Gulf Journal of Mathematics
Vol 2, Issue 2 (2014) 88-93

SOME PROPERTIES ON CERTAIN NEW SUBCLASSES OF ANALYTIC FUNCTIONS

LI XIAOFEI[1] AND FENG JIANZHONG[2*]

Abstract. In this article, we define two new subclasses of univalent and analytic functions in the open disc $U=\{z : |z| < 1\}$. For functions belonging to these classes, we determine coefficient inequalities, inclusion relation and convolution property.

1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \tag{1.1}$$

which are analytic and univalent in the open dis $U=\{z \in C : |z| < 1\}$. The subclass $\mathcal{S}_1^*(b)$ of $\mathcal{A}$ is the class of functions satisfying

$$\mathrm{Re}\{1 + \tfrac{1}{b}[\tfrac{zf'(z)}{f(z)} - 1]\} > 0 \ (z \in U, b \in R - \{0\}).$$

The subclass $\mathcal{C}_1(b)$ of $\mathcal{A}$ is the class of functions satisfying

$$\mathrm{Re}\{1 + \tfrac{1}{b}\tfrac{zf''(z)}{f'(z)}\} > 0 \ (z \in U, b \in R - \{0\}).$$

These classes and their subclasses are studied by authors(see [1,2,3,6,8,9]).

Recently, H Mahzoon have defined two new sublcasses of $\mathcal{A}$. The subclass $\mathcal{H}(\alpha, \beta, b)$ denote the subclass of $\mathcal{A}$ consisting of functions f satisfying the inequality

$$\mathrm{Re}\{1 + \tfrac{1}{b}[\tfrac{zf'(z)}{f(z)} - 1]\} > \beta|\tfrac{1}{b}[\tfrac{zf'(z)}{f(z)} - 1]| + \alpha \ (z \in U, b \in R - \{0\})$$

for some $\alpha(0 \leq \alpha < 1)$, and for some $\beta(\beta \geq 0)$. The subclass $\mathscr{T}(\alpha, \beta, b)$ denote the subclass of $\mathcal{A}$ consisting of functions f satisfying the inequality

$$\mathrm{Re}\{1 + \tfrac{1}{b}\tfrac{zf''(z)}{f'(z)}\} > \beta|\tfrac{1}{b}\tfrac{zf''(z)}{f'(z)}| + \alpha \ (z \in U, b \in R - \{0\})$$

for some $\alpha(0 \leq \alpha < 1)$, and for some $\beta(\beta \geq 0)$(for detail, see [4]).

In this paper, we define two new subclasses $\mathcal{F}(\alpha, \beta, b)$ and $\mathcal{L}(\alpha, \beta, b)$ similar to above and obtain the coefficient inequalities, inclusion relation and convolution

Date: Received: Jul 11, 2013; Accepted: Aug 28, 2013.

* Corresponding author.

2010 *Mathematics Subject Classification.* Primary 30C45; Secondary 26D15.

Key words and phrases. Coefficient inequalities, Inclusion relation, Convolution property.

property.

Definition 1.1. Let $\mathcal{F}(\alpha,\beta,b)$ denote the subclass of $\mathcal{A}$ consisting of functions f satisfying the inequality

$$\mathrm{Re}\{1+\tfrac{1}{b}[\tfrac{zf'(z)}{f(z)}-1]\}<\beta|\tfrac{1}{b}[\tfrac{zf'(z)}{f(z)}-1]|+\alpha\ (z\in U, b\in R-\{0\})$$

for some $\alpha(\alpha>1)$, and for some $\beta(\beta\leq 0)$.

Definition 1.2. Let $\mathcal{L}(\alpha,\beta,b)$ denote the subclass of $\mathcal{A}$ consisting of functions f satisfying the inequality

$$\mathrm{Re}\{1+\tfrac{1}{b}\tfrac{zf''(z)}{f'(z)}\}<\beta|\tfrac{1}{b}\tfrac{zf''(z)}{f'(z)}|+\alpha\ (z\in U, b\in R-\{0\})$$

for some $\alpha(\alpha>1)$, and for some $\beta(\beta\leq 0)$.

We note that $f(z)\in\mathcal{L}(\alpha,\beta,b)$ if and only if $zf'(z)\in\mathcal{F}(\alpha,\beta,b)$. The classes $\mathcal{F}(\alpha,\beta,1)$ and $\mathcal{L}(\alpha,\beta,1)$ were studied by J Nishiwaki et al[7].

2. Main results

2.1. **Coefficient inequalities for the classes $\mathcal{F}(\alpha,\beta,b)$ and $\mathcal{L}(\alpha,\beta,b)$.** In this section, we derive sufficient conditions for functions to be in the classes $\mathcal{F}(\alpha,\beta,b)$ and $\mathcal{L}(\alpha,\beta,b)$ in terms of coefficient inequalities.

Theorem 2.1. *If $f(z)\in\mathcal{A}$ defined by (1.1) satisfies*

$$\sum_{j=2}^{\infty}\{|j+2b-b\alpha-1|+|j-b\alpha-1|-2\beta(j-1)\}|a_j|\leq|b|\alpha-|2b-b\alpha|, \quad (2.1)$$

then $f(z)\in\mathcal{F}(\alpha,\beta,b)$.

Proof. Let us suppose that

$$\sum_{j=2}^{\infty}\{|j+2b-b\alpha-1|+|j-b\alpha-1|-2\beta(j-1)\}|a_j|\leq|b|\alpha-|2b-b\alpha|$$

for $f(z)\in\mathcal{A}$. It suffices to show that

$$|\tfrac{1+\frac{1}{b}[\frac{zf'(z)}{f(z)}-1]-\beta|\frac{1}{b}[\frac{zf'(z)}{f(z)}-1]|-\alpha+1}{1+\frac{1}{b}[\frac{zf'(z)}{f(z)}-1]-\beta|\frac{1}{b}[\frac{zf'(z)}{f(z)}-1]|-\alpha-1}|<1\ (z\in U).$$

We have

$$\begin{aligned}&|\tfrac{1+\frac{1}{b}[\frac{zf'(z)}{f(z)}-1]-\beta|\frac{1}{b}[\frac{zf'(z)}{f(z)}-1]|-\alpha+1}{1+\frac{1}{b}[\frac{zf'(z)}{f(z)}-1]-\beta|\frac{1}{b}[\frac{zf'(z)}{f(z)}-1]|-\alpha-1}|\\ &=|\frac{(2b-b\alpha)+\sum_{j=2}^{\infty}(j+2b-b\alpha-1)a_jz^{j-1}-\beta e^{i\theta}|\sum_{j=2}^{\infty}(j-1)a_jz^{j-1}|}{-[b\alpha-\sum_{j=2}^{\infty}(j-b\alpha-1)a_jz^{j-1}+\beta e^{i\theta}|\sum_{j=2}^{\infty}(j-1)a_jz^{j-1}|]}|\\ &\leq\frac{|2b-b\alpha|+\sum_{j=2}^{\infty}|j+2b-b\alpha-1||a_j|-\beta\sum_{j=2}^{\infty}(j-1)|a_j|}{|b|\alpha-\sum_{j=2}^{\infty}|j-b\alpha-1||a_j|+\beta\sum_{j=2}^{\infty}(j-1)|a_j|}.\end{aligned}$$

The last expression is bounded above by 1, if

$$|2b-b\alpha|+\sum_{j=2}^{\infty}|j+2b-b\alpha-1||a_j|-\beta\sum_{j=2}^{\infty}(j-1)|a_j|$$
$$\leq |b|\alpha-\sum_{j=2}^{\infty}|j-b\alpha-1||a_j|+\beta\sum_{j=2}^{\infty}(j-1)|a_j|,$$

which is equivalent to our condition (2.1). This completes the proof of our Theorem 2.1. □

By using Theorem 2.1, we have

Theorem 2.2. *If $f(z)\in\mathcal{A}$ defined by (1.1) satisfies*

$$\sum_{j=2}^{\infty}j\{|j+2b-b\alpha-1|+|j-b\alpha-1|-2\beta(j-1)\}|a_j|\leq |b|\alpha-|2b-b\alpha|, \tag{2.2}$$

then $f(z)\in\mathcal{L}(\alpha,\beta,b)$.

Proof. From $f(z)\in\mathcal{L}(\alpha,\beta,b)$ if and only if $zf'(z)\in\mathcal{F}(\alpha,\beta,b)$, replacing a_j by ja_j in Theorem 2.1, we have the Theorem 2.2. □

2.2. **Relation for $\mathcal{F}^*(\alpha,\beta,b)$ and $\mathcal{L}^*(\alpha,\beta,b)$.** By Theorem 2.1, the class $\mathcal{F}^*(\alpha,\beta,b)$ is considered as the subclass of $\mathcal{F}(\alpha,\beta,b)$ consisting of $f(z)$ satisfying

$$\sum_{j=2}^{\infty}\{|j+2b-b\alpha-1|+|j-b\alpha-1|-2\beta(j-1)\}|a_j|\leq |b|\alpha-|2b-b\alpha| \tag{2.3}$$

for some $\alpha(\alpha>1)$, and for some $\beta(\beta\leq 0)$. The class $\mathcal{L}^*(\alpha,\beta,b)$ is also considered as the subclass of $\mathcal{L}(\alpha,\beta,b)$ consisting of $f(z)$ which satisfy

$$\sum_{j=2}^{\infty}j\{|j+2b-b\alpha-1|+|j-b\alpha-1|-2\beta(j-1)\}|a_j|\leq |b|\alpha-|2b-b\alpha| \tag{2.4}$$

for some $\alpha(\alpha>1)$, and for some $\beta(\beta\leq 0)$. By the coefficient inequalities for the classes $\mathcal{F}^*(\alpha,\beta,b)$ and $\mathcal{L}^*(\alpha,\beta,b)$, we see

Theorem 2.3. *If $f(z)\in\mathcal{A}$, then*

$$\mathcal{F}^*(\alpha,\beta_1,b)\subset\mathcal{F}^*(\alpha,\beta_2,b)$$

for some β_1 and $\beta_2(\beta_1\leq\beta_2\leq 0)$.
Proof. For $\beta_1\leq\beta_2\leq 0$, we obtain

$$\sum_{j=2}^{\infty}\{|j+2b-b\alpha-1|+|j-b\alpha-1|-2\beta_2(j-1)\}|a_j|\leq\sum_{j=2}^{\infty}\{|j+2b-b\alpha-1|+|j-b\alpha-1|-2\beta_1(j-1)\}|a_j|.$$

Therefore, if $f(z)\in\mathcal{F}^*(\alpha,\beta_1,b)$, then $f(z)\in\mathcal{F}^*(\alpha,\beta_2,b)$. Hence we get the required result. □

By using Theorem 2.3, we also have

Theorem 2.4. *If $f(z) \in \mathcal{A}$, then*

$$\mathcal{L}^*(\alpha, \beta_1, b) \subset \mathcal{L}^*(\alpha, \beta_2, b)$$

for some β_1 and $\beta_2(\beta_1 \leq \beta_2 \leq 0)$.

2.3. **Convolution of the class $\mathcal{F}^*(\alpha, \beta, b)$ and $\mathcal{L}^*(\alpha, \beta, b)$.** In this section, we obtain the results about convolution of the classes $\mathcal{F}^*(\alpha, \beta, b)$ and $\mathcal{L}^*(\alpha, \beta, b)$.

Definition 2.5. For analytic functions $f_k(z)$ $(k = 1, 2)$ given by

$$f_k(z) = z + \sum_{j=2}^{\infty} a_{j,k} z^j \ (k = 1, 2),$$

the Hadamard product (or convolution) of $f_1(z), f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z + \sum_{j=2}^{\infty} a_{j,1} a_{j,2} z^j.$$

Thus we have

Theorem 2.6. *If $f_1 \in \mathcal{F}^*(\alpha_1, \beta, b)$ and $f_2 \in \mathcal{F}^*(\alpha_2, \beta, b)$ for some β such that*

$$-7\beta^2 + 4\beta - 3 + b^2 - 2b - b^2\alpha_1 - b^2\alpha_2 + b^2\alpha_1\alpha_2 + b\alpha_1 + b\alpha_2 \leq 0,$$

for given $b, \alpha_1, \alpha_2 (b > 0, 1 < \alpha_1, \alpha_2 \leq 2)$, then $(f_1 * f_2) \in \mathcal{F}^*(\alpha, \beta, b)$, where

$$\alpha = 1 + \frac{b(\alpha_1 - 1)(\alpha_2 - 1)(2 - \beta)}{(1 + b - \beta - b\alpha_1)(1 + b - \beta - b\alpha_2) + b^2(\alpha_1 - 1)(\alpha_2 - 1)}.$$

Proof. From (2.3), for $f(z) \in \mathcal{F}^*(\alpha, \beta, b)$ with $1 < \alpha_1, \alpha_2 \leq 2$, we have

$$\sum_{j=2}^{\infty} \{(j + 2b - b\alpha - 1) + (j - b\alpha - 1) - 2\beta(j - 1)\}|a_j|$$

$$\leq \sum_{j=2}^{\infty} \{|j + 2b - b\alpha - 1| + |j - b\alpha - 1| - 2\beta(j - 1)\}|a_j|$$

$$\leq b\alpha - (2b - b\alpha) = 2b\alpha - 2b,$$

that is, if $f(z) \in \mathcal{F}^*(\alpha, \beta, b)$,then

$$\sum_{j=2}^{\infty} \frac{j(1-\beta)+b+\beta-b\alpha-1}{b(\alpha-1)} |a_j| \leq 1. \tag{2.5}$$

Conversely, if $f(z)$ satisfies

$$\sum_{j=2}^{\infty} \frac{j(1-\beta)+b+\beta-b\alpha}{b(\alpha-1)} |a_j| \leq 1, \tag{2.6}$$

then $f(z) \in \mathcal{F}^*(\alpha, \beta, b)$. From (2.5), if $f_1(z) \in \mathcal{F}^*(\alpha_1, \beta, b)$, then

$$\sum_{j=2}^{\infty} \frac{j(1-\beta)+b+\beta-b\alpha_1-1}{b(\alpha_1-1)} |a_{j,1}| \leq 1, \tag{2.7}$$

and also if $f_2(z) \in \mathcal{F}^*(\alpha_2, \beta, b)$, then

$$\sum_{j=2}^{\infty} \frac{j(1-\beta)+b+\beta-b\alpha_2-1}{b(\alpha_2-1)}|a_{j,2}| \leq 1. \tag{2.8}$$

Applying the Schwarz's inequality, we have the following inequality

$$\sum_{j=2}^{\infty} \sqrt{\frac{[j(1-\beta)+b+\beta-b\alpha_1-1][j(1-\beta)+b+\beta-b\alpha_2-1]}{b^2(\alpha_1-1)(\alpha_2-1)}}\sqrt{|a_{j,1}||a_{j,2}|} \leq 1. \tag{2.9}$$

From (2.9), we obtain that

$$\sqrt{|a_{j,1}||a_{j,2}|} \leq \sqrt{\frac{b^2(\alpha_1-1)(\alpha_2-1)}{[j(1-\beta)+b+\beta-b\alpha_1-1][j(1-\beta)+b+\beta-b\alpha_2-1]}}. \tag{2.10}$$

From (2.6) and (2.9), if the following inequality

$$\sum_{j=2}^{\infty} \frac{j(1-\beta)+b+\beta-b\alpha}{b(\alpha-1)}|a_{j,1}||a_{j,2}|$$

$$\leq \sum_{j=2}^{\infty} \sqrt{\frac{[j(1-\beta)+b+\beta-b\alpha_1-1][j(1-\beta)+b+\beta-b\alpha_2-1]}{b^2(\alpha_1-1)(\alpha_2-1)}}\sqrt{|a_{j,1}||a_{j,2}|}$$

is satisfied, then we say that $f(z) \in \mathcal{F}^*(\alpha, \beta, b)$. This inequality holds true if

$$\frac{j(1-\beta)+b+\beta-b\alpha}{b(\alpha-1)}\sqrt{|a_{j,1}||a_{j,2}|} \leq \sqrt{\frac{[j(1-\beta)+b+\beta-b\alpha_1-1][j(1-\beta)+b+\beta-b\alpha_2-1]}{b^2(\alpha_1-1)(\alpha_2-1)}}$$

for all $j \geq 2$. That is

$$\sqrt{|a_{j,1}||a_{j,2}|} \leq \frac{b(\alpha-1)}{j(1-\beta)+b+\beta-b\alpha}\sqrt{\frac{[j(1-\beta)+b+\beta-b\alpha_1-1][j(1-\beta)+b+\beta-b\alpha_2-1]}{b^2(\alpha_1-1)(\alpha_2-1)}}. \tag{2.11}$$

From (2.10) and (2.11), we need only to prove that

$$\sqrt{\frac{b^2(\alpha_1-1)(\alpha_2-1)}{[j(1-\beta)+b+\beta-b\alpha_1-1][j(1-\beta)+b+\beta-b\alpha_2-1]}} \leq \frac{b(\alpha-1)}{j(1-\beta)+b+\beta-b\alpha}\sqrt{\frac{[j(1-\beta)+b+\beta-b\alpha_1-1][j(1-\beta)+b+\beta-b\alpha_2-1]}{b^2(\alpha_1-1)(\alpha_2-1)}},$$

which is equivalent to

$$\alpha \geq 1 + \frac{b(\alpha_1-1)(\alpha_2-1)[j(1-\beta)+\beta]}{[j(1-\beta)+b+\beta-b\alpha_1-1][j(1-\beta)+b+\beta-b\alpha_2-1]+b^2(\alpha_1-1)(\alpha_2-1)} = A(j)$$

for $j = 2, 3, \cdots$. Let us define $H(j)$ by the numerator of $A'(j)$. Then $H(j)$ give us that

$$H(j) = -j^2(\beta-1)^2 - j\beta(\beta+2) + (b^2-2b+1-b^2\alpha_1-b^2\alpha_2+b^2\alpha_1\alpha_2+b\alpha_1+b\alpha_2-\beta^2).$$

It is easy to find that $-\frac{\beta(\beta+2)}{2(\beta-1)^2} < 2$, so if

$$H(2) = -7\beta^2 + 4\beta - 3 + b^2 - 2b - b^2\alpha_1 - b^2\alpha_2 + b^2\alpha_1\alpha_2 + b\alpha_1 + b\alpha_2 \leq 0,$$

then it implies that $A(j)$ is decreasing for $j \geq 2$. Thus $A(2)$ is the maximum of $A(j)$. This completes the proof of the theorem. □

With the same method, we can obtain convolution of the class $\mathcal{L}^*(\alpha, \beta, b)$.

Acknowledgement. This work is supported by research development fund of Engineering and Technology College Yangtze University(13J0802) and scientific research fund of Hubei provincial education department(B2013281). The authors would like to thank the referees for their helpful suggestions which essentially improved the presentation of this paper.

References

1. O. Altintas, H. Irmak, S. Owa, and H. Srivastava, *Coefficients bounds for some families of starlike and convex functions of complex order*, Appl. Math. Lett. **20** (2000), 1218-1222.
2. O. Altintas, H.M.Srivastava, *Some majorization problems associated with p-valently starlike and convex functions of complex order*, East Asian Math. J. **17** (2001), 175-183.
3. A. W. Goodman, *On uniformly starlike functions*, J. Math. Anal. Appl. **151** (1991), 364-370.
4. H. Mahzoon, *On a certain new subclass of analytic functions*, American Journal of Sci. Res. **31** (2011), 116-120.
5. P. T. Mocanu, *Some starlikeness conditions for analytic functions*, Rev. Roumaine Math. Pure Appl. **33** (1988), 1-2.
6. M. A. Nasr, M. K. Aouf, Starlike functions of complex order, J. Natur. Sci. Math. 25(1985), 1-12.
7. J. Nishiwaki, S. Owa, *Certain classes of anaytic functions concerned with uniformly starlike and convex functions*, Appl. Math. Compu. **187** (2007), 350-355.
8. S. Shams, S. R. Kulkarni, J. M. Jahangiri, *Classes of uniformly starlike and convex functions*, Int. J. Math. Sci. **55** (2004), 2959-2961.
9. M. S. Robertson, *On the theory of univalent functions*, Annalysis of Math. (Ser.1) **37** (1936), 374-408.

[1] Engineering and Technology College, Yangtze University, Jingzhou 434020, P. R. China .
E-mail address: lxfei0828@163.com

[2] School of Information and Mathematics, Yangtze University, Jingzhou 434023, P. R. China .
E-mail address: 19495791@qq.com