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LOCAL DERIVATIONS AND ROTA-BAXTER OPERATORS OF QUANTUM LOTKA-VOLTERRA ALGEBRAS ON $\mathbb{M}_2(\mathbb{C})$

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ABSTRACT. In this paper, we investigate in detail the local and 2-local derivations of the flow of quantum genetic Lotka-Volterra algebra belonging to $\mathbb{M}_2(\mathbb{C})$. It aims to reveal how the different values of the parameter affect the different characteristic properties of these derivations. Our findings furnish structural details of the FQGLV-A algebras and develop the gap in actualizing local and 2-local derivations. We show that each local derivation is a derivation, and a set with 2-local derivations is a set of derivations. Further, we investigate the characterization of the Rota-Baxter operators in one of the considered scenarios with derivations to get better insights into the role and consequences of the algebraic structure of FQGLV-A.

1. Introduction and preliminaries

In this paper, we are going to continue our initial investigation started in [20], where a quantum analog of Lotka-Volterra algebras has been initiated by employing a coalgebra scheme. It is noticed that such a framework allows us to look at the genetic algebras from other perspective, and provides new treatment of this kind of algebras. In the current work, we are going to describe local derivatives of the flow of quantum genetic Lotka-Volterra algebras (FQGLV-A). We thoroughly examine the algebraic properties of a FQGLV-A on $\mathbb{M}_2(\mathbb{C})$. The notation and definitions used therein will be consistently applied in the current study, negating the need for restating them here. The reader is directed to consult our prior work for these details. Here, we recall that each genetic algebra can be considered as coalgebra (see [22]). It is known [16, 27] that genetic algebra $(\mathbb{R}^n, \circ)$ is defined via a cubic matrix $(p_{ij,k})_{i,j,k=1}^n$ satisfying the following conditions:

$$p_{ij,k} \geq 0, \quad p_{ij,k} = p_{ji,k}, \ \sum_{k=1}^n p_{ij,k} = 1, \quad \text{for all} \quad i, j, k.$$

Let us denote by ℓ_∞^n the vector space $\mathbb{R}^n$ with a norm $\|\mathbf{x}\| = \max_{1\leq k\leq n} |x_k|$. The space ℓ_∞^n is endowed with natural order. Now, we may define a mapping $\Delta : \ell_\infty^n \to$

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$\ell_\infty^n \times \ell_\infty^n$ by

$$\Delta(\mathbf{x})_{ij} = \sum_{k=1}^{n} p_{ij,k} x_k, \quad \mathbf{x} = (x_k) \in \ell_\infty^n.$$

One can see that Δ is positive. Here, the positivity means that $\Delta(\mathbf{x}) \geq 0$ if $\mathbf{x} \geq 0$. Moreover, we have

$$\mathbf{x} \circ \mathbf{y} = \Delta^*(\mathbf{x}, \mathbf{y}),$$

where

$$\Delta^*(\mathbf{x}, \mathbf{y})_k = \sum_{i,j=1}^{n} p_{ij,k} x_i y_j, \quad \mathbf{x} = (x_k), \mathbf{y} = (y_k) \in \ell_1^n.$$

Consequently, the genetic algebra $(\mathbb{R}^n, \circ)$ may be equivalently viewed as a coalgebra (ℓ_∞^n, Δ) (see [1]). It is noteworthy that when $k \notin \{i, j\}$ implies $p_{ij,k} = 0$, then resulting genetic algebra is called *Lotka-Volterra* (LV) algebra [7]. The significance of LV algebras has been highlighted in [13, 15]. We stress that Holgate's contributions [11, 12] offer a biological interpretation of the derivations in genetic algebras, a notion further explored in [9, 14, 18, 23]. Specifically, the set of derivations of LV algebras were characterized up to dimension three (see [3, 7]). Additional studies, such as [6], delve into the derivation space of four-dimensional LV algebras. Therefore, in [20] a quantum version of the proposed coalgebra approach was utilized to define a quantum analog of Lotka-Voltera algebras. Hence, it is natural to further investigate local derivatives of FQGLV-A.

It is worth mentioning that several papers have been devoted to studying the relationships between non-associative algebra and genetic algebra; for example, see [5, 21, 26].

On the other hand, Rota-Baxter operators were first introduced by G. Baxter as a solution to a problem in probability, which arises from the theory of associative algebras [4]. The study of algebraic structures has recently stressed the importance of Rota-Baxter operators and their applications. In [10],it has been investigated the extension of Rota-Baxter operators in the field of cocommutative Hopf algebras. It offers fresh perspectives on their algebraic characteristics and interactions.In [2], it was established a relationship between Rota-Baxter groups and skew left braces. It looks deeper into their connection with the Yang-Baxter equation and provides an association between group theory and set-theoretic solutions to this important equation in mathematical physics.

In the current paper, using the description of the derivations of FQGLV-A, we are going to characterize local and 2-local derivations of FQGLV-A. Namely, it will be established that each local derivation is a derivation, as a consequence, the set of all 2-local derivations coincides with the set of derivations. Furthermore, we investigate the Rota-Baxter operators of FQGLV-A.

2. Quantum quadratic operators on $\mathbb{M}_2(\mathbb{C})$ and quantum genetic Lotka-Volterra algebras

This section recalls some necessary definitions, taken from [17]. Hereafter, $\mathbb{M}_2(\mathbb{C})$ denotes an algebra of 2×2 matrices over the complex field $\mathbb{C}$. The tensor product $\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ is interpreted as an algebra of 4×4 matrices, $\mathbb{M}_4(\mathbb{C})$, over $\mathbb{C}$. Subsequently, $\mathbb{I}$ represents an identity matrix. Furthermore, $\mathbb{M}_2(\mathbb{C})^*_+$ is defined as the set of all positive functionals on $\mathbb{M}_2(\mathbb{C})$, and $S(\mathbb{M}_2(\mathbb{C}))$ denotes the set of all states (i.e., linear positive functionals that evaluate to 1 at $\mathbb{I}$) defined on $\mathbb{M}_2(\mathbb{C})$. It is also important to recall that the identity matrix $\mathbb{I}$, along with the Pauli matrices $\{\sigma_1, \sigma_2, \sigma_3\}$, forms a basis for $\mathbb{M}_2(\mathbb{C})$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence, every matrix $x \in \mathbb{M}_2(\mathbb{C})$ can be expressed as $x = v_0\mathbb{I} + \mathbf{v}\sigma$ with $v_0 \in \mathbb{C}$, $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{C}^3$, here $\mathbf{v}\sigma = v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3$. In the following section, we frequently use notation $\overline{\mathbf{v}} = (\overline{v_1}, \overline{v_2}, \overline{v_3})$.

Lemma 2.1. [25] *Let $x \in \mathbb{M}_2(\mathbb{C})$. Then the following assertions hold:*

(a) *x is self-adjoint if and only if $v_0, \mathbf{v}$ are reals;*
(b) *$x \geq 0$ if and only if $\|\mathbf{v}\| \leq v_0$, where $\|\mathbf{v}\| = \sqrt{|v_1|^2 + |v_2|^2 + |v_3|^2}$;*
(c) *A linear functional φ on $\mathbb{M}_2(\mathbb{C})$ is a state if and only if it can be written as:*
$$\varphi(v_0\mathbb{I} + \mathbf{v}\sigma) = v_0 + \langle \mathbf{v}, \mathbf{f}\rangle, \tag{2.1}$$
where $\mathbf{f} = (f_1, f_2, f_3) \in \mathbb{R}^3$ such that $\|\mathbf{f}\| \leq 1$. Here as before $\langle\cdot,\cdot\rangle$ stands for the scalar product in $\mathbb{C}^3$.

One can observe that a basis of $\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ can be established by the following system
$$\{\mathbb{I}\otimes\mathbb{I}, \sigma_i\otimes\mathbb{I}, \mathbb{I}\otimes\sigma_i, \sigma_i\otimes\sigma_j\}_{i,j=1}^3.$$
A linear operator $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ is termed a *quantum quadratic operator (q.q.o.)* if it satisfies two conditions: it is unital (i.e., $\Delta\mathbb{I} = \mathbb{I}\otimes\mathbb{I}$) and positive (meaning $\Delta x \geq 0$ for any non-negative x). Consider the operator $U : \mathbb{M}_2(\mathbb{C})\otimes\mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C})\otimes\mathbb{M}_2(\mathbb{C})$, defined as a twist map where $U(x\otimes y) = y\otimes x$ for all x, y within $\mathbb{M}_2(\mathbb{C})$. A q.q.o. Δ is classified as *symmetric* if it complies with the condition $U\Delta = \Delta$. In the subsequent discussion, we focus exclusively on symmetric q.q.o.s. Therefore, any symmetric q.q.o. $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C})\otimes\mathbb{M}_2(\mathbb{C})$ can be represented as follows (refer to [17, Proposition 3.2]):

$$\Delta(x) = (v_0 + \langle \mathbf{c}, \overline{\mathbf{v}}\rangle)\mathbb{I}\otimes\mathbb{I} + \mathbb{I}\otimes\mathbf{C}\mathbf{v}\cdot\sigma + \mathbf{C}\mathbf{v}\cdot\sigma\otimes\mathbb{I} + \sum_{m,l=1}^{3}\langle \mathbf{c}_{ml}, \overline{\mathbf{v}}\rangle\sigma_m\otimes\sigma_l, \tag{2.2}$$

where $\mathbf{c} = (c_1, c_2, c_3)$, $\mathbf{c}_{ml} = (c_{ml,1}, c_{ml,2}, c_{ml,3})$, and $\mathbf{C} = (c_{ij})_{i,j=1}^3$ are real. Here as before $\langle\cdot,\cdot\rangle$ stands for the standard dot product in $\mathbb{C}^3$. We emphasize that to construct q.q.o. via (2.2) is not easy task. In [19, 20] we established that the next operator

$$\begin{aligned}\Delta_t(v_0\mathbb{I} + v\sigma) \ &= \ \left[v_0 + a(1 - \frac{1}{2^t})v_3 + \frac{a}{2^{t+1}}v_3\right]\mathbb{I}\otimes\mathbb{I} + \lambda^{t+1}v_1(\sigma_1\otimes\mathbb{I} + \mathbb{I}\otimes\sigma_1) \\ &\quad + \mu^{t+1}v_2(\sigma_2\otimes\mathbb{I} + \mathbb{I}\otimes\sigma_2) + \frac{v_3}{2^{t+1}}(\sigma_3\otimes\mathbb{I} + \mathbb{I}\otimes\sigma_3) \\ &\quad - \frac{a}{2^{t+1}}v_3(\sigma_3\otimes\sigma_3).\end{aligned} \tag{2.3}$$

is a q.q.o. if and only if

$$\max\{|\lambda|, |\mu|\} \leq \frac{\sqrt{1-|a|}}{2}. \tag{2.4}$$

where $\lambda, \mu \in \mathbb{R}$ and $a \in [-1, 1]$.

By means of Δ_t, the next binary operation on $\mathbb{M}_2(\mathbb{C})^*$ is defined by

$$(\varphi \circ_t \psi)(x) = (\varphi \otimes \psi)(\Delta_t(x)), \quad \varphi, \psi \in \mathbb{M}_2(\mathbb{C})^*, x \in \mathbb{M}_2(\mathbb{C}), \tag{2.5}$$

Here $\mathbb{M}_2(\mathbb{C})^*$ is the dual of $\mathbb{M}_2(\mathbb{C})$. The *flow of quantum genetic Lotka-Volterra algebras (FQGLV-A)* is defined by the triple $(\mathbb{M}_2(\mathbb{C})^*, \circ_t, \Delta_t)$. In the sequel, for the sake of shortness, it is denoted by $(A_t, \circ_t)$. Now, by Lemma 2.1, we infer that any functional $\varphi \in \mathbb{M}_2(\mathbb{C})^*$ is described by a vector $\mathbf{f} = (f_0, f_1, f_2, f_3)$ here $f_0 = \varphi(\mathbb{I})$, $f_k = \varphi(\sigma_k)$. In what follows, we always use such a description. Hence, due to (2.5) and (2.3) the above given binary operation reduces to

$$\begin{aligned}\mathbf{f} \circ_t \mathbf{p} = \Big(\quad & f_0 p_0, \lambda^{t+1}(f_1 p_0 + f_0 p_1), \mu^{t+1}(f_2 p_0 + f_0 p_2), f_0 p_0 g(a, t) \\ & + \frac{1}{2^{t+1}}(f_3 p_0 + f_0 p_3) - \frac{a}{2^{t+1}}(f_3 p_3)\Big).\end{aligned} \tag{2.6}$$

where $\mathbf{f} = (f_0, f_1, f_2, f_3)$, $\mathbf{p} = (p_0, p_1, p_2, p_3)$ and $g(a, t) := a\big(1 - \frac{1}{2^{t+1}}\big)$.

Remark 2.2. It is noticed that if one takes $\lambda = \mu = 0$, then the quantum genetic Lotka-Volterra algebras reduce to the classical Lotka-Volterra coalgebras [1, 7].

Recall that a linear mapping $d : A \to A$ of algebra $(A, \circ)$ is called *derivation* if

$$d(x \circ y) = d(x) \circ y + x \circ d(y), \tag{2.7}$$

for all $x, y \in A$.

Theorem 2.3. [20] *The derivations of $(A, \circ_t)$ are given as follows:*

(1) *If $a \neq 0, \lambda \neq \mu$, then* $D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11} & 0 & 0 \\ 0 & 0 & d_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

(2) *If $a \neq 0, \lambda \neq 0, \lambda = \mu$, then* $D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11} & d_{12} & 0 \\ 0 & d_{21} & d_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(3) *If $a = 0, \lambda = \mu = \frac{1}{2}, t = 0$, then* $D = \begin{pmatrix} 0 & 0 & 0 & d_{03} \\ 0 & d_{11} & d_{12} & d_{13} \\ 0 & d_{21} & d_{22} & d_{23} \\ 0 & d_{31} & d_{32} & d_{33} \end{pmatrix}$

(4) *If $a = 0, \lambda = \mu \neq \frac{1}{2}, t = 0$, then* $D = \begin{pmatrix} 0 & 0 & 0 & d_{03} \\ 0 & d_{11} & d_{12} & 0 \\ 0 & d_{21} & d_{22} & 0 \\ 0 & 0 & 0 & d_{33} \end{pmatrix}$

(5) *If* $a = 0, \lambda = \frac{1}{2}, \mu \neq \frac{1}{2}, t = 0,$ *then* $D = \begin{pmatrix} 0 & 0 & 0 & d_{03} \\ 0 & d_{11} & 0 & d_{13} \\ 0 & 0 & d_{22} & 0 \\ 0 & d_{31} & 0 & d_{33} \end{pmatrix}$

(6) *If* $a = 0, \lambda \neq \frac{1}{2}, \mu = \frac{1}{2}, t = 0,$ *then* $D = \begin{pmatrix} 0 & 0 & 0 & d_{03} \\ 0 & d_{11} & 0 & 0 \\ 0 & 0 & d_{22} & d_{23} \\ 0 & 0 & d_{32} & d_{33} \end{pmatrix}$

(7) *If* $a = 0, \lambda \neq \mu, \lambda \neq \frac{1}{2}, \mu \neq \frac{1}{2}, t = 0,$ *then* $D = \begin{pmatrix} 0 & 0 & 0 & d_{03} \\ 0 & d_{11} & 0 & 0 \\ 0 & 0 & d_{22} & 0 \\ 0 & 0 & 0 & d_{33} \end{pmatrix}$

(8) *If* $a = 0, \lambda = \mu = \frac{1}{2}, t \neq 0,$ *then* $D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11} & d_{12} & d_{13} \\ 0 & d_{21} & d_{22} & d_{23} \\ 0 & d_{31} & d_{32} & d_{33} \end{pmatrix}$

(9) *If* $a = 0, \lambda = \mu \neq \frac{1}{2}, t \neq 0,$ *then* $D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11} & d_{12} & 0 \\ 0 & d_{21} & d_{22} & 0 \\ 0 & 0 & 0 & d_{33} \end{pmatrix}$

(10) *If* $a = 0, \lambda = \frac{1}{2}, \mu \neq \frac{1}{2}, t \neq 0,$ *then* $D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11} & 0 & d_{13} \\ 0 & 0 & d_{22} & 0 \\ 0 & d_{31} & 0 & d_{33} \end{pmatrix}$

(11) *If* $a = 0, \lambda \neq \frac{1}{2}, \mu = \frac{1}{2}, t \neq 0,$ *then* $D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11} & 0 & 0 \\ 0 & 0 & d_{22} & d_{23} \\ 0 & 0 & d_{32} & d_{33} \end{pmatrix}$

(12) *If* $a = 0, \lambda \neq \mu, \lambda \neq \frac{1}{2}, \mu \neq \frac{1}{2}, t \neq 0,$ *then* $D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11} & 0 & 0 \\ 0 & 0 & d_{22} & 0 \\ 0 & 0 & 0 & d_{33} \end{pmatrix}$

3. Main Result

3.1. **Local derivation of quantum genetic Lotka-Volterra algebra on $\mathbb{M}_2(\mathbb{C})$.** This subsection looks deeper into the FQGLV-A algebra to understand its local derivation space in detail. We will check whether the local derivation space of such algebras is the same as the standard derivation space. The answer to this question is important in view of understanding the structure and properties of the FQGLV-A algebras in a better way.

Definition 3.1. A linear mapping $D : A_t \to A_t$ is called local derivation if for any $x \in A_t$ there exist a derivation d depending on x such that $D(x) = d_x(x)$.

The upcoming theorem will reveal the characteristics of the local derivation space in the FQGLV-A algebra. It will also investigate whether or not these local derivations qualify as derivations.

Theorem 3.2. *Let $D : A_t \to A_t$ be any local derivation then D is a derivation.*

Proof. Referring to Theorem 2.3, we shall consider all cases of the derivation depending on the values of parameters. First notice that $\{e_0, e_1, e_2, e_3\}$ is the basis for $M_2(\mathbb{C})^*$, where

$$e_0 = (1,0,0,0), \quad e_1 = (0,1,0,0), \quad e_2 = (0,0,1,0), \quad e_3 = (0,0,0,1).$$

Assume that D is local derivation. Then

$$D(e_i) = \sum_{k=0}^{3} D_{ik} e_k, \quad i \in \{0,1,2,3\}.$$

Case 1. Let $a \neq 0, \lambda \neq \mu$. Then

$$D(e_0) = \sum_{k=0}^{3} D_{0k} e_k = d_{e_0}(e_0) = 0.$$

Hence, $D_{0k} = 0$ for all $k \in \{0,1,2,3\}$. Next, consider

$$D(e_1) = \sum_{k=0}^{3} D_{1k} e_k = d_{e_1}(e_1) = d_{11}^{(e_1)} e_1.$$

Comparing both sides one gets $D_{10} = D_{12} = D_{13} = 0$, and $D_{11} = d_{11}^{(e_1)}$. Similarly, consider

$$D(e_2) = \sum_{k=0}^{3} D_{2k} e_k = d_{e_2}(e_2) = d_{22}^{(e_2)} e_2.$$

Thus, $D_{20} = D_{21} = D_{23} = 0$, and $D_{22} = d_{22}^{(e_2)}$. Finally, take

$$D(e_3) = \sum_{k=0}^{3} D_{3k} e_k = d_{e_3}(e_3) = 0,$$

which implies that, $D_{3k} = 0$ for all $k \in \{0,1,2,3\}$. Therefore, the matrix representation of D is given by

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11}^{(e_1)} & 0 & 0 \\ 0 & 0 & d_{22}^{(e_2)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, Theorem 2.3 implies that D is a derivation.

Case 2. Let $a \neq 0, \lambda \neq 0, \lambda = \mu$. Then,

$$D(e_0) = \sum_{k=0}^{3} D_{0k} e_k = d_{e_0}(e_0) = 0.$$

Hence, $D_{0k} = 0$ for all $k \in \{0, 1, 2, 3\}$. Now, let us delve into

$$D(e_1) = \sum_{k=0}^{3} D_{1k}e_k = d_{e_1}(e_1) = d_{11}^{(e_1)}e_1 + d_{12}^{(e_1)}e_2.$$

By aligning the terms on each side, we deduce $D_{10} = D_{13} = 0, D_{11} = d_{11}^{(e_1)}$, and $D_{12} = d_{12}^{(e_1)}$. Moving forward, let us examine

$$D(e_2) = \sum_{k=0}^{3} D_{2k}e_k = d_{e_2}(e_2) = d_{21}^{(e_2)}e_1 + d_{22}^{(e_2)}e_2.$$

Aligning both sides term by term, we conclude $D_{20} = D_{23} = 0, D_{21} = d_{21}^{(e_2)}$, and $D_{22} = d_{22}^{(e_2)}$ Finally,

$$D(e_3) = \sum_{k=0}^{3} D_{3k}e_k = d_{e_3}(e_3) = 0.$$

Hence, $D_{3k} = 0$ for all $k \in \{0, 1, 2, 3\}$. Therefore, the matrix form of D is given by

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11}^{(e_1)} & d_{12}^{(e_1)} & 0 \\ 0 & d_{21}^{(e_2)} & d_{22}^{(e_2)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which by Theorem 2.3 yields that D is a derivation.

Case 3. Let $a = 0, \lambda = \mu = \frac{1}{2}, t = 0$. Then

$$D(e_0) = \sum_{k=0}^{3} D_{0k}e_k = d_{e_0}(e_0) = d_{03}^{(e_0)}e_3.$$

Hence, $D_{00} = D_{01} = D_{02} = 0$, and $D_{03} = d_{03}^{(e_0)}$. Following this, we explore

$$D(e_1) = \sum_{k=0}^{3} D_{1k}e_k = d_{e_1}(e_1) = d_{11}^{(e_1)}e_1 + d_{12}^{(e_1)}e_2 + d_{13}^{(e_1)}e_3.$$

A comparison of the terms yields $D_{10} = 0, D_{11} = d_{11}^{(e_1)}, D_{12} = d_{12}^{(e_1)}$ and $D_{13} = d_{13}^{(e_1)}$. Next, consider

$$D(e_2) = \sum_{k=0}^{3} D_{2k}e_k = d_{e_2}(e_2) = d_{21}^{(e_2)}e_1 + d_{22}^{(e_2)}e_2 + d_{23}^{(e_2)}e_3.$$

Comparing both sides one gets $D_{20} = 0, D_{21} = d_{21}^{(e_2)}, D_{22} = d_{22}^{(e_2)}$ and $D_{23} = d_{23}^{(e_2)}$ Finally, we have

$$D(e_3) = \sum_{k=0}^{3} D_{3k}e_k = d_{e_3}(e_3) = d_{31}^{(e_3)}e_1 + d_{32}^{(e_3)}e_2 + d_{33}^{(e_3)}e_3.$$

Equating terms from both sides reveals $D_{30}=0, D_{31}=d_{31}^{(e_3)}, D_{32}=d_{32}^{(e_3)}$ and $D_{33}=d_{33}^{(e_3)}$ Therefore,

$$D=\begin{pmatrix}0 & 0 & 0 & d_{13}^{(e_0)}\\ 0 & d_{11}^{(e_1)} & d_{12}^{(e_1)} & d_{13}^{(e_1)}\\ 0 & d_{21}^{(e_2)} & d_{22}^{(e_2)} & d_{23}^{(e_2)}\\ 0 & d_{31}^{(e_3)} & d_{32}^{(e_3)} & d_{33}^{(e_3)}\end{pmatrix}$$

which by Theorem 2.3 implies that D is a derivation.

Case 4. Let $a=0, \lambda=\mu\neq\frac{1}{2}, t=0$. Then,

$$D(e_0)=\sum_{k=0}^{3}D_{0k}e_k=d_{e_0}(e_0)=d_{03}^{(e_0)}e_3.$$

Hence, $D_{00}=D_{01}=D_{02}=0$, and $D_{03}=d_{03}^{(e_0)}$. Now, consider

$$D(e_1)=\sum_{k=0}^{3}D_{1k}e_k=d_{e_1}(e_1)=d_{11}^{(e_1)}e_1+d_{12}^{(e_1)}e_2.$$

Comparing both sides one has that $D_{10}=D_{13}=0, D_{11}=d_{11}^{(e_1)}$, and $D_{12}=d_{12}^{(e_1)}$. Next, consider

$$D(e_2)=\sum_{k=0}^{3}D_{2k}e_k=d_{e_2}(e_2)=d_{21}^{(e_2)}e_1+d_{22}^{(e_2)}e_2.$$

Matching corresponding terms on both sides, we ascertain $D_{20}=D_{23}=0, D_{21}=d_{21}^{(e_2)}$, and $D_{22}=d_{22}^{(e_2)}$. Finally,

$$D(e_3)=\sum_{k=0}^{3}D_{3k}e_k=d_{e_3}(e_3)=d_{33}^{(e_3)}e_3.$$

Aligning both sides term by term, we conclude $D_{30}=D_{31}=D_{32}=0$, and $D_{33}=d_{33}^{(e_3)}$ Therefore,

$$D=\begin{pmatrix}0 & 0 & 0 & d_{13}^{(e_0)}\\ 0 & d_{11}^{(e_1)} & d_{12}^{(e_1)} & 0\\ 0 & d_{21}^{(e_2)} & d_{22}^{(e_2)} & 0\\ 0 & 0 & 0 & d_{33}^{(e_3)}\end{pmatrix}$$

so, Theorem 2.3 yields D is a derivation.

Case 5. Let $a=0, \lambda=\frac{1}{2}, \mu\neq\frac{1}{2}, t=0$. Then

$$D(e_0)=\sum_{k=0}^{3}D_{0k}e_k=d_{e_0}(e_0)=d_{03}^{(e_0)}e_3.$$

Hence, $D_{00}=D_{01}=D_{02}=0$, and $D_{03}=d_{03}^{(e_0)}$. Now, consider

$$D(e_1)=\sum_{k=0}^{3}D_{1k}e_k=d_{e_1}(e_1)=d_{11}^{(e_1)}e_1+d_{13}^{(e_1)}e_3.$$

Comparing both sides one finds $D_{10} = D_{12} = 0, D_{11} = d_{11}^{(e_1)}$, and $D_{13} = d_{13}^{(e_1)}$. Moving forward, let us examine

$$D(e_2) = \sum_{k=0}^{3} D_{2k}e_k = d_{e_2}(e_2) = d_{22}^{(e_2)}e_2.$$

From the equivalence of both sides, it follows that $D_{20} = D_{21} = D_{23} = 0$, and $D_{22} = d_{22}^{(e_2)}$. Moreover, we have

$$D(e_3) = \sum_{k=0}^{3} D_{3k}e_k = d_{e_3}(e_3) = d_{31}^{(e_3)}e_1 + d_{33}^{(e_3)}e_3.$$

Aligning both sides term by term, we conclude $D_{30} = D_{32} = 0,\ D_{31} = d_{31}^{(e_3)}$ and $D_{33} = d_{33}^{(e_3)}$ Therefore,

$$D = \begin{pmatrix} 0 & 0 & 0 & d_{13}^{(e_0)} \\ 0 & d_{11}^{(e_1)} & 0 & d_{13}^{(e_1)} \\ 0 & 0 & d_{22}^{(e_2)} & 0 \\ 0 & d_{31}^{(e_3)} & 0 & d_{33}^{(e_3)} \end{pmatrix}.$$

By Theorem 2.3, one gets D is a derivation.

Case 6. Let $a = 0, \lambda \neq \frac{1}{2}, \mu = \frac{1}{2}, t = 0$. Then

$$D(e_0) = \sum_{k=0}^{3} D_{0k}e_k = d_{e_0}(e_0) = d_{03}^{(e_0)}e_3.$$

Hence, $D_{00} = D_{01} = D_{02} = 0$, and $D_{03} = d_{03}^{(e_0)}$. Now, consider

$$D(e_1) = \sum_{k=0}^{3} D_{1k}e_k = d_{e_1}(e_1) = d_{11}^{(e_1)}e_1.$$

Matching corresponding terms on both sides, we ascertain $D_{10} = D_{12} = D_{13} = 0$, and $D_{11} = d_{11}^{(e_1)}$. Next, consider

$$D(e_2) = \sum_{k=0}^{3} D_{2k}e_k = d_{e_2}(e_2) = d_{22}^{(e_2)}e_2 + d_{23}^{(e_2)}e_3.$$

Comparing both sides one gets $D_{20} = D_{21} = 0, D_{22} = d_{22}^{(e_2)}$, and $D_{23} = d_{23}^{(e_2)}$. Finally, take

$$D(e_3) = \sum_{k=0}^{3} D_{3k}e_k = d_{e_3}(e_3) = d_{32}^{(e_3)}e_2 + d_{33}^{(e_3)}e_3.$$

From the equivalence of both sides, it follows that $D_{30} = D_{31} = 0,\ D_{32} = d_{32}^{(e_3)}$ and $D_{33} = d_{33}^{(e_3)}$ Therefore, D has the following matrix representation:

$$D = \begin{pmatrix} 0 & 0 & 0 & d_{13}^{(e_0)} \\ 0 & d_{11}^{(e_1)} & 0 & 0 \\ 0 & 0 & d_{22}^{(e_2)} & d_{23}^{(e_2)} \\ 0 & 0 & d_{32}^{(e_3)} & d_{33}^{(e_3)} \end{pmatrix}.$$

Hence, it is a derivation by Theorem 2.3.

Case 7. Let $a = 0, \lambda \neq \mu, \lambda \neq \frac{1}{2}, \mu \neq \frac{1}{2}, t = 0$. Then

$$D(e_0) = \sum_{k=0}^{3} D_{0k} e_k = d_{e_0}(e_0) = d_{03}^{(e_0)} e_3.$$

Thus, $D_{00} = D_{01} = D_{02} = 0$, and $D_{03} = d_{03}^{(e_0)}$. Now, consider

$$D(e_1) = \sum_{k=0}^{3} D_{1k} e_k = d_{e_1}(e_1) = d_{11}^{(e_1)} e_1.$$

Comparing both sides one gets $D_{10} = D_{12} = D_{13} = 0$, and $D_{11} = d_{11}^{(e_1)}$. Next, consider

$$D(e_2) = \sum_{k=0}^{3} D_{2k} e_k = d_{e_2}(e_2) = d_{22}^{(e_2)} e_2.$$

A comparison of the terms yields $D_{20} = D_{21} = D_{23} = 0$, and $D_{22} = d_{22}^{(e_2)}$. Finally, Take

$$D(e_3) = \sum_{k=0}^{3} D_{3k} e_k = d_{e_3}(e_3) = d_{33}^{(e_3)} e_3.$$

By correlating elements from each side, we determine $D_{30} = D_{31} = D_{32} = 0$, and $D_{33} = d_{33}^{(e_3)}$. Therefore, the matrix form of D has the following form:

$$D = \begin{pmatrix} 0 & 0 & 0 & d_{13}^{(e_0)} \\ 0 & d_{11}^{(e_1)} & 0 & 0 \\ 0 & 0 & d_{22}^{(e_2)} & 0 \\ 0 & 0 & 0 & d_{33}^{(e_3)} \end{pmatrix}.$$

By Theorem 2.3, we conclude that D is a derivation.

Case 8. Let $a = 0, \lambda = \mu = \frac{1}{2}, t \neq 0$. Then

$$D(e_0) = \sum_{k=0}^{3} D_{0k} e_k = d_{e_0}(e_0) = 0.$$

which implies that, $D_{00} = D_{01} = D_{02} = D_{03} = 0$. Now, we proceed to analyse

$$D(e_1) = \sum_{k=0}^{3} D_{1k}e_k = d_{e_1}(e_1) = d_{11}^{(e_1)}e_1 + d_{12}^{(e_1)}e_2 + d_{13}^{(e_1)}e_3.$$

A comparison of the terms yields $D_{10} = 0, D_{11} = d_{11}^{(e_1)}, D_{12} = d_{12}^{(e_1)}$ and $D_{13} = d_{13}^{(e_1)}$. Next, consider

$$D(e_2) = \sum_{k=0}^{3} D_{2k}e_k = d_{e_2}(e_2) = d_{21}^{(e_2)}e_1 + d_{22}^{(e_2)}e_2 + d_{23}^{(e_2)}e_3.$$

Comparing both sides one gets $D_{20} = 0, D_{21} = d_{21}^{(e_2)}, D_{22} = d_{22}^{(e_2)}$ and $D_{23} = d_{23}^{(e_2)}$ Finally, Take

$$D(e_3) = \sum_{k=0}^{3} D_{3k}e_k = d_{e_3}(e_3) = d_{31}^{(e_3)}e_1 + d_{32}^{(e_3)}e_2 + d_{33}^{(e_3)}e_3.$$

Aligning both sides term by term, we conclude $D_{30} = 0, D_{31} = d_{31}^{(e_3)}, D_{32} = d_{32}^{(e_3)}$ and $D_{33} = d_{33}^{(e_3)}$ Therefore, D has the following form:

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11}^{(e_1)} & d_{12}^{(e_1)} & d_{13}^{(e_1)} \\ 0 & d_{21}^{(e_2)} & d_{22}^{(e_2)} & d_{23}^{(e_2)} \\ 0 & d_{31}^{(e_3)} & d_{32}^{(e_3)} & d_{33}^{(e_3)} \end{pmatrix}.$$

By Theorem 2.3, we infer that D is a derivation.

Case 9. Let $a = 0, \lambda = \mu \neq \frac{1}{2}, t \neq 0$. Then

$$D(e_0) = \sum_{k=0}^{3} D_{0k}e_k = d_{e_0}(e_0) = 0.$$

Thus, $D_{00} = D_{01} = D_{02} = D_{03} = 0$. Now, consider

$$D(e_1) = \sum_{k=0}^{3} D_{1k}e_k = d_{e_1}(e_1) = d_{11}^{(e_1)}e_1 + d_{12}^{(e_1)}e_2.$$

Comparing both sides, one finds $D_{10} = D_{13} = 0, D_{11} = d_{11}^{(e_1)}$, and $D_{12} = d_{12}^{(e_1)}$. Next, consider

$$D(e_2) = \sum_{k=0}^{3} D_{2k}e_k = d_{e_2}(e_2) = d_{21}^{(e_2)}e_1 + d_{22}^{(e_2)}e_2.$$

From the equivalence of both sides, it follows that $D_{20} = D_{23} = 0, D_{21} = d_{21}^{(e_2)}$, and $D_{22} = d_{22}^{(e_2)}$. Finally, Take

$$D(e_3) = \sum_{k=0}^{3} D_{3k}e_k = d_{e_3}(e_3) = d_{33}^{(e_3)}e_3.$$

Aligning both sides term by term, we conclude $D_{30} = D_{31} = D_{32} = 0$, and $D_{33} = d_{33}^{(e_3)}$ Therefore, the matrix representation of D as follows:

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11}^{(e_1)} & d_{12}^{(e_1)} & 0 \\ 0 & d_{21}^{(e_2)} & d_{22}^{(e_2)} & 0 \\ 0 & 0 & 0 & d_{33}^{(e_3)} \end{pmatrix}.$$

Which is clear a derivation by Theorem 2.3.

Case 10. Let $a = 0, \lambda = \frac{1}{2}, \mu \neq \frac{1}{2}, t \neq 0$. Then

$$D(e_0) = \sum_{k=0}^{3} D_{0k}e_k = d_{e_0}(e_0) = 0.$$

Hence, $D_{00} = D_{01} = D_{02} = D_{03} = 0$. Now, consider

$$D(e_1) = \sum_{k=0}^{3} D_{1k}e_k = d_{e_1}(e_1) = d_{11}^{(e_1)}e_1 + d_{13}^{(e_1)}e_3.$$

Matching corresponding terms on both sides, we ascertain $D_{10} = D_{12} = 0, D_{11} = d_{11}^{(e_1)}$, and $D_{13} = d_{13}^{(e_1)}$. Next, consider

$$D(e_2) = \sum_{k=0}^{3} D_{2k}e_k = d_{e_2}(e_2) = d_{22}^{(e_2)}e_2.$$

From the equivalence of both sides, it follows that $D_{20} = D_{21} = D_{23} = 0$, and $D_{22} = d_{22}^{(e_2)}$. Finally, Take

$$D(e_3) = \sum_{k=0}^{3} D_{3k}e_k = d_{e_3}(e_3) = d_{31}^{(e_3)} + d_{33}^{(e_3)}e_3.$$

Comparing both sides, one gets $D_{30} = D_{32} = 0,\ D_{31} = d_{31}^{(e_3)}$ and $D_{33} = d_{33}^{(e_3)}$ Therefore, the form of D is given by:

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11}^{(e_1)} & 0 & d_{13}^{(e_1)} \\ 0 & 0 & d_{22}^{(e_2)} & 0 \\ 0 & d_{31}^{(e_3)} & 0 & d_{33}^{(e_3)} \end{pmatrix}.$$

Using Theorem 2.3, one infers that D is a derivation.

Case 11. Let $a = 0, \lambda \neq \frac{1}{2}, \mu = \frac{1}{2}, t \neq 0$. Then

$$D(e_0) = \sum_{k=0}^{3} D_{0k}e_k = d_{e_0}(e_0) = 0.$$

Consequently, $D_{00} = D_{01} = D_{02} = D_{03} = 0$. Now, consider

$$D(e_1) = \sum_{k=0}^{3} D_{1k}e_k = d_{e_1}(e_1) = d_{11}^{(e_1)}e_1.$$

Comparing both sides, one gets $D_{10} = D_{12} = D_{13} = 0$, and $D_{11} = d_{11}^{(e_1)}$. Next, consider

$$D(e_2) = \sum_{k=0}^{3} D_{2k} e_k = d_{e_2}(e_2) = d_{22}^{(e_2)} e_2 + d_{23}^{(e_2)} e_3.$$

Aligning both sides term by term, we conclude that $D_{20} = D_{21} = 0, D_{22} = d_{22}^{(e_2)}$, and $D_{23} = d_{23}^{(e_2)}$. Finally, Take

$$D(e_3) = \sum_{k=0}^{3} D_{3k} e_k = d_{e_3}(e_3) = d_{32}^{(e_3)} e_2 + d_{33}^{(e_3)} e_3.$$

Equating terms from both sides reveals $D_{30} = D_{31} = 0,\ D_{32} = d_{32}^{(e_3)}$ and $D_{33} = d_{33}^{(e_3)}$ Therefore, D has the following matrix form:

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11}^{(e_1)} & 0 & 0 \\ 0 & 0 & d_{22}^{(e_2)} & d_{23}^{(e_2)} \\ 0 & 0 & d_{32}^{(e_3)} & d_{33}^{(e_3)} \end{pmatrix}.$$

Again, by Theorem 2.3, we find that D is a derivation.

Case 12. Let $a = 0, \lambda \neq \mu, \lambda \neq \frac{1}{2}, \mu \neq \frac{1}{2}, t \neq 0$. Then

$$D(e_0) = \sum_{k=0}^{3} D_{0k} e_k = d_{e_0}(e_0) = 0.$$

Hence, $D_{00} = D_{01} = D_{02} = D_{03} = 0$. Now, consider

$$D(e_1) = \sum_{k=0}^{3} D_{1k} e_k = d_{e_1}(e_1) = d_{11}^{(e_1)} e_1.$$

Comparing both sides, one gets $D_{10} = D_{12} = D_{13} = 0$, and $D_{11} = d_{11}^{(e_1)}$. Next, consider

$$D(e_2) = \sum_{k=0}^{3} D_{2k} e_k = d_{e_2}(e_2) = d_{22}^{(e_2)} e_2.$$

Equating terms from both sides reveals $D_{20} = D_{21} = D_{23} = 0$, and $D_{22} = d_{22}^{(e_2)}$. Finally, Take

$$D(e_3) = \sum_{k=0}^{3} D_{3k} e_k = d_{e_3}(e_3) = d_{33}^{(e_3)} e_3.$$

A comparison of the terms yields $D_{30} = D_{31} = D_{32} = 0$, and $D_{33} = d_{33}^{(e_3)}$. Therefore,

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11}^{(e_1)} & 0 & 0 \\ 0 & 0 & d_{22}^{(e_2)} & 0 \\ 0 & 0 & 0 & d_{33}^{(e_3)} \end{pmatrix}.$$

Which is clear a derivation.

Hence, across all 12 cases, it has been demonstrated that every local derivation indeed is a derivation. Therefore, any local derivation $D : A_t \to A_t$ is a derivation, thereby concluding the proof. $\square$

3.2. **2-Local derivation of Quantum genetic Lotka-Volterra algebra on $\mathbb{M}_2(\mathbb{C})$..** In this section, we investigate thee space of 2-Local derivations of quantum genetic Lotka-Volterra algebra on $\mathbb{M}_2(\mathbb{C})$. In particular, we aim to determine whether every 2-local derivation is, in fact, derivation.

Definition 3.3. A mapping $\nabla : A_t \to A_t$ (not necessary linear) is called $2-$local derivation of algebra A_t if for every $x, y \in A_t$ there exist a derivation $d^{(x,y)}$ such that

$$\nabla(x) = d^{(x,y)}(x),\ \nabla(y) = d^{(x,y)}(y).$$

Theorem 3.4. *Any $2-$local derivation of quantum genetic Lotka-Volterra algebra on $\mathbb{M}_2(\mathbb{C})$ is a derivation.*

Proof. We will focus on three specific cases, with the understanding that the remaining cases will be addressed similarly.

Case 1. Let $a \neq 0, \lambda \neq \mu$. Assume that ∇ is $2-$local derivation, then

$$\nabla(x) = d^{(x,y)}(x) = \alpha_1 d_{11}^{(x,y)} e_1 + \alpha_2 d_{22}^{(x,y)} e_2$$

for each $x, y \in A_t$. Then for any $y' \in A_t$ on has that

$$\nabla(x) = d^{(x,y')}(x) = \alpha_1 d_{11}^{(x,y')} e_1 + \alpha_2 d_{22}^{(x,y')} e_2.$$

This implies that $d^{(x,y)}(x) = d^{(x,y')}(x)$. Thus, $d_{11}^{(x,y)} = d_{11}^{(x,y')}$, and $d_{22}^{(x,y)} = d_{22}^{(x,y')}$ for any $y, y' \in A_t$. This means that $d_{11}^{(x,y)} := d_{11}^{(x)}$, and $d_{22}^{(x,y)} := d_{22}^{(x)}$. Hence,

$$\nabla(x) = \alpha_1 d_{11}^{(x)} e_1 + \alpha_2 d_{22}^{(x)} e_2.$$

Therefore,

$$\nabla(x) = \alpha_1 d_{11}^{(x)} e_1 + \alpha_2 d_{22}^{(x)} e_2 = d^{(x)}(x) = D(x).$$

Then ∇ is a local derivation. Hence, using Theorem 3.2 (case 1), one has that ∇ is a derivation.

Case 2. Let $a \neq 0, \lambda \neq 0, \lambda = \mu$. Suppose that ∇ is a $2-$local derivation, then there exist a derivation $d^{(u,v)}$ such that $\nabla(u) = d^{(u,v)}(u)$ and $\nabla(v) = d^{(u,v)}(v)$. Let $u \in A_t$ such that $u = \lambda_0 e_0 + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in A_t$. If $\lambda_1 = \lambda_2 = 0$, then

$$\nabla(u) = d^{(u,v)}(u) = \lambda_0 d^{(u,v)}(e_0) + \lambda_3 d^{(u,v)}(e_3) = 0.$$

If at least one of λ_1 and λ_2 is non-zero, then

$$\nabla(u) = d^{(u,v)}(u) = (\lambda_1 d_{11}^{(u,v)} + \lambda_2 d_{21}^{(u,v)})e_1 + (\lambda_1 d_{12}^{(u,v)} + \lambda_2 d_{22}^{(u,v)})e_2.$$

On the other hand, one can write $\nabla(u)$ as follows:

$$\nabla(u) = d^{(u,v')}(u) = (\lambda_1 d_{11}^{(u,v')} + \lambda_2 d_{21}^{(u,v')})e_1 + (\lambda_1 d_{12}^{(u,v')} + \lambda_2 d_{22}^{(u,v)})e_2.$$

Comparing the last to expressions of $\nabla(u)$ one has that

$$\lambda_1(d_{11}^{(u,v)} - d_{11}^{(u,v')}) + \lambda_2(d_{21}^{(u,v)} - d_{21}^{(u,v')}) = 0$$
$$\lambda_1(d_{12}^{(u,v)} - d_{12}^{(u,v')}) + \lambda_2(d_{22}^{(u,v)} - d_{22}^{(u,v')}) = 0.$$

Due to arbitrariness of v, v' we have the following solution of the above system

$$d_{11}^{(u,v)} = d_{11}^{(u,v')},\ d_{21}^{(u,v)} = d_{21}^{(u,v')},\ d_{12}^{(u,v)} = d_{12}^{(u,v')},\ d_{22}^{(u,v)} = d_{22}^{(u,v')}.$$

Which implies that ∇ depend on u only. Hence, we may write $\nabla(u)$ as follows:

$$\nabla(u) = (\lambda_1 d_{11}^{(u)} + \lambda_2 d_{21}^{(u)})e_1 + (\lambda_1 d_{12}^{(u)} + \lambda_2 d_{22}^{(u)})e_2 = d^{(u)}(u) = D(u).$$

Then by Theorem 3.2 we have that D is a derivation. Thus, ∇ is a derivation.

Case 3. Assume that $a = 0, \lambda = \mu = \frac{1}{2}, t = 0$. Suppose that ∇ is 2−local derivation, then there exist a derivation $d^{(u,v)}$ such that $\nabla(u) = d^{(u,v)}(u)$ and $\nabla(v) = d^{(u,v)}(v)$. Let $u \in A_t$ such that $u = \lambda_0 e_0 + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in A_t$ then

$$\begin{aligned}\nabla(u) = d^{(u,v)}(u) &= (\lambda_1 d_{11}^{(u,v)} + \lambda_2 d_{21}^{(u,v)} + \lambda_3 d_{31}^{(u,v)})e_1 + (\lambda_1 d_{12}^{(u,v)} + \lambda_2 d_{22}^{(u,v)} + \lambda_3 d_{32}^{(u,v)})e_2 \\ &+ (\lambda_0 d_{03}^{(u,v)} + \lambda_1 d_{13}^{(u,v)} + \lambda_2 d_{23}^{(u,v)} + \lambda_3 d_{33}^{(u,v)})e_3.\end{aligned}$$

On the other hand,

$$\begin{aligned}\nabla(u) = d^{(u,v')}(u) &= (\lambda_1 d_{11}^{(u,v')} + \lambda_2 d_{21}^{(u,v')} + \lambda_3 d_{31}^{(u,v')})e_1 + (\lambda_1 d_{12}^{(u,v')} + \lambda_2 d_{22}^{(u,v')} + \lambda_3 d_{32}^{(u,v)})e_2 \\ &+ (\lambda_0 d_{03}^{(u,v')} + \lambda_1 d_{13}^{(u,v')} + \lambda_2 d_{23}^{(u,v')} + \lambda_3 d_{33}^{(u,v')})e_3.\end{aligned}$$

Due to the arbitrariness of v, v' we have the following solution of the above system

$$d11^{(u,v)} = d_{11}^{(u,v')},\ d_{21}^{(u,v)} = d_{21}^{(u,v')},\ d_{31}^{(u,v)} = d_{31}^{(u,v')}$$
$$d_{12}^{(u,v)} = d_{12}^{(u,v')},\ d_{22}^{(u,v)} = d_{22}^{(u,v')},\ d_{32}^{(u,v)} = d_{32}^{(u,v')}$$
$$d_{13}^{(u,v)} = d_{13}^{(u,v')},\ d_{23}^{(u,v)} = d_{23}^{(u,v')},\ d_{33}^{(u,v)} = d_{33}^{(u,v')},\ d_{03}^{(u,v)} = d_{03}^{(u,v')}$$

which implies that ∇ depends on u only. Hence, we may write $\nabla(u)$ as follows:

$$\begin{aligned}\nabla(u) &= (\lambda_1 d_{11}^{(u)} + \lambda_2 d_{21}^{(u)} + \lambda_3 d_{31}^{(u)})e_1 + (\lambda_1 d_{12}^{(u)} + \lambda_2 d_{22}^{(u)} + \lambda_3 d_{32}^{(u)})e_2 \\ &+ (\lambda_0 d_{03}^{(u)} + \lambda_1 d_{13}^{(u)} + \lambda_2 d_{23}^{(u)} + \lambda_3 d_{33}^{(u)})e_3 = d^{(u)}(u)D(u).\end{aligned}$$

Then by Theorem 3.2 we have that D is a derivation. Thus, ∇ is a derivation.

The proof of the reset cases will process in the same manner. $\square$

3.3. **Rota-Baxter operators.** In this section, we are going to investigate Rota-Baxter operator which is defined on quantum genetic Lotka-Volterra algebra on $\mathbb{M}_2(\mathbb{C})$.
Recall that a linear operator $R : A \to A$ is called Rota-Baxter operator if

$$R(x)R(y) = R[R(x)y + xR(y) + \theta xy], \text{for all } x, y \in A.$$

The pair (A, R) is called Rota–Baxter algebra of weight θ.

Remark 3.5. Let R be a Rota–Baxter operator of weight θ then $-\theta\mathbb{I} - R$ is also a Rota–Baxter operator. Further, if σ in the grounded field then σR is a Rota–Baxter operator of weight $\sigma\theta$.

Now, we are going to investigate some properties related to a Rota-Baxter operator defined on the quantum genetic Lotka-Volterra algebras on $\mathbb{M}_2(\mathbb{C})$.

Proposition 3.6. *Let A be a quantum genetic Lotka-Volterra algebra and let $R : A \to A$ be a Rota-Baxter operator. Then $\varphi \circ R \circ \varphi^{-1}$ is also Rota-Baxter operator, where φ is an automorphism.*

Proof. Set $\hat{R} = \varphi \circ R \circ \varphi^{-1}$. We are going to show that

$$\hat{R}(x)\hat{R}(y) = \hat{R}\left[\hat{R}(x)y + x\hat{R}(y) + \theta xy\right].$$

Consider

$$\hat{R}(x)\hat{R}(y) = \left(\varphi R\left(\varphi^{-1}(x)\right)\right)\left(\varphi R\left(\varphi^{-1}(y)\right)\right) = \varphi\left(R\left(\varphi^{-1}(x)\right)R\left(\varphi^{-1}(y)\right)\right).$$

Now, since R is a Rota-Baxter operator then

$$\varphi\left(R\left(\varphi^{-1}(x)\right)R\left(\varphi^{-1}(y)\right)\right) = \varphi\left[R\left[R\left(\varphi^{-1}(x)\right)\varphi^{-1}(y) + \varphi^{-1}(x)R\left(\varphi^{-1}(y)\right) + \theta\varphi^{-1}(x)\varphi^{-1}(y)\right]\right].$$

But φ is an automorphism, then last expression becomes as follows:

$$\begin{aligned}\varphi\left(R\left(\varphi^{-1}(x)\right)R\left(\varphi^{-1}(y)\right)\right) &= \varphi\left[R\left[R\left(\varphi^{-1}(x)\right)\varphi^{-1}(y)\right]\right] + \varphi\left[R\left[\varphi^{-1}(x)R\left(\varphi^{-1}(y)\right)\right]\right] \\ &+ \theta\varphi\left[R\left[\varphi^{-1}(xy)\right]\right].\end{aligned}$$

Let us take Right hand side of last expression one by one.

$$\varphi\left[R\left[R\left(\varphi^{-1}(x)\right)\varphi^{-1}(y)\right]\right] = \varphi(R\varphi^{-1})\varphi(R(\varphi^{-1}(x))\varphi(\varphi^{-1}(y)) = \hat{R}(\hat{R}(x)y).$$

The second term

$$\varphi\left[R\left[\varphi^{-1}(x)R\left(\varphi^{-1}(y)\right)\right]\right] = \varphi R\varphi^{-1}\varphi(\varphi^{-1}(x)R\left(\varphi^{-1}(y)\right)) = \hat{R}(x\hat{R}(y)).$$

The last team as follows:

$$\theta\varphi\left[R\left[\varphi^{-1}(xy)\right]\right] = \theta\hat{R}(xy).$$

Adding last three expressions one has

$$\hat{R}(x)\hat{R}(y) = \hat{R}(\hat{R}(x)y) + \hat{R}(x\hat{R}(y)) + \theta\hat{R}(xy) = \hat{R}\left[\hat{R}(x)y + x\hat{R}(y) + \theta xy\right].$$

Thus, $\hat{R}$ is a Rota-Baxter operator. $\square$

In what follows, for the sake of simplicity, we are going to describe the Rota-Baxter operators on quantum genetic Lotka-Volterra algebra $\mathcal{A}$ when $a = 0$, $\lambda = \mu = \frac{1}{2}$ and $t = 0$. Then, by (2.6) the table of multiplication of $\mathcal{A}$ is given by

$$e_0e_0 = e_0, \quad e_0e_j = \frac{1}{2}e_j, \ j \in \{1, 2, 3\}. \tag{3.1}$$

Assume that

$$R(e_i) = \sum_{k=0}^{3} \alpha_{ki}e_k.$$

Proposition 3.7. *Let R be a linear operator define on the algebra $\mathcal{A}$. Then the following are true:*

(1) $R(e_i)R(e_j) = \frac{\alpha_{0i}}{2}R(e_j) + \frac{\alpha_{0j}}{2}R(e_i)$
(2) *If $j \neq 0$, then $R(e_i)e_j = \frac{\alpha_{0i}}{2}e_j$.*
(3) $e_0R(e_j) = \frac{\alpha_{0j}}{2}e_0 + \frac{1}{2}R(e_j)$.
(4) *If $e_ie_j = 0$, then $R(e_i)R(e_j) = R(R(e_i)e_j + e_iR(e_j))$, is always true.*

Proof. (1) Now, using the table (3.1), we obtain

$$\begin{aligned} R(e_i)R(e_j) &= \left(\sum_{k=0}^{3} \alpha_{ki}e_k\right)\left(\sum_{m=0}^{3} \alpha_{mj}e_m\right) = \sum_{k=0}^{3}\left[\alpha_{ki}e_k \sum_{m=0}^{3} \alpha_{mj}e_m\right] \\ &= \alpha_{0i}e_0 \sum_{m=0}^{3} \alpha_{mj}e_m + \sum_{k=1}^{3}\left[\alpha_{ki}e_k \sum_{m=0}^{3} \alpha_{mj}e_m\right] \\ &= \alpha_{0i}\alpha_{0j}e_0 + \sum_{m=1}^{3} \frac{\alpha_{0i}\alpha_{mj}}{2}e_m + \sum_{m=1}^{3} \frac{\alpha_{0j}\alpha_{mi}}{2}e_m \\ &= \frac{\alpha_{0i}\alpha_{0j}}{2}e_0 + \frac{\alpha_{0i}\alpha_{0j}}{2}e_0 + \sum_{m=1}^{3} \frac{\alpha_{0i}\alpha_{mj}}{2}e_m + \sum_{m=1}^{3} \frac{\alpha_{0j}\alpha_{mi}}{2}e_m \\ &= \frac{\alpha_{0i}}{2}\sum_{m=0}^{3} \alpha_{mj}e_m + \frac{\alpha_{0j}}{2}\sum_{m=0}^{3} \alpha_{mi}e_m \\ &= \frac{\alpha_{0i}}{2}R(e_j) + \frac{\alpha_{0j}}{2}R(e_i). \end{aligned}$$

Note that if $i = j$, then $R(e_i)R(e_i) = \alpha_{0i}R(e_i)$.
(2) Consider $R(e_i)e_j$, since $j \neq 0$ then

$$R(e_i)e_j = \left(\sum_{k=0}^{3} \alpha_{ki}e_k\right)e_j = \alpha_{0i}e_0e_j = \frac{\alpha_{0i}}{2}e_j$$

(3) Similarly, one finds

$$\begin{aligned} e_0R(e_j) = e_0\sum_{k=0}^{3}\alpha_{kj}e_k &= \alpha_{0j}e_0 + \frac{\alpha_{1j}}{2}e_1 + \frac{\alpha_{2j}}{2}e_2 + \frac{\alpha_{3j}}{2}e_3 \\ &= \frac{\alpha_{0j}}{2}e_0 + \frac{\alpha_{0j}}{2}e_0 + \frac{\alpha_{1j}}{2}e_1 + \frac{\alpha_{2j}}{2}e_2 + \frac{\alpha_{3j}}{2}e_3 \\ &= \frac{\alpha_{0j}}{2}e_0 + \frac{1}{2}R(e_j). \end{aligned}$$

(4) By (1) the left hand side $R(e_i)R(e_j) = \frac{\alpha_{0i}}{2}R(e_j) + \frac{\alpha_{0j}}{2}R(e_i)$. Let us now consider the right hand side $R(R(e_i)e_j + e_iR(e_j))$. By (2) $R(e_i)e_j = \frac{\alpha_{0i}}{2}e_j$ and $e_iR(e_j) = \frac{\alpha_{0j}}{2}e_i$. Plugging these values into $R(R(e_i)e_j + e_iR(e_j))$, one gets

$$\begin{aligned} R(R(e_i)e_j + e_iR(e_j)) &= R\left(\frac{\alpha_{0i}}{2}e_j + \frac{\alpha_{0j}}{2}e_i\right) \\ &= \frac{\alpha_{0i}}{2}R(e_j) + \frac{\alpha_{0j}}{2}R(e_i) \\ &= R(e_i)R(e_j). \end{aligned}$$

The proof is complete. □

Due to (4) of Proposition (3.7), to describe all Rota-Baxter operators we only need to investigate the multiplication e_0e_j. Let us define the following notations. The linear operator R has the matrix form as follows $R = (\alpha_{ij})_{i,j=0}^{3}$. Let

$$K = \begin{pmatrix} R(e_0) \\ R(e_1) \\ R(e_2) \\ R(e_3) \end{pmatrix}$$

Theorem 3.8. *Let $R : \mathcal{A} \to \mathcal{A}$ be a linear operator. Then the following statements are true:*

(i) *R is an invertible Rota-Baxter operator if and only if $R = -\theta\mathbb{I}_{4\times 4}$;*
(ii) *R is not invertible Rota-Baxter operator if and only if the following system is satisfied*

$$(\theta\mathbb{I} + R^T)K = 0.$$

Proof. Let R be a linear operator and consider $R(e_0)R(e_j)$ where $j \in \{0,1,2,3\}$. From (1) Proposition (3.7), we get

$$R(e_0)R(e_j) = \frac{\alpha_{00}}{2}R(e_j) + \frac{\alpha_{0j}}{2}R(e_0). \tag{3.2}$$

On the other hand, let us find the form of the expression $R\left[R(e_0)e_j + e_0R(e_j) + \frac{\theta}{2}e_j\right]$. Using (2) and (3) of Proposition (3.7), we have

$$\begin{aligned} R\left[R(e_0)e_j + e_0R(e_j) + \frac{\theta}{2}e_j\right] &= R\left[\frac{\alpha_{00}}{2}e_j + \frac{\alpha_{0j}}{2}e_0 + \frac{1}{2}R(e_j) + \frac{\theta}{2}e_j\right] \\ &= \frac{\alpha_{00}}{2}R(e_j) + \frac{\alpha_{0j}}{2}R(e_0) + \frac{1}{2}R(R(e_j)) + \frac{\theta}{2}R(e_j) \end{aligned}$$

Equalizing (3.2) with the last equality we arrive at

$$R\left[R(e_j)+\theta e_j\right]=0.$$

(i). Now, if R is invertible then $R(e_j)+\theta e_j=0$. Hence, $R(e_j)=-\theta e_j$ this implies the required assertion.

Conversely, assume that $R(e_i)=-\theta e_i$ then again from (1) Proposition (3.7), we have

$$R(e_0)R(e_j)=(-\theta e_0)(-\theta e_1)=\frac{\theta^2}{2}e_j. \tag{3.3}$$

On the other hand,

$$\begin{aligned} & R\left[R(e_0)e_j+e_0R(e_j)+\theta e_0e_j\right] \\ = \quad & R\left[\frac{-\theta}{2}e_j-\frac{-\theta}{2}e_j+\frac{\theta}{2}e_j\right]=R\left(\frac{-\theta}{2}e_j\right)=\frac{\theta^2}{2}e_j. \end{aligned} \tag{3.4}$$

Comparing (3.3) with (3.4), one has that R is invertible Rota-Baxter operator.This completes the proofs of (1).

(ii). From the above equalities we infer that R is not invertible Rota-Baxter operator if and only if

$$R\left[R(e_j)+\theta e_j\right]=0.$$

$$R\left[R(e_j)+\theta e_j\right]=\sum_{k=0}^{3}\alpha_{kj}R(e_k)+\theta R(e_j)=(\theta+\alpha_{jj})R(e_j)+\sum_{j\neq k=0}^{3}\alpha_{kj}R(e_k)=0.$$

Then, for $j=0$, we obtain

$$\begin{aligned} & (\theta+\alpha_{00})R(e_0)+\alpha_{10}R(e_1)+\alpha_{20}R(e_2)+\alpha_30)R(e_3) \\ & =\begin{bmatrix}(\theta+\alpha_{00})R(e_0) & \alpha_{10} & \alpha_{20} & \alpha_{30}\end{bmatrix}K=0. \end{aligned}$$

Similarly, for $j=1,2,3$ one has that

$$\begin{aligned} \begin{bmatrix}\alpha_{01} & (\theta+\alpha_{11}) & \alpha_{21} & \alpha_{31}\end{bmatrix}K&=0 \\ \begin{bmatrix}\alpha_{02} & \alpha_{12} & (\theta+\alpha_{22}) & \alpha_{32}\end{bmatrix}K&=0 \\ \begin{bmatrix}\alpha_{03} & \alpha_{13} & \alpha_{23} & (\theta+\alpha_{33})\end{bmatrix}K&=0. \end{aligned}$$

Hence,

$$(\theta\mathbb{I}+R^T)K=0.$$

This completes the proof. $\square$

Let us now give the full description of the non-invertible Rota-Baxter operator in the considered case. Let M be a vector space generated by $\{R(e_0),R(e_1),R(e_2),R(e_3)\}$. Since R is not invertible, then the $dim(M)$ is either $1,2$ or 3.

The present investigation only addresses a case where the dimension of M is equal to one. The cases where the dimension of M is equal to two and three will be handled in a similar manner. In addition, we only take into account the case when $R(e_i)\neq 0$ for any $0\leq i\leq 3$.

Case 1. If $\dim(M) = 1$, then, without loss of generality, we may assume that $R(e_0)$ is the basis of M, then $R(e_j) = A_j R(e_0)$. Using (2) of Theorem (3.8), one gets

$$\begin{cases} \alpha_{00} + \theta + A_1\alpha_{10} + A_2\alpha_{20} + A_3\alpha_{30} = 0, \\ \alpha_{01} + A_1(\alpha_{11} + \theta) + A_2\alpha_{21} + A_3\alpha_{31} = 0, \\ \alpha_{02} + A_1\alpha_{12} + A_2(\alpha_{22} + \theta) + A_3\alpha_{32} = 0, \\ \alpha_{03} + A_1\alpha_{13} + A_2\alpha_{23} + A_3(\alpha_{33} + \theta) = 0, \\ \alpha_{ij} = A_j\alpha_{i0}. \end{cases} \tag{3.5}$$

Now, Let us define the following Matrix:

$$R^{[i]}_{jk} = \begin{cases} 1 & \text{if } j = 0 \text{ and } k = i, \\ A_1 & \text{if } j = 1 \text{ and } k = i, \\ A_2 & \text{if } j = 2 \text{ and } k = i, \\ A_3 & \text{if } j = 3 \text{ and } k = i, \\ 0 & \text{otherwise.} \end{cases}$$

To solve the System (3.5), we should consider 16 cases which depends on the value of the $\alpha_{00}, \alpha_{10}, \alpha_{20}$, and α_{30}.

Theorem 3.9. *Let R is not invertible Rota-Baxter operator which is defined as (2) in Theorem (3.8). Then the following statement hold true:*

(1) *If* $(\alpha_{00}, \alpha_{10}, \alpha_{20}, \alpha_{30}) = (\alpha_{00}, 0, 0, 0)$, *then* $R = -\theta R^{[0]}_{jk}$.
(2) *If* $(\alpha_{00}, \alpha_{10}, \alpha_{20}, \alpha_{30}) = (\alpha_{00}, 0, 0, \alpha_{30})$, *then* $R = \alpha_{00}R^{[0]}_{jk} - \left(\frac{\alpha_{00}+\theta}{A_3}\right) R^{[3]}_{jk}$.
(3) *If* $(\alpha_{00}, \alpha_{10}, \alpha_{20}, \alpha_{30}) = (\alpha_{00}, 0, \alpha_{20}, 0)$, *then* $R = \alpha_{00}R^{[0]}_{jk} - \left(\frac{\alpha_{00}+\theta}{A_2}\right) R^{[2]}_{jk}$.
(4) *If* $(\alpha_{00}, \alpha_{10}, \alpha_{20}, \alpha_{30}) = (\alpha_{00}, 0, \alpha_{20}, \alpha_{30})$, *then* $R = \alpha_{00}R^{[0]}_{jk} + \alpha_{20}R^{[2]}_{jk} - \left(\frac{\alpha_{00}+\theta+A_2\alpha_{20}}{A_3}\right) R^{[3]}_{jk}$.
(5) *If* $(\alpha_{00}, \alpha_{10}, \alpha_{20}, \alpha_{30}) = (\alpha_{00}, \alpha_{10}, 0, 0)$, *then* $R = \alpha_{00}R^{[0]}_{jk} - \left(\frac{\alpha_{00}+\theta}{A_1}\right) R^{[1]}_{jk}$.
(6) *If* $(\alpha_{00}, \alpha_{10}, \alpha_{20}, \alpha_{30}) = (\alpha_{00}, \alpha_{10}, 0, \alpha_{30})$, *then* $R = \alpha_{00}R^{[0]} + \alpha_{10}R^{[1]} - \frac{\alpha_{00}+\theta+A_1\alpha_{10}}{A_3} R^{[3]}$.
(7) *If* $(\alpha_{00}, \alpha_{10}, \alpha_{20}, \alpha_{30}) = (\alpha_{00}, \alpha_{10}, \alpha_{20}, 0)$, *then* $R = \alpha_{00}R^{[0]} + \alpha_{10}R^{[1]} - \frac{\alpha_{00}+\theta+A_1\alpha_{10}}{A_2} R^{[2]}$.
(8) *If* $(\alpha_{00}, \alpha_{10}, \alpha_{20}, \alpha_{30}) = (\alpha_{00}, \alpha_{10}, \alpha_{20}, \alpha_{30})$, *then* $R = \alpha_{00}R^{[0]} + \alpha_{10}R^{[1]} + \alpha_{20}R^{[2]} - \frac{\alpha_{00}+\theta+A_1\alpha_{10}+A_2\alpha_{30}}{A_3} R^{[3]}$.

Proof. We are going to proof only (1) the rest proof of the items can be processed in the same manner. Let $(\alpha_{00}, \alpha_{10}, \alpha_{20}, \alpha_{30}) = (\alpha_{00}, 0, 0, 0)$. Putting these values in the last equation of the System (3.5), on gets

$$\alpha_{11} = \alpha_{21} = \alpha_{31} = 0, \ \alpha_{12} = \alpha_{22} = \alpha_{32} = 0, \ \alpha_{13} = \alpha_{23} = \alpha_{33} = 0.$$

Form first equation of the same system, we have $\alpha_{00} = -\theta$. Then, from second equation, one has that $\alpha_{01} = -A_1\theta$, from third equation, we have $\alpha_{02} = -A_2\theta$ and from fourth equation, one gets $\alpha_{03} = -A_3\theta$. Hence, $R = -\theta R^{[0]}_{j,k}$.

□

Remark 3.10. The case when $\alpha_{00} = 0$ can be easily derived by symmetry from the cases considered in the Theorem (3.9).

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Conflicts of Interest

The authors declare that they have no conflicts of interest.

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