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THE RESOLVENT OPERATOR OF SINGULAR CONFORMABLE FRACTIONAL DIRAC SYSTEM

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ABSTRACT. We discuss the resolvent operator of a singular conformable fractional Dirac system. We obtain integral representations for the resolvent of this system in terms of the spectral function.

1. INTRODUCTION

Today conformable fractional calculus is one of the most actual fields of fractional calculus, since conformable fractional derivative allows for many extensions of classical properties in ordinary calculus. It was initiated with the work [20] in which the authors defined a conformable fractional derivative. After that, the right and left conformable fractional derivatives, the fractional chain rule and fractional integrals of higher orders have been achieved by Abdeljawad [1]. Many recent results concerned with conformable fractional calculus are achieved (see [1-4, 10-20, 25-26]).

In 1910, H. Weyl [30] first proved the integral representation of the resolvent for the singular Sturm-Liouville equation. Similar theorems were proved in [28, 6, 7, 8]. The integral representation of the resolvent for the classical Dirac systems was obtained by B. M. Levitan and I. S. Sargsjan (see [23]).

On the other hand, analysis of the resolvent of (1.1) is important not only for the sake of further development of the operator theory, but for practical reasons too. In fact, the study of Dirac systems has become an important area of research due to the fact that such systems arise in a variety of quantum physics problems such as in the study of the existence of antimatter, a description of the electron spin: see Thaller [27]. Recently, in [18], Gulsen et al. studied the conformable fractional Dirac system with separated boundary conditions on an arbitrary time scale $\mathbb{T}$. They extended some basic spectral properties of the classical Dirac system to the conformable fractional case. In [9], the authors studied the conformable fractional Dirac system (1.1). They proved an existence and uniqueness theorem for this system and formulated a self-adjoint boundary value problem.

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In this paper, we are concerned with the resolvent operator of a conformable fractional (CF) Dirac system

$$\begin{pmatrix} 0 & -T_\alpha \\ T_\alpha & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{1.1}$$

where $p(.)$ and $r(.)$ are real-valued and conformable fractional locally integrable functions on $[0, \infty)$. We will investigate the integral representation of the resolvent operator of this system.

The rest of this paper is organized as follows. In Section 2, we present a brief background of the conformable fractional calculus and operator theory. In Section 3, we consider the CF-Dirac system. In Section 4, we study the resolvent of the CF-Dirac system. Finally, we give the integral representation of the resolvent CF-Dirac operator in terms of the spectral function in Section 5.

2. Preliminary facts

In this section, we give the basic concepts and necessary results from the conformable fractional calculus and operator theory. For more details, the reader may want to consult [1], [20], [24], [22]. Throughout this paper, we will fix $\alpha \in (0, 1)$.

Definition 2.1 ([20])**.** A function $f : [0, \infty) \to \mathbb{R}$ the conformable fractional derivative of order α of f at $x > 0$ was defined by

$$T_\alpha f(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}, \tag{1}$$

and

the fractional derivative at 0 is defined

$$(T_\alpha f)(0) = \lim_{x \to 0^+} (T_\alpha f)(x).$$

Definition 2.2 ([20])**.** The left conformable fractional derivative starting from a of a function $f : [a, \infty) \to \mathbb{R}$ of order α is defined by

$$(T_\alpha^a f)(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon (x - a)^{1-\alpha}) - f(x)}{\varepsilon}, \ 0 < \alpha \leq 1. \tag{2}$$

When $a = 0$ we write T_α. If $(T_\alpha f)(x)$ exists on (a, b) then

$$(T_\alpha^a f)(a) = \lim_{x \to a^+} (T_\alpha^a f)(x).$$

Definition 2.3 ([20])**.** The right conformable fractional derivative of f is defined by

$$({}_\alpha^b T f)(x) = -\lim_{\varepsilon \to 0} \frac{f(x + \varepsilon (b - x)^{1-\alpha}) - f(x)}{\varepsilon}, \tag{3}$$

where f is terminating at b and $({}^b T_\alpha f)(x) = \lim_{x \to b^-} ({}^b T_\alpha f)(x)$.

In what follows, we present some important properties of the conformable derivative.

Lemma 2.4 ([20])**.** *Let f, g be conformable differentiable of order α at a point x. Then*

(i) $T_\alpha(\lambda f+\beta g)=\lambda T_\alpha(f)+\beta T_\alpha(g),\ \lambda,\beta\in\mathbb{R}$,
(ii) $T_\alpha(fg)=fT_\alpha(g)+gT_\alpha(f)$,
(iii) $T_\alpha\left(\frac{f}{g}\right)=\frac{gT_\alpha(f)-fT_\alpha(g)}{g^2}$,
(iv) f *is differentiable then* $T_\alpha^a(f)(x)=(x-a)^{1-\alpha}f'(x)$.
(v) $T_\alpha(x^n)=nx^{n-\alpha}$, *for all* $n\in\mathbb{R}$.

The concept of the conformable fractional integral of the function f can be defined as follows.

Definition 2.5 ([1])**.** The conformable fractional integral starting from a of a function f is defined by

$$\left(I_\alpha^a f\right)(x)=\int_a^x f(t)d\alpha(t,a)=\int_a^x (t-a)^{\alpha-1}f(t)dt.$$

Similarly, in the right case we have

$$\left({}^bI_\alpha f\right)(x)=\int_x^b f(t)d\alpha(b,t)=\int_x^b (b-t)^{\alpha-1}f(t)dt.$$

Lemma 2.6 ([1])**.** *Assume that* f *is a continuous function on* $[a,\infty)$*. Then, we have*

$$T_\alpha^a I_\alpha^a f(x)=f(x),$$

for all $x>a$.

Theorem 2.7 ([1])**.** *Let* $f,g:[a,b]\to\mathbb{R}$ *be two functions such that* f *and* g *are* α*-differentiable. Then, we have*

$$\int_a^b f(x)T_\alpha^a(g)(x)\,d\alpha(x,a)+\int_a^b g(x)\,T_\alpha^a(f)(x)\,d\alpha(x,a)$$

$$=f(b)\,g(b)-f(a)\,g(a).$$

Let $L_\alpha^2(0,\infty)$ be the space of all complex-valued functions defined on $[0,\infty)$ such that

$$\|f\|:=\left(\int_0^\infty |f(x)|^2\,d\alpha(x)\right)^{1/2}=\left(\int_0^\infty |f(x)|^2\,x^{\alpha-1}dx\right)^{1/2}<\infty.$$

The space $L_\alpha^2(0,\infty)$ is a Hilbert space (see [29]) with the inner product

$$(f,g):=\int_0^\infty f(x)\,\overline{g(x)}d\alpha(x),\ \ f,g\in L_\alpha^2(0,\infty).$$

Now we define the *Wronskian* of y and z by

$$W(y,z)(x)=y_1(x)z_2(x)-z_1(x)y_2(x), \tag{2.1}$$

where

$$y(x)=\begin{pmatrix} y_1(x)\\ y_2(x)\end{pmatrix},\ z(x)=\begin{pmatrix} z_1(x)\\ z_2(x)\end{pmatrix},\ x\in[0,\infty).$$

Definition 2.8. A matrix valued function $M(x,t)$ in $E:=\mathbb{C}^2$ of two variables with $0\leq x,t\leq b$ is called the $\alpha-$Hilbert-Schmidt kernel if

$$\int_0^b\int_0^b \|M(x,t)\|_E^2\, d\alpha(x)\, d\alpha(t)<\infty.$$

Theorem 2.9. *([24]) If*

$$\sum_{i,k=1}^{\infty}|a_{ik}|^2<\infty, \tag{2.2}$$

then the operator A defined by the formula

$$A\{y_i\}=\{z_i\},$$

where

$$z_i=\sum_{k=1}^{\infty}a_{ik}y_k,\, i=1,2,... \tag{2.3}$$

is compact in the sequence space l^2.

Definition 2.10. A function f defined on an interval $[a,b]$ is said to be of *bounded variation* if there is a constant $C>0$ such that

$$\sum_{k=1}^{n}|f(x_k)-f(x_{k-1})|\leq C$$

for every partition

$$a=x_0<x_1<...<x_n=b \tag{2.4}$$

of $[a,b]$ by points of subdivision $x_0,x_1,...,x_n$ (see [22]).

Definition 2.11. Let f be a function of bounded variation. Then, by the *total variation* of f on $[a,b]$, denoted by $\overset{b}{\underset{a}{V}}(f)$, we mean the quantity

$$\overset{b}{\underset{a}{V}}(f):=\sup\sum_{k=1}^{n}|f(x_k)-f(x_{k-1})|,$$

where the least upper bound is taken over all (finite) partitions (2.4) of the interval $[a,b]$ (see [22]).

Now, we recall that the following well-known theorems of Helly's.

Theorem 2.12. *([22]) Let $(w_n)_{n\in\mathbb{N}}$ be a uniformly bounded sequence of real nondecreasing function on a finite interval $a\leq\lambda\leq b$. Then there exists a subsequence $(w_{n_k})_{k\in\mathbb{N}}$ and a nondecreasing function w such that*

$$\lim_{k\to\infty}w_{n_k}(\lambda)=w(\lambda),\ a\leq\lambda\leq b.$$

Theorem 2.13. *([22]) Assume $(w_n)_{n\in\mathbb{N}}$ is a real, uniformly bounded, sequence of nondecreasing function on a finite interval $a\leq\lambda\leq b$, and suppose*

$$\lim_{n\to\infty}w_n(\lambda)=w(\lambda),\ a\leq\lambda\leq b.$$

If f is any continuous function on $a \leq \lambda \leq b$, then

$$\lim_{n\to\infty}\int_a^b f(\lambda)\,dw_n(\lambda) = \int_a^b f(\lambda)\,dw(\lambda).$$

3. The CF-Dirac system

Let us consider the regular CF-Dirac system

$$-T_\alpha y_2 + (p(x) - \lambda)\,y_1 = 0 \tag{3.1}$$

$$T_\alpha y_1 + (r(x) - \lambda)\,y_2 = 0 \tag{3.2}$$

with the boundary conditions

$$y_2(0,\lambda)\cos\alpha + y_1(0,\lambda)\sin\alpha = 0, \tag{3.3}$$

$$y_2(\zeta,\lambda)\cos\beta + y_1(\zeta,\lambda)\sin\beta = 0,\ \ \alpha,\beta\in\mathbb{R},\ \ \zeta\in(0,\infty), \tag{3.4}$$

where λ is a complex eigenvalue parameter, p and r are real-valued functions defined on $[0,\infty)$ and $p, r \in L^1_{\alpha,loc}(0,\infty)$, where

$$L^1_{\alpha,loc}(0,\infty) := \left\{ f : [0,\infty) \to \mathbb{C} : \int_0^b |f(x)|\,d\alpha(x) < \infty,\ \forall b \in (0,\infty) \right\}.$$

We will denote by $\phi(x,\lambda) = \begin{pmatrix} \phi_1(x,\lambda) \\ \phi_2(x,\lambda) \end{pmatrix}$ and $\psi(x,\lambda) = \begin{pmatrix} \psi_1(x,\lambda) \\ \psi_2(x,\lambda) \end{pmatrix}$, the solution of the system (3.1)-(3.2) which satisfy the initial conditions

$$\phi_1(\zeta,\lambda) = \sin\alpha,\ \ \phi_2(\zeta,\lambda) = \cos\alpha,\ \ \psi_1(0,\lambda) = \cos\alpha,\ \ \psi_2(0,\lambda) = -\sin\alpha. \tag{3.5}$$

In order to go over from system to operators, we introduce convenient Hilbert space $\mathcal{H} := L^2_\alpha((0,\zeta);E)$ of vector-valued functions using the inner product

$$(f,g) := \int_0^\infty (f(x),g(x))_E\,d\alpha(x),$$

where $(.,.)_E$ denotes the standard inner product in E :

$$(\xi,\gamma)_E = \sum_{j=1}^{2} \xi_j \overline{\gamma}_j.$$

Now, we will denote by $\phi(x,\lambda) + m_\zeta(\lambda)\,\psi(x,\lambda)$ the solution of the equation (3.1)-(3.2) which satisfy the boundary condition

$$(\phi_1(\zeta,\lambda) + m_\zeta(\lambda)\,\psi_1(\zeta,\lambda))\sin\beta$$

$$+(\phi_2(\zeta,\lambda) + m_\zeta(\lambda)\,\psi_2(\zeta,\lambda))\cos\beta = 0.$$

Then, $m_\zeta(\lambda)$ satisfies the relation

$$m_\zeta(\lambda) = -\frac{\phi_1(\zeta,\lambda) + \phi_2(\zeta,\lambda)\cot\beta}{\psi_1(\zeta,\lambda) + \psi_2(\zeta,\lambda)\cot\beta}.$$

It is clear that $m_\zeta(\lambda)$ is a meromorphic function of λ, since $\phi(x,\lambda)$ and $\psi(x,\lambda)$ are entire functions of λ. Furthermore, since the eigenvalues of the regular problem are real, all poles of $m_\zeta(\lambda)$ are real and simple. The function m_ζ is called the

Titchmarsh-Weyl function of the regular problem (3.1)-(3.4). If $\cot\beta$ is replaced by a complex variable z, then we have

$$m_{\zeta}(\lambda, z) = -\frac{\phi_1(\zeta,\lambda) + \phi_2(\zeta,\lambda) z}{\psi_1(\zeta,\lambda) + \psi_2(\zeta,\lambda) z}. \tag{3.6}$$

For every λ, the equality (3.6) is a one-to-one conformal mapping in z, which follows from the theory of Möbius transformations [21]. Hence, if $\operatorname{Im}\lambda \neq 0$, then $m_{\zeta}(\lambda, z)$ varies on a circle $C_{\zeta}(\lambda)$ with a finite radius in the m_{ζ}-plane as z varies over the real axis of the z-plane.

Using this notation we now state the result from [5].

Theorem 3.1. *Let $\phi(x,\lambda), \psi(x,\lambda)$ be two linearly independent solution of the system (3.1)-(3.2) satisfying the initial conditions (3.5) Then, the solution*

$$\sigma(x,\lambda) = \phi(x,\lambda) + m_{\zeta}(\lambda)\psi(x,\lambda)$$

satisfies the boundary condition

$$(\phi_1(\zeta,\lambda) + m_{\zeta}(\lambda)\psi_1(\zeta,\lambda))\sin\beta$$

$$+(\phi_2(\zeta,\lambda) + m_{\zeta}(\lambda)\psi_2(\zeta,\lambda))\cos\beta = 0$$

if and only if $m_{\zeta}(\lambda)$ is on C_{ζ} with

$$\lim_{\zeta\to\infty} W(\sigma,\overline{\sigma})(\zeta,\lambda) = 0.$$

If $\zeta \to \infty$, then C_{ζ} tends either to limit-circle C_{∞} or to the limit-point m_{∞}. In the first case, all solutions of the system are in the space $\mathcal{H}$. In the second case, if $\operatorname{Im}\lambda \neq 0$, one linearly independent solution is in the space $\mathcal{H}$. In the limit-circle case, a point is on C_{∞} if and only if

$$\lim_{\zeta\to\infty} W(\sigma,\overline{\sigma})(\zeta,\lambda) = 0.$$

The function $m(\lambda) := \lim_{\zeta\to\infty} m_{\zeta}(\lambda)$ is called the *Titchmars-Weyl* function, and $\omega(x,\lambda) := \phi(x,\lambda) + m(\lambda)\psi(x,\lambda)$ is called the *Weyl solution* of the singular system.

Lemma 3.2. *Let us define*

$$\omega_{\zeta}(x,\lambda) := \phi(x,\lambda) + m_{\zeta}(\lambda)\psi(x,\lambda),\ x \in (0,\infty). \tag{3.7}$$

Then, for each nonreal λ, we have

$$\omega_{\zeta}(x,\lambda) \to \omega(x,\lambda),\ \zeta \to \infty,$$

$$\int_0^{\zeta} \|\omega_{\zeta}(x,\lambda)\|_E^2\, d\alpha(x) \to \int_0^{\infty} \|\omega(x,\lambda)\|_E^2\, d\alpha(x),\ \zeta \to \infty.$$

Proof. It is immediate that

$$\omega_{\zeta}(x,\lambda) = \omega(x,\lambda) + \{m_{\zeta}(\lambda) - m(\lambda)\}\psi(x,\lambda),$$

where $\omega(x,\lambda) \in \mathcal{H}$ and $m_\zeta(\lambda)$ is a point of the circle. According to [23], [5]

$$|m_\zeta(\lambda) - m(\lambda)| \leq 2r_\zeta(\lambda) = \left[|v| \left(\int_0^\zeta \|\omega_\zeta(x,\lambda)\|_E^2 \, d\alpha(x) \right) \right]^{-1}, \ \operatorname{Im}\lambda = v \neq 0.$$

Since $r_\zeta(\lambda) \to 0$, we have $\omega_\zeta(x,\lambda) \to \omega(x,\lambda)$ $(\zeta \to \infty)$.

Moreover, we get

$$\int_0^\zeta \|\{m_\zeta(\lambda) - m(\lambda)\} \psi(x,\lambda)\|_E^2 \, d\alpha(x)$$

$$= |m_\zeta(\lambda) - m(\lambda)|^2 \int_0^\zeta \|\psi(x,\lambda)\|_E^2 \, d\alpha(x)$$

$$\leq \left(|v|^2 \int_0^\zeta \|\psi(x,\lambda)\|_E^2 \, d\alpha(x) \right)^{-1}.$$

Then, we have

$$\int_0^\zeta \|\omega_\zeta(x,\lambda)\|_E^2 \, d\alpha(x) \to \int_0^\infty \|\omega(x,\lambda)\|_E^2 \, d\alpha(x), \ \zeta \to \infty.$$

□

4. Construction of the resolvent operator

Here, we shall construct a resolvent operator for the regular CF-Dirac system

$$-T_\alpha y_2 + p(x) y_1 = \lambda y_1 + f_1(x) \tag{4.1}$$

$$T_\alpha y_1 + r(x) y_2 = \lambda y_2 + f_2(x) \tag{4.2}$$

with the boundary conditions (3.3)-(3.4). Later, we will show that this system has a compact resolvent, thus it has a purely discrete spectrum.

Let us introduce the resolvent operator for the regular CF-Dirac system (3.1)-(3.4) as

$$(R_\zeta f)(x,\lambda) = y(x,\lambda) = \int_0^\zeta G_\zeta(x,t,\lambda) f(t) \, d\alpha(t), \ \lambda \in \mathbb{C}, \tag{4.3}$$

where

$$f(.) = \begin{pmatrix} f_1(.) \\ f_2(.) \end{pmatrix} \in L_\alpha^2((0,\zeta);\mathbb{R}^2),$$

and

$$G_\zeta(x,t,\lambda) = \begin{cases} \omega_\zeta(x,\lambda)\psi^T(t,\lambda), & t \leq x \\ \psi(x,\lambda)\omega_\zeta^T(t,\lambda), & t > x \end{cases}$$

$$= \begin{cases} \begin{pmatrix} \omega_{\zeta 1}(x,\lambda)\psi_1(t,\lambda) & \omega_{\zeta 1}(x,\lambda)\psi_2(t,\lambda) \\ \omega_{\zeta 2}(x,\lambda)\psi_1(t,\lambda) & \omega_{\zeta 2}(x,\lambda)\psi_2(t,\lambda) \end{pmatrix}, & t \leq x \\ \begin{pmatrix} \psi_1(x,\lambda)\omega_{\zeta 1}(t,\lambda) & \psi_1(x,\lambda)\omega_{\zeta 2}(t,\lambda) \\ \psi_2(x,\lambda)\omega_{\zeta 1}(t,\lambda) & \psi_2(x,\lambda)\omega_{\zeta 2}(t,\lambda) \end{pmatrix}, & x < t. \end{cases} \tag{4.4}$$

Now, we will show that the resolvent operator satisfies the system (4.1)-(4.2) and the boundary conditions (3.3)-(3.4).

It follows now from (4.4) that

$$G_{\zeta}(x,t,\lambda) f(t) = \begin{cases} \begin{pmatrix} \omega_{\zeta 1}(x,\lambda)\psi_1(t,\lambda) f_1(t) + \omega_{\zeta 1}(x,\lambda)\psi_2(t,\lambda) f_2(t) \\ \omega_{\zeta 2}(x,\lambda)\psi_1(t,\lambda) f_1(t) + \omega_{\zeta 2}(x,\lambda)\psi_2(t,\lambda) f_2(t) \end{pmatrix}, & t \le x \\ \begin{pmatrix} \psi_1(x,\lambda)\omega_{\zeta 1}(t,\lambda) f_1(t) + \psi_1(x,\lambda)\omega_{\zeta 2}(t,\lambda) f_2(t) \\ \psi_2(x,\lambda)\omega_{\zeta 1}(t,\lambda) f_1(t) + \psi_2(x,\lambda)\omega_{\zeta 2}(t,\lambda) f_2(t) \end{pmatrix}, & x < t. \end{cases}$$

From (4.3), we conclude that

$$y_1(x,\lambda) = \omega_{\zeta 1}(x,\lambda)\int_0^x (\psi_1(t,\lambda) f_1(t) + \psi_2(t,\lambda) f_2(t))\, d\alpha(t) + \psi_1(x,\lambda)\int_x^{\zeta} (\omega_{\zeta 1}(t,\lambda) f_1(t) + \omega_{\zeta 2}(t,\lambda) f_2(t))\, d\alpha(t), \tag{4.5}$$

$$y_2(x,\lambda) = \omega_{\zeta 2}(x,\lambda)\int_0^x (\psi_1(t,\lambda) f_1(t) + \psi_2(t,\lambda) f_2(t))\, d\alpha(t) + \psi_2(x,\lambda)\int_x^{\zeta} (\omega_{\zeta 1}(t,\lambda) f_1(t) + \omega_{\zeta 2}(t,\lambda) f_2(t))\, d\alpha(t). \tag{4.6}$$

It follows now from (4.5) that

$$T_\alpha y_1(x,\lambda)$$

$$= T_\alpha \omega_{\zeta 1}(x,\lambda)\int_0^x (\psi_1(t,\lambda) f_1(t) + \psi_2(t,\lambda) f_2(t))\, d\alpha(t)$$

$$+ T_\alpha \psi_1(x,\lambda)\int_x^\zeta (\omega_{\zeta 1}(t,\lambda) f_1(t) + \omega_{\zeta 2}(t,\lambda) f_2(t))\, d\alpha(t)$$

$$+ W(\psi,\omega_\zeta) f_2(x) = \{\lambda - r(x)\}\,\omega_{\zeta 2}(x,\lambda)$$

$$\times \int_0^x (\psi_1(t,\lambda) f_1(t) + \psi_2(t,\lambda) f_2(t))\, d\alpha(t) + \{\lambda - r(x)\}\,\psi_2(x,\lambda)$$

$$\times \int_x^\zeta (\omega_{\zeta 1}(t,\lambda) f_1(t) + \omega_{\zeta 2}(t,\lambda) f_2(t))\, d\alpha(t) + f_2(x)$$

$$= \{\lambda - r(x)\}\,\omega_{\zeta 2}(x,\lambda)$$

$$\times \int_0^x (\psi_1(t,\lambda) f_1(t) + \psi_2(t,\lambda) f_2(t))\, d\alpha(t)$$

$$\{\lambda - r(x)\}\,\psi_2(x,\lambda)$$

$$\times \int_x^\zeta (\Omega_{\zeta 1}(t,\lambda) f_1(t) + \Omega_{\zeta 2}(t,\lambda) f_2(t))\, d\alpha(t) + f_2(x)$$

$$= \{\lambda - r(x)\}\, y_2(x) + f_2(x).$$

The validity of (4.1) is proved similarly. Hence the vector-valued function

$$y(x,\lambda) = \begin{pmatrix} y_1(x,\lambda) \\ y_2(x,\lambda) \end{pmatrix}$$

in (4.3) is the solution of the system (4.1)-(4.2). We check at once that (4.3) satisfies the boundary conditions (3.3)-(3.4).

In the following results, without restriction of generality, we assume that $\lambda = 0$ is not an eigenvalue of the problem (4.1)-(4.2), (3.3)-(3.4).

Theorem 4.1. *The matrix $G(x,t)$ defined by*

$$G(x,t) = G(x,t,0) = \begin{cases} \frac{\omega(x)\psi^T(t)}{W(\psi,\omega)}, & 0 \le t < x \\ \frac{\psi(x)\omega^T(t)}{W(\psi,\omega)}, & x < t \le \zeta \end{cases} \tag{4.7}$$

is a α-Hilbert-Schmidt kernel.

Proof. By the upper half of the formula (4.7), we have

$$\int_0^\zeta d\alpha(x) \int_0^x \|G(x,t)\|_E^2 \, d\alpha(t) < \infty;$$

and by the lower half of (4.7), we have

$$\int_0^\zeta d\alpha(x) \int_x^\zeta \|G(x,t)\|_E^2 \, d\alpha(t) < \infty$$

since the inner integral exists and is a linear combination of the products $\psi_i(x)\,\omega_j(t)$ $(i,j = 1,2)$, and these products belong to $\mathcal{H} \times \mathcal{H}$ because each of the factors belongs to $\mathcal{H}$. Hence, we conclude that

$$\int_0^\zeta \int_0^\zeta \|G(x,t)\|_E^2 \, d\alpha(x)\, d\alpha(t) < \infty, \tag{4.8}$$

i.e., $G(x,t)$ is a α-Hilbert-Schmidt kernel. □

We present the following result which will be essential to our purposes.

Theorem 4.2. *The operator $\mathcal{K}$ defined by the formula*

$$g(x) := (\mathcal{K}f)(x) = \int_0^\zeta G(x,t)\, f(t)\, d\alpha(t)$$

is compact and self-adjoint in space $\mathcal{H}$.

Proof. Let $\theta_i = \theta_i(x)$ $(i = 1,2,3,...)$ be a complete, orthonormal basis of $\mathcal{H}$. Since $G(x,t)$ is a Hilbert-Schmidt kernel, we can define

$$x_i = (f, \theta_i) = \int_0^\zeta (f(t), \theta_i(t))_E d\alpha(t),$$

$$y_i = (g, \theta_i) = \int_0^\zeta ((g(t), \theta_i(t))_E d\alpha(t),$$

$$a_{ik} = \int_0^\zeta \int_0^\zeta (G(x,t)\,\theta_i(x), \theta_k(t))_E d\alpha(x)\, d\alpha(t).$$

Then, $\mathcal{H}$ is mapped isometrically l^2. By this mapping, our integral operator transforms into the operator defined by the formula (2.3) in the space l^2. The condition (4.8) is also translated into the condition (2.2). By virtue of Theorem 2.9, this operator is compact. Therefore, the original operator is compact.

Let $g, h \in \mathcal{H}$. As $G(x,t) = G^T(t,x)$ and $G(x,t)$ is a matrix-valued function in E defined on $[0,\zeta] \times [0,\zeta]$, we have

$$(\mathcal{K}g, h) = \int_0^\zeta (\mathcal{K}g(x), h(x))_E d\alpha(x)$$

$$= \int_0^\zeta \left(\int_0^\zeta G(x,t)\, g(t) d\alpha(t), h(x) \right)_E d\alpha(x)$$

$$= \int_0^\zeta \left(g(t), \int_0^\zeta G(x,t)\, h(x)\, d\alpha(x) \right)_E d\alpha(t) = (g, \mathcal{K}h).$$

This completes the proof. □

5. Integral representation of the resolvent operator

In this section, we will give the integral representation of resolvent operator of the CF-Dirac system.

Let $\lambda_{m,\zeta}$ and

$$\psi_{m,\zeta}(x) = \begin{pmatrix} \psi_{m,\zeta 1}(x) \\ \psi_{m,\zeta 2}(x) \end{pmatrix} := \begin{pmatrix} \psi_{\zeta 1}(x, \lambda_m) \\ \psi_{\zeta 2}(x, \lambda_m) \end{pmatrix}$$
$$(m \in \mathbb{Z} := \{0, \pm 1, \pm 2, ...\}$$

be the eigenvalues and eigenfunctions of the boundary-value problem (3.1)-(3.4) and

$$\alpha_{m,\zeta}^2 = \int_0^\zeta \|\psi_{m,\zeta}(x)\|_E^2\, d\alpha(x).$$

If

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}, \quad \int_0^\zeta \left(f_1^2(x) + f_2^2(x)\right) d\alpha(x) < \infty,$$

i.e., $f(.) \in L_\alpha^2\left((0,\zeta); \mathbb{R}^2\right)$. Then it follows from Theorem 4.1 and the Hilbert-Schmidt theorem ([22]) we have

$$\int_0^\zeta \left(f_1^2(x) + f_2^2(x)\right) d\alpha(x)$$
$$= \sum_{m=-\infty}^{\infty} \frac{1}{\alpha_{m,\zeta}^2} \left\{ \int_0^\zeta f^T(x)\, \psi_{m,\zeta}(x)\, d\alpha(x) \right\}^2 \tag{5.1}$$

which is called *Parseval equality*.

Now, define the nondecreasing step function ϱ_ζ on $[0,\infty)$ by

$$\varrho_\zeta(\lambda) = \begin{cases} -\sum_{\lambda < \lambda_{m,\zeta} < 0} \frac{1}{\alpha_{m,\zeta}^2}, & \text{for } \lambda \le 0 \\ \sum_{0 \le \lambda_{m,\zeta} < \lambda} \frac{1}{\alpha_{m,\zeta}^2} & \text{for } \lambda \ge 0. \end{cases}$$

Then the Parsaval equality (5.1) can be written as

$$\int_0^{\zeta}\left(f_1^2(x)+f_2^2(x)\right)d\alpha(x)=\int_{-\infty}^{\infty}F^2(\lambda)\,d\varrho_\zeta(\lambda),$$

where

$$F(\lambda)=\int_0^{\zeta}f^T(x)\,\psi(x,\lambda)\,d\alpha(x).$$

Lemma 5.1. *For any positive h, there is a positive constant $M=M(h)$ not depending on ζ such that*

$$\overset{h}{\underset{-h}{V}}\{\varrho_\zeta(\lambda)\}=\sum_{-h\leq\lambda_{m,\zeta}<h}\frac{1}{\alpha_{m,\zeta}^2}=\varrho_\zeta(h)-\varrho_\zeta(-h)<M. \tag{5.2}$$

Proof. Let $\sin\alpha\neq0$. Since $\psi_2(x,\lambda)$ is continuous at zero. By condition $\psi_2(x,\lambda)=-\sin\alpha$, there is a positive number δ and near by 0 such that

$$\frac{1}{\delta}\left(\int_0^{\delta}\psi_2(x,\lambda)\,d\alpha(x)\right)^2>\frac{1}{2}\sin^2\alpha. \tag{5.3}$$

Let us define

$$f_\delta(x)=\begin{pmatrix}f_{1\delta}(x)\\ f_{2\delta}(x)\end{pmatrix}$$

by

$$f_{1\delta}(x)=0,\;\; f_{2\delta}(x)=\begin{cases}\frac{1}{\delta}, & 0\leq x\leq\delta\\ 0, & x>\delta.\end{cases}$$

It follows from the Parseval equality and (5.3) that

$$\int_0^{\delta}\left(f_{1\delta}^2(x)+f_{2\delta}^2(x)\right)d\alpha(x)=\frac{1}{\delta}=\int_{-\infty}^{\infty}\left(\frac{1}{\delta}\int_0^{\delta}\psi_2(x,\lambda)\,d\alpha(x)\right)^2d\varrho_\zeta(\lambda)$$

$$\geq\int_{-h}^{h}\left(\frac{1}{\delta}\int_0^{\delta}\psi_2(x,\lambda)\,d\alpha(x)\right)^2d\varrho_\zeta(\lambda)$$

$$>\frac{1}{2}\sin^2\alpha\left\{\varrho_\zeta(h)-\varrho_\zeta(-h)\right\}.$$

Hence

$$\left\{\varrho_\zeta(h)-\varrho_\zeta(-h)\right\}<\frac{2\delta}{\sin^2\alpha}=M.$$

This proves the lemma.

If $\sin\alpha=0$, then we define the function $f_\delta(x)=\begin{pmatrix}f_{1\delta}(x)\\ f_{2\delta}(x)\end{pmatrix}$ as

$$f_{1\delta}(x)=\begin{cases}\frac{1}{\delta^2}, & 0\leq x\leq\delta\\ 0, & x>\delta\end{cases},\;\; f_{2\delta}(x)=0.$$

Applying the Parseval equality, we get the desires result. □

Now, we shall obtain an expansion into a Fourier series of resolvent if one knows the expansion of the function $f(.)$. By conformable fractional integration by parts, we get

$$\int_0^{\zeta}\left[\begin{pmatrix} 0 & -T_\alpha \\ T_\alpha & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}+\begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right]^T$$

$$\times \psi_{m,\zeta}(x)\, d\alpha(x)$$

$$= \int_0^{\zeta}\left[-T_\alpha y_2 + p(x) y_1\right]\psi_{m,\zeta 1}(x)\, d\alpha(x)$$

$$+ \int_0^{\zeta}\left[T_\alpha y_1 + p(x) y_2\right]\psi_{m,\zeta 2}(x)\, d\alpha(x)$$

$$= \int_0^{\zeta}\left[-T_\alpha \psi_{m,\zeta 2}(x) + p(x) y_1 \psi_{m,\zeta 1}(x)\right] y_2 d\alpha(x)$$

$$+ \int_0^{\zeta}\left[T_\alpha \psi_{m,\zeta 1}(x) + p(x) \psi_{m,\zeta 2}(x)\right] y_1 d\alpha(x)$$

$$= \lambda_{m,\zeta}\int_0^{\zeta}\psi_{m,\zeta}^T(x)\, y(x,\lambda)\, d\alpha(x) = \lambda_{m,\zeta} t_m(\lambda). \tag{5.4}$$

Set

$$y(x,\lambda) = \sum_{m=-\infty}^{\infty} t_m(\lambda)\, \psi_{m,\zeta}(x),$$

$$a_m = \int_0^{\zeta} f^T(x)\, \psi_{m,\zeta}(x)\, d\alpha(x), \ m \in \mathbb{Z}.$$

Since $y(x,\lambda)$ satisfies the system (4.1)-(4.2), we have

$$a_m = \int_0^{\zeta} f^T(x)\, \psi_{m,\zeta}(x)\, d\alpha(x)$$

$$= \int_0^{\zeta}\left[\begin{matrix} \begin{pmatrix} 0 & -T_\alpha \\ T_\alpha & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ +\begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \end{matrix}\right]^T \psi_{m,\zeta}(x)\, d\alpha(x)$$

$$-\lambda\int_0^{\zeta} y^T(x)\, \psi_{m,\zeta}(x)\, d\alpha(x) = (\lambda_{m,\zeta}-\lambda)\, t_m(\lambda).$$

Then, we get

$$t_m(\lambda) = -\frac{a_m}{\lambda - \lambda_{m,\zeta}},$$

and

$$y(x,\lambda) = \int_0^\zeta G_\zeta(x,t,\lambda) f(t) \, d_q t = -\sum_{m=0}^{\infty} \frac{a_m}{\lambda - \lambda_{m,\zeta}} \psi_{m,\zeta}(x).$$

Therefore, the expansion of the resolvent is

$$(R_\zeta f)(x,z)$$

$$= -\sum_{m=-\infty}^{\infty} \frac{\psi_{m,\zeta}(x) f^T(x) \psi_{m,\zeta}(x) \, d\alpha(x)}{\alpha_{m,\zeta}^2 (z - \lambda_{m,\zeta})} \tag{5.5}$$

$$= \int_{-\infty}^{\infty} \frac{\psi(x,\lambda)}{\lambda - z} \left\{ \int_0^\zeta f^T(x) \psi(x,\lambda) \, d\alpha(x) \right\} d\varrho_\zeta(\lambda). \tag{5.6}$$

Lemma 5.2. *Let z be a non real number and x be a fixed number. Then we have*

$$\int_{-\infty}^{\infty} \left| \frac{\psi(x,\lambda)}{z - \lambda} \right|^2 d\varrho_\zeta(\lambda) < K. \tag{5.7}$$

Proof. Since the eigenfunctions $\psi_{m,\zeta}(x)$ are orthogonal, if we put $f(x) = \psi_{m,\zeta}(x)$ in (5.5), then we obtain

$$\frac{1}{\alpha_{m,\zeta}} \int_0^\zeta G_\zeta(x,t,z) \psi_{m,\zeta}(t) \, d\alpha(x) = -\frac{\psi_{m,\zeta}(x)}{\alpha_{m,\zeta}(z - \lambda_{m,\zeta})}. \tag{5.8}$$

Applications of (5.8) and the Parseval equality to $G_\zeta(x,t,z)$ yield

$$\int_0^\zeta G_\zeta^2(x,t,z) \, d\alpha(x) = \sum_{m=-\infty}^{\infty} \frac{\|\psi_{m,\zeta}(.)\|^2}{\alpha_{m,\zeta}^2 |z - \lambda_{m,\zeta}|^2}$$

$$= \int_{-\infty}^{\infty} \left| \frac{\psi(x,\lambda)}{z - \lambda} \right|^2 d\varrho_\zeta(\lambda).$$

Since the last integral convergent by Lemma 3.2, the statement of lemma follows. □

By Lemma 5.1, the set$\{\varrho_\zeta(\lambda)\}$ is bounded. Using Theorems 2.12 and 2.13, we can find a sequence $\{\zeta_k\}$ such that the function $\varrho_{\zeta_k}(\lambda)$ converge to a monotone function $\varrho(\lambda)$. Then we have a lemma

Lemma 5.3. *Let z be a non real number and x be a fixed number. Then we have*

$$\int_{-\infty}^{\infty} \left| \frac{\psi(x,\lambda)}{z - \lambda} \right|^2 d\varrho(\lambda) \le K. \tag{5.9}$$

Proof. Using (5.7), for arbitrary $\eta > 0$, we deduce that

$$\int_{-\eta}^{\eta} \left| \frac{\psi(x,\lambda)}{z-\lambda} \right|^2 d\varrho_\zeta(\lambda) < K.$$

Letting $\eta \to \infty$ and $\zeta \to \infty$, we obtain the desired result. □

Lemma 5.4. *For arbitrary $\eta > 0$, we have the following inequalities.*

$$\int_{-\infty}^{-\eta} \frac{d\varrho(\lambda)}{\lambda^2} < \infty, \quad \int_{\eta}^{\infty} \frac{d\varrho(\lambda)}{\lambda^2} < \infty. \tag{5.10}$$

Proof. Since $\|\psi_{m,\zeta}(0,\lambda)\|_E^2 \neq 0$, putting $x = 0$ in (5.9), we have

$$\int_{-\infty}^{\infty} \frac{d\varrho(\lambda)}{|z-\lambda|^2} < \infty,$$

and the statement of lemma follows. □

Lemma 5.5. *Let $f(.) \in L_\alpha^2((0,\infty);\mathbb{R}^2)$, and let*

$$(Rf)(x,z) = \int_0^\infty G(x,t,z) f(t)\, d\alpha(x),$$

where

$$G(x,t,z) = \begin{cases} \omega(x,z)\psi^T(t,z), & t \le x \\ \psi(x,z)\omega^T(t,z), & t > x. \end{cases}$$

Then

$$\int_0^\infty \|(Rf)(x,z)\|_E^2\, d\alpha(x) \le \frac{1}{v^2} \int_0^\infty \|f(x)\|_E^2\, d\alpha(x), \quad z = u + iv.$$

Proof. It follows from (5.5) and the Parseval equality that

$$\int_0^\zeta \|(Rf)(x,z)\|_E^2\, d\alpha(x)$$

$$= \sum_{m=-\infty}^{\infty} \frac{1}{\alpha_{m,\zeta}^2 |z-\lambda_{m,\zeta}|^2} \left\{ \int_0^\zeta f^T(x)\psi_{m,\zeta}(x)\, d\alpha(x) \right\}^2$$

$$\le \frac{1}{v^2} \sum_{m=-\infty}^{\infty} \frac{1}{\alpha_{m,\zeta}^2} \left\{ f^T(x)\psi_{m,\zeta}(x)\, d\alpha(x) \right\}^2$$

$$= \frac{1}{v^2} \int_0^\zeta \|f(x)\|_E^2\, d\alpha(x).$$

Letting $\zeta \to \infty$, we obtain the desired result. □

The main result of this paper is the following theorem.

Theorem 5.6. *For every nonreal z and for each $f(.) \in L^2_\alpha((0,\infty);\mathbb{R}^2)$, one has the following equality*

$$(Rf)(x,z) = \int_{-\infty}^{\infty} \frac{\psi(x,\lambda)}{\lambda - z} F(\lambda)\, d\varrho(\lambda), \tag{5.11}$$

where

$$F(\lambda) = \lim_{\xi\to\infty} \int_0^{\xi} f^T(x)\,\psi(x,\lambda)\, d\alpha(x).$$

Proof. Let the vector-valued function $f_\xi(x)$ vanishes outside the interval $[0,\xi]$, $\xi < \zeta$ and satisfies the boundary condition (3.3) and let a be an arbitrary positive number. Set

$$F_\xi(\lambda) = \int_0^{\xi} f_\xi^T(x)\,\psi(x,\lambda)\, d\alpha(x).$$

It follows from (5.6) that

$$\begin{aligned}
&(R_\zeta f_\xi)(x,z) \\
&= \int_{-\infty}^{\infty} \frac{\psi(x,\lambda)}{\lambda - z} F_\xi(\lambda)\, d\varrho_\zeta(\lambda) = \int_{-\infty}^{-a} \frac{\psi(x,\lambda)}{\lambda - z} F_\xi(\lambda)\, d\varrho_\zeta(\lambda) \\
&+ \int_{-a}^{a} \frac{\psi(x,\lambda)}{\lambda - z} F_\xi(\lambda)\, d\varrho_\zeta(\lambda) + \int_{a}^{\infty} \frac{\psi(x,\lambda)}{\lambda - z} F_\xi(\lambda)\, d\varrho_\zeta(\lambda) \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{5.12}$$

Now, we shall estimate I_1. By virtue of (5.5), we deduce that

$$\begin{aligned}
|I_1| &= \left| \int_{-\infty}^{-a} \frac{\psi(x,\lambda)}{\lambda - z} F_\xi(\lambda)\, d\varrho_\zeta(\lambda) \right| \\
&= \left| \sum_{\lambda_{k,\zeta} < -a} \frac{\psi_{k,\zeta}(x) \int_0^{\xi} f_\xi^T(x)\,\psi_{k,\zeta}(x)\, d\alpha(x)}{\alpha_{k,\zeta}^2 (\lambda_{k,\zeta} - z)} \right| \\
&\leq \left(\sum_{\lambda_{k,\zeta} < -a} \frac{\|\psi_{k,\zeta}(x)\|_E^2}{\alpha_{k,\zeta}^2 |z - \lambda_{k,\zeta}|^2} \right)^{1/2} \\
&\times \left(\sum_{\lambda_{k,\zeta} < -a} \frac{1}{\alpha_{k,\zeta}^2} \left[\int_0^{\xi} f_\xi^T(x)\,\psi_{k,\zeta}(x)\, d\alpha(x) \right]^2 \right)^{1/2}.
\end{aligned} \tag{5.13}$$

Using conformable fractional integration by parts, we find

$$\int_0^{\xi} f_{\xi}^T(x)\,\psi_{k,\zeta}(x)\,d\alpha(x)$$

$$= \frac{1}{\lambda_{k,\zeta}}\int_0^{\xi} f_{\xi 1}(x)\left\{-T_\alpha \psi_{k,\zeta 2}(x) + p(x)\,\psi_{k,\zeta 1}(x)\right\}d\alpha(x)$$

$$+ \frac{1}{\lambda_{k,\zeta}}\int_0^{\xi} f_{\xi 2}(x)\left\{T_\alpha \psi_{k,\zeta 1}(x) + r(x)\,\psi_{k,\zeta 2}(x)\right\}d\alpha(x)$$

$$= \frac{1}{\lambda_{k,\zeta}}\int_0^{\xi} \psi_{k,\zeta 1}(x)\left\{-T_\alpha f_{\xi 2}(x) + p(x)\,f_{\xi 1}(x)\right\}d\alpha(x)$$

$$+ \frac{1}{\lambda_{k,\zeta}}\int_0^{\xi} \psi_{k,\zeta 2}(x)\left\{T_\alpha f_{\xi 1}(x) + r(x)\,f_{\xi 2}(x)\right\}d\alpha(x). \tag{5.14}$$

By Lemma 5.2, we obtain

$$|I_1| \leq \frac{K^{1/2}}{a}\left(\sum_{\lambda_{k,\zeta}<-a}\frac{1}{\alpha_{k,\zeta}^2}\left[\int_0^{\xi} h_{\xi}^T(x)\,\psi_{k,\zeta}(x)\,d\alpha(x)\right]^2\right)^{1/2},$$

where

$$h_{\xi}(x) = \begin{pmatrix} -T_\alpha f_{\xi 2}(x) + p(x)\,f_{\xi 1}(x) \\ T_\alpha f_{\xi 1}(x) + r(x)\,f_{\xi 2}(x) \end{pmatrix}.$$

Using Bessel inequality, we get

$$|I_1| \leq \frac{K^{1/2}}{a}\left[\int_0^{\xi}\left\|h_{\xi}^T(x)\right\|_E^2 d\alpha(x)\right]^{1/2} = \frac{C_1}{a}.$$

Similarly, one can prove that $|I_3| \leq \frac{C_2}{a}$. Then I_1 and I_3 tend to zero as $a \to \infty$, uniformly in ζ. Using Theorems 2.12 and 2.13 in (5.12), we deduce that

$$(Rf_{\xi})(x,z) = \int_{-\infty}^{\infty}\frac{\psi(x,\lambda)}{\lambda - z}F_{\xi}(\lambda)\,d\varrho(\lambda). \tag{5.15}$$

As is known, if $f(.) \in L_\alpha^2\left((0,\infty);\mathbb{R}^2\right)$, then one can find a sequence $\{f_{\xi}(x)\}_{\xi=1}^{\infty}$ which satisfy the previous conditions and tend to $f(x)$ as $\xi \to \infty$. By the Parseval equality, the sequence of Fourier transform converges to the transform of $f(.)$. By Lemmas 5.3 and 5.5, we can pass to the limit $\xi \to \infty$ in (5.15). So, we get the assertion of the theorem. □

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