (3.11)$$

Proof. Consider a regular solution, then Differentiate $V_2(\cdot)$ and using integration by part, we have

$$\begin{aligned}
\frac{d}{dt}V_2(t) &= 2h \int_0^1 (1-\mu) u_x(0, t-h\mu) \partial_t u_x(0, t-h\mu) d\mu \\
&= -2 \int_0^1 (1-\mu) u_x(0, t-h\mu) \partial_\mu u_x(0, t-h\mu) d\mu \\
&= -[(1-\mu) u_x^2(0, t-h\mu)]_0^1 - \int_0^1 u_x^2(0, t-h\mu) d\mu \\
&= u_x^2(0, t) - \int_0^1 u_x^2(0, t-h\mu) d\mu.
\end{aligned} \quad (3.12)$$

Therefore

$$\frac{d}{dt}V_2(t) = - \int_0^1 u_x^2(0, t-h\mu) d\mu + u_x^2(0, t).$$

□

Now, we can present and demonstrate the primary result of this section.

Theorem 3.4. Assume that $a = a(x) \in L^\infty[0, L]$ satisfying (1.4), and $L < \pi\sqrt{3}$. Moreover suppose that the assumptions (2.3) is satisfied. Then there exists $r > 0$, such that for every $(u_0, z_0) \in \mathcal{H}$ satisfying $\|(u_0, z_0)\|_{\mathcal{H}} \leq r$, there exists $\delta > 0$ and $M > 0$ so that

$$E(t) \leq M e^{-2\delta t} E(0), \quad \forall t > 0. \quad (3.13)$$

where for λ and γ sufficiently small, the two positive constants δ and M satisfy the following inequality:

$$\delta \leq \min \left\{ \frac{(9\pi^2 - 3L^2 - 2L^{\frac{3}{2}} r \pi^2)}{3L^2(1 + 2L\lambda)} \lambda, \frac{\gamma}{h(2\gamma + |\beta|)} \right\} \quad (3.14)$$

and

$$M \leq 1 + \max \left\{ L\lambda, \frac{2\gamma}{|\beta|} \right\}.$$

Remark 3.5. The Lyapunov function $V(\cdot)$ and the energy $E(\cdot)$ are equivalent. Indeed,

$$E(t) \leq V(t) \leq M_1 E(t) \quad \forall t > 0, \quad (3.15)$$

where $M_1 = 1 + \max \left\{ L\lambda, \frac{2\gamma}{|\beta|} \right\} > 0$. Due to the inequality (3.15), to establish the exponential stability of system (2.1), it is sufficient to show that for all $\delta > 0$,

$$\frac{d}{dt}V(t) + 2\delta V(t) \leq 0.$$

Proof. Using Lemma 3.1 from (3.7), (3.11), we get

$$\begin{aligned} \frac{d}{dt}V(t) &\leq \frac{1}{2}Y^T M_1 Y + \lambda \int_0^L u^2 dx + \lambda L \alpha^2 u_x^2(0, t) \\ &\quad + 2L\lambda\alpha\beta u_x(0, t)u_x(0, t-h) + L\lambda\beta^2 u_x^2(0, t-h) \\ &\quad - 3\lambda \int_0^L u_x^2 dx - 2 \int_0^L x \text{sat}(au) u dx \\ &\quad - \gamma \int_0^1 u_x^2(0, t-h\mu) d\mu + \gamma u_x^2(0, t) \end{aligned} \quad (3.16)$$

Since, $x \in [0, L]$ and $\text{sat}(au)u \geq 0$, then $\int_0^L x \text{sat}(au) u dx \geq 0$. Therefore

$$\begin{aligned} \frac{d}{dt}V(t) &\leq Y^T \left[\frac{1}{2}M_1 + M_2 \right] Y + \lambda \int_0^L u^2 dx \\ &\quad - 3\lambda \int_0^L u_x^2 dx - \gamma \int_0^1 u_x^2(0, t-h\mu) d\mu, \end{aligned} \quad (3.17)$$

where $Y = \begin{pmatrix} u_x(0, t) \\ u_x(0, t-h\mu) \end{pmatrix}$ and $M_2 = \begin{pmatrix} L\lambda\alpha^2 + \gamma & L\lambda\alpha\beta \\ L\lambda\alpha\beta & L\lambda\beta^2 \end{pmatrix}$.

Now we calculate $2\delta V(t)$, we have

$$\begin{aligned} 2\delta V(t) &= 2\delta E(t) + 2\delta\lambda V_1(t) + 2\delta\gamma V_2(t) \\ &= \delta \int_0^L u^2 dx + \delta|\beta|h \int_0^1 u_x^2(0, t-h\mu) d\mu + 2\delta\lambda \int_0^L x u^2 dx \\ &\quad + 2\delta\gamma h \int_0^1 u_x^2(0, t-h\mu) d\mu - 2\delta\gamma h \int_0^1 \mu u_x^2(0, t-h\mu) d\mu \\ &\leq \delta \int_0^L u^2 dx + \delta|\beta|h \int_0^1 u_x^2(0, t-h\mu) d\mu \\ &\quad + 2\delta\lambda L \int_0^L u^2 dx + 2\delta\gamma h \int_0^1 u_x^2(0, t-h\mu) d\mu \end{aligned} \quad (3.18)$$

We know that the under the assumption (2.3), the matrix M_1 is negative definite. Subsequently, utilizing the continuity properties of the trace and determinant, we

infer that for sufficiently small λ and γ , the matrix $\frac{1}{2}M_1 + M_2$ is negative definite. Hence from (3.16) and (3.18), we deduce that

$$\begin{aligned}
\frac{d}{dt}V(t) + 2\delta V(t) &\leq Y^T \left[\frac{1}{2}M_1 Y + M_2 \right] - 3\lambda \int_0^L u_x^2 dx \\
&\quad + (\lambda + \delta + 2L\lambda\delta) \int_0^L u^2 dx \\
&\quad + (\delta|\beta|h + 2\gamma\delta h - \gamma) \int_0^1 u_x^2(0, t - h\mu) d\mu \\
&\leq -3\lambda \int_0^L u_x^2 dx + (\lambda + \delta + 2L\lambda\delta) \int_0^L u^2 dx \\
&\quad + (\delta|\beta|h + 2\gamma\delta h - \gamma) \int_0^1 u_x^2(0, t - h\mu) d\mu
\end{aligned} \tag{3.19}$$

By using the Poincaré inequality, we have

$$\begin{aligned}
\frac{d}{dt}V(t) + 2\delta V(t) &\leq \left(\frac{L^2}{\pi^2} (\lambda + \delta + 2L\lambda\delta) - 3\lambda \right) \int_0^L u_x^2 dx \\
&\quad + (\delta|\beta|h + 2\gamma\delta h - \gamma) \int_0^1 u_x^2(0, t - h\mu) d\mu
\end{aligned} \tag{3.20}$$

By assumption $L < \pi\sqrt{3}$, then according to [2], it is possible to choose r satisfying that $r < \frac{3(3\pi^2 - L^2)}{2L^{\frac{3}{2}}\pi^2}$. Consequently, we choose $\delta > 0$ so that (3.14) holds to achieve that $\frac{L^2}{\pi^2} (\lambda + \delta + 2L\lambda\delta) - 3\lambda \leq 0$ and $\delta|\beta|h + 2\gamma\delta h - \gamma(1 - d) \leq 0$.
therefore

$$\frac{d}{dt}V(t) + 2\delta V(t) \leq 0 \quad \forall t \geq 0,$$

hence we deduce

$$V(t) \leq Ce^{-2\delta t}V(0) \quad \forall t \geq 0.$$

Using (3.15), we get

$$E(t) \leq Ce^{-2\delta t}E(0) \quad \forall t \geq 0.$$

Using the density of $D(A)$, We conclude the proof by generalizing the result to any initial condition within $\mathcal{H}$.

□

4. CONCLUSION

In this paper, we proved the existence and uniqueness and the exponential stability of the linear KdV equation with time-delay on the boundary feedback with saturated source term. The well-posedness is obtained under some conditions. The proof of the stabilization result is essentially based on the use of an appropriate Lyapunov functional with an estimate of the decay rate, but the length of the spatial domain L , satisfies the condition $L < \pi\sqrt{3}$.

The exponential stability of the Korteweg-de Vries equation with time-delay on boundary feedback in the presence of cone-bounded nonlinearity as a source term may be a potential area for future research.

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