Gulf Journal of Mathematics
Vol 2, Issue 4 (2014) 32-42

GORENSTEIN $\mathcal{X}$-FLAT DIMENSION AND TATE HOMOLOGY

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ABSTRACT. In this paper, we study a Gorenstein $\mathcal{X}$-flat dimension and investigate Tate homology $\widehat{\mathcal{X}tor}$ of modules of finite Gorenstein $\mathcal{X}$-flat dimension. We prove that if R is a right $G\mathcal{X}F$-closed ring and M a R-module admitting a Tate $\mathcal{X}$-flat resolution, and if N is a left R-module admitting a Tate $\mathcal{X}$-flat resolution T_N, then for any $i \in \mathbb{Z}$, $\widehat{\mathcal{X}tor}_i^R(M, N) \cong H_i(M \otimes_R T_N)$.

1. Introduction and preliminaries

Throughout this paper, R will always denote an associative ring with identity element. All modules will be unital right R-modules unless otherwise is specified.

The notion of Tate homology for modules over group algebras has a natural extension to modules over Gorenstein rings and, more generally, to modules of finite Gorenstein projective dimension over any ring. This theory was introduced by John Tate in 1952 for group cohomology arising from class field theory [13]. It has been generalized through works of, in chronological order, Buchweitz [2], Avramov and Martsinkovsky [1], and Veliche [14] into a cohomology theory for modules with complete resolutions. In particular, Avramov and Martsinkovsky [1] extended the definition, based on Tate projective resolutions (or complete resolutions), so that it can work well for finite modules of finite G-dimension over a Noetherian ring. Veliche [14] and Christensen and Jørgensen [3] studied a Tate cohomology theory for complexes and Sather-Wagstaff et al. [11] constructed a theory of Tate cohomology in any abelian category $\mathcal{A}$ based on a so-called Tate $\mathcal{W}$-resolution, where $\mathcal{W}$ is a class of objects of $\mathcal{A}$. The parallel theory of Tate homology has been treated by Iacob [9] and Christensen and Jørgensen [3, 4]. Recently, Li Liang [10] develop a theory of Tate homology based on so-called Tate flat resolutions.

The notion of Gorenstein flat modules was introduced by Enochs, Jenda, and Torrecillas in [6]. An R-module M is called Gorenstein flat (G-flat for short) if there is an exact sequence

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

Date: Received: Jul 23, 2014; Accepted: Sep 4, 2014.
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2010 *Mathematics Subject Classification.* Primary 16D10, 13D05; Secondary 13D07, 18G25.
Key words and phrases. Gorenstein $\mathcal{X}$-flat module, Gorenstein $\mathcal{X}$-flat dimension, Tate $\mathcal{X}$-flat resolution, Tate homology.

of flat right R-modules such that $M \cong im(F_0 \to F^0)$ and such that $- \otimes I$ leaves the above sequence exact whenever I is an injective left R-module. Selvaraj et al. [12] introduced the concept of Gorenstein $\mathcal{X}$-flat modules and their properties. Using this concept, in this paper we investigate Tate homology $\widehat{\mathcal{X}tor}$ of modules of finite Gorenstein $\mathcal{X}$-flat dimension.

A right R-module M is called $\mathcal{X}$-flat [12] if $Tor_1^R(M, X) = 0$ holds for all left R-modules $X \in \mathcal{X}$ and a left R-module G is called $\mathcal{X}$-injective if $Ext_R^1(X, G) = 0$ holds for all left R-modules $X \in \mathcal{X}$. The $\mathcal{X}$-flat dimension of a right R-module M, denoted by $\mathcal{X}$-dim(M) ($\mathcal{X}$-$fd_R(M)$), is defined as $inf\{m$: there is a $\mathcal{X}$-flat resolution of the form $0 \to F_m \to \cdots \to F_0 \to M \to 0$ of $M\}$. If there is no such m, set $\mathcal{X}$-$fd_R(M) = \infty$.

A complex

$$\cdots \to X_1 \xrightarrow{\delta_1^X} X_0 \xrightarrow{\delta_0^X} X_{-1} \to \cdots$$

of R-modules will be denoted by (X, δ^X) or simply X. We frequently (and without warning) identify R-modules with complexes concentrated in degree 0. The nth cycle (resp., homology) of X is defined as $\ker \delta_n^X$ (resp., $\ker \delta_n^X / im\delta_{n+1}^X$) and it is denoted by $Z_n(X)$ (resp., $H_n(X)$). For any $m \in \mathbb{Z}$, $\Sigma^m X$ denotes the complex with the degree-n term $(\Sigma^m X)_n = X_{n-m}$ and whose boundary operators are $(-1)^m \delta_{n-m}^X$. We set $\Sigma M = \Sigma^1 M$.

If X and Y are both complexes, then by a morphism $\alpha : X \to Y$ we mean a sequence $\alpha_n : X_n \to Y_n$ such that $\alpha_{n-1}\delta_n^X = \delta_n^Y \alpha_n$ for each $n \in \mathbb{Z}$. A quasi isomorphism, indicated by the symbol " $\simeq$ ", is a morphism of complexes that induces an isomorphism in homology.

Let $\mathscr{C}$ be a class of left R-modules. Following [5], we say that a map $f \in Hom_R(C, M)$ with $C \in \mathscr{C}$ is a $\mathscr{C}$-precover of M, if the group homomorphism $Hom_R(C', f) : Hom_R(C', C) \to Hom_R(C', M)$ is surjective for each $C' \in \mathscr{C}$. A $\mathscr{C}$-precover $f \in Hom_R(C, M)$ of M is called a $\mathscr{C}$-cover of M if f is right minimal. That is, if $fg = f$ implies that g is an automorphism for each $g \in End_R(C)$. $\mathscr{C} \subseteq R - Mod$ is a precovering class (resp. covering class) provided that each module has a $\mathscr{C}$-precover (resp. $\mathscr{C}$-cover). Dually, we have the definition of $\mathscr{C}$ preenvelope (resp. $\mathscr{C}$ envelope).

This paper is divided into three sections. In the second section, we introduce the Gorenstein $\mathcal{X}$-flat dimension and prove that over a right $G\mathcal{X}F$-closed ring, for any R-module M and any integer $m \geq 0$, $G\mathcal{X}$-$fd_R(M) \leq m$ if and only if for each non-negative integer t with $0 \leq t \leq m$, there is an exact sequence of R-modules $0 \to C_m \to \cdots \to C_t \to \cdots \to C_1 \to C_0 \to M \to 0$ such that C_t is Gorenstein $\mathcal{X}$-flat and all C_i are $\mathcal{X}$-flat for $i \neq t$.

In the final section, we introduce Tate $\mathcal{X}$-flat resolutions (see Definition 3.2). More precisely, given a left R-module N and an R-module M admitting a Tate $\mathcal{X}$-flat resolution T (see Proposition 3.5 for the existence of Tate $\mathcal{X}$-flat resolutions), the Tate homology of M with coefficients in N is defined as $\widehat{\mathcal{X}tor}_i^R(M, N) := H_i(T \otimes_R N)$. Finally, we prove that if R is a right $G\mathcal{X}F$-closed ring and M a R-module admitting a Tate $\mathcal{X}$-flat resolution, and if N is a left R-module admitting a Tate $\mathcal{X}$-flat resolution T_N, then for any $i \in \mathbb{Z}$, $\widehat{\mathcal{X}tor}_i^R(M, N) \cong H_i(M \otimes_R T_N)$.

2. Gorenstein $\mathcal{X}$-flat dimension

We recall the following definition

Definition 2.1. [12] An R-module M is called Gorenstein $\mathcal{X}$-flat ($G\mathcal{X}$-flat for short) if there is an exact sequence

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

of $\mathcal{X}$-flat right R-modules such that $M \cong im(F_0 \to F^0)$ and such that $- \otimes X$ leaves the above sequence exact whenever X is an $\mathcal{X}$-injective left R-module.

Definition 2.2. [12] A ring R is called a right $G\mathcal{X}F$-closed if the class $\mathcal{GXF}$ of all Gorenstein $\mathcal{X}$-flat right R-modules is closed under extensions.

Proposition 2.3. *The class $\mathcal{GXF}$ of all Gorenstein $\mathcal{X}$-flat R-modules is closed under arbitrary direct sums.*

Proof. Sum of complete $\mathcal{X}$-flat resolutions again is a complete $\mathcal{X}$-flat resolution since tensor product commutes with sums. □

Now we introduce the definition of Gorenstein $\mathcal{X}$-flat dimension as follows:

Definition 2.4. The Gorenstein $\mathcal{X}$-flat dimension, $G\mathcal{X}\text{-}fd_R(M)$, of an R-module M is defined by $G\mathcal{X}\text{-}fd_R(M) \leq m$ if and only if M has a resolution by Gorenstein $\mathcal{X}$-flat modules of length m. If there is no such m, set $G\mathcal{X}\text{-}fd_R(M) = \infty$.

Proposition 2.5. *Let R be a right $G\mathcal{X}F$-closed ring and suppose that $0 \to K \to G_1 \xrightarrow{f} G_0 \to M \to 0$ is an exact sequence of R-modules with G_0, G_1 Gorenstein $\mathcal{X}$-flat. Then we have the following exact sequences:*

$$0 \to K \to F \to G \to M \to 0,$$

and

$$0 \to K \to H \to Q \to M \to 0$$

where F, Q are $\mathcal{X}$-flat and G, H are Gorenstein $\mathcal{X}$-flat.

Proof. Since G_1 is Gorenstein $\mathcal{X}$-flat, there exists a short exact sequence $0 \to G_1 \to F \to G_2 \to 0$ with F $\mathcal{X}$-flat and G_2 Gorenstein $\mathcal{X}$-flat. Then we have the following pushout diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & G_1 \arrow[r] \arrow[d] & im(f) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & F \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & 0 \\
 & & G_2 \arrow[r, equal] \arrow[d] & G_2 \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

We consider the following pushout diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & im(f) \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & D \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & G_2 \arrow[r, equal] \arrow[d] & G_2 \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

In the sequence $0 \to G_0 \to G \to G_2 \to 0$, both G_0 and G_2 are Gorenstein $\mathcal{X}$-flat, then so is G since R is right $G\mathcal{X}F$-closed. Assembling the sequences $0 \to K \to F \to D \to 0$ and $0 \to D \to G \to M \to 0$, we obtain the exact sequence $0 \to K \to F \to G \to M \to 0$ of R-modules with F $\mathcal{X}$-flat and G Gorenstein $\mathcal{X}$-flat, as desired.

Next we shall prove the second desired exact sequence. Since G_0 is Gorenstein $\mathcal{X}$-flat, there exists an exact sequence $0 \to G_3 \to Q \to G_0 \to 0$ of R-modules with Q $\mathcal{X}$-flat and G_3 Gorenstein $\mathcal{X}$-flat. Dually, one easily gets the desired result by taking pull-back diagram. $\square$

Lemma 2.6. *Let M be an R-module, and consider the two exact sequences of R-modules,*

$$0 \to G_m \to G_{m-1} \to \cdots \to G_0 \to M \to 0,$$

and

$$0 \to H_m \to H_{m-1} \to \cdots \to H_0 \to M \to 0,$$

where $G_0, \cdots, G_{m-1}$ and $H_0, \cdots, H_{m-1}$ are Gorenstein $\mathcal{X}$-flat. If R is right $G\mathcal{X}F$-closed, then G_m is Gorenstein $\mathcal{X}$-flat if and only if H_m is Gorenstein $\mathcal{X}$-flat.

Proof. It follows from Proposition 2.3, [12, Theorem 2.13] and [12, Corollary 2.14]. $\square$

Theorem 2.7. *Let R be a right $G\mathcal{X}F$-closed ring, and let M be an R-module. Then, the following are equivalent for a positive integer m:*

(1) *$G\mathcal{X}\text{-}fd_R(M) \le m$;*
(2) *For every exact sequence of R-modules $0 \to K_m \to G_{m-1} \to \cdots \to G_m \to M \to 0$, if each G_i is Gorenstein $\mathcal{X}$-flat, then so is K_m.*

Proof. It follows from Lemma 2.6 and by the definition of the Gorenstein $\mathcal{X}$-flat dimension. $\square$

Theorem 2.8. *Let R be a right $G\mathcal{X}F$-closed ring. For any R-module M and any integer $m \ge 0$, the following are equivalent:*

(1) *$G\mathcal{X}\text{-}fd_R(M) \le m$.*

(2) *For each non-negative integer t with $0 \leq t \leq m$, there is an exact sequence of R-modules $0 \to C_m \to \cdots \to C_t \to \cdots \to C_1 \to C_0 \to M \to 0$ such that C_t is Gorenstein $\mathcal{X}$-flat and all C_i are $\mathcal{X}$-flat for $i \neq t$.*

Proof. (2) $\Rightarrow$ (1). It is obvious.

(1) $\Rightarrow$ (2). We proceed by induction on m. If $G\mathcal{X}\text{-}fd_R(M) \leq 1$, then there exists an exact sequence $0 \to G_1 \to G_0 \to M \to 0$ of R-modules with G_0 and G_1 Gorenstein $\mathcal{X}$-flat. Since G_0 is Gorenstein $\mathcal{X}$-flat, then we get a short exact sequence $0 \to G_2 \to F_0 \to G_0 \to 0$, where F_0 is $\mathcal{X}$-flat and G_2 is Gorenstein $\mathcal{X}$-flat. We consider the following pullback diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & G_2 \arrow[r, equal] \arrow[d] & G_2 \arrow[d] & & \\
0 \arrow[r] & G_1' \arrow[r] \arrow[d] & F_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G_1 \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}$$

Since both G_1 and G_2 are Gorenstein $\mathcal{X}$-flat, G_1' is also Gorenstein $\mathcal{X}$-flat by R is a $G\mathcal{X}F$-closed ring. Thus we get the exact sequence $0 \to G_1' \to F_0 \to M \to 0$ with F_0 $\mathcal{X}$-flat and G_1' Gorenstein $\mathcal{X}$-flat. Because G_1 is Gorenstein $\mathcal{X}$-flat, there exists an exact sequence $0 \to G_1 \to F_1 \to G_3 \to 0$ with F_1 $\mathcal{X}$-flat and G_3 Gorenstein $\mathcal{X}$-flat. Consider the pushout of $G_1 \to G_0$ and $G_1 \to F_1$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & G_1 \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F_1 \arrow[r] \arrow[d] & G_0' \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & G_3 \arrow[r, equal] \arrow[d] & G_3 \arrow[d] & & \\
 & 0 & 0. & &
\end{tikzcd}$$

In the sequence $0 \to G_0 \to G_0' \to G_3 \to 0$, both G_0 and G_3 are Gorenstein $\mathcal{X}$-flat, hence so is G_0' by R is a $G\mathcal{X}F$-closed ring. Therefore, one has the exact sequence $0 \to F_1 \to G_0' \to M \to 0$, with F_1 $\mathcal{X}$-flat and G_0' Gorenstein $\mathcal{X}$-flat.

Next we suppose $m \geq 2$. Then there is an exact sequence of R-modules,

$$0 \to G_m \to \cdots \to G_1 \to G_0 \to M \to 0 \tag{2.1}$$

where all G_i are Gorenstein $\mathcal{X}$-flat for $1 \leq i \leq m$. Put $L = coker(G_3 \to G_2)$, and we have the exact sequence $0 \to L \to G_1 \to G_0 \to M \to 0$. By Proposition 2.5, one gets the exactness of $0 \to L \to G_1' \to F_0 \to M \to 0$ with F_0 $\mathcal{X}$-flat and G_1' Gorenstein $\mathcal{X}$-flat. Hence we obtain the following exact sequence of R-modules

$$0 \to G_m \to \cdots \to G_2 \to G_1' \to F_0 \to M \to 0.$$

Set $N = im(G_1' \to F_0)$. It is clear that $G\mathcal{X}\text{-}fd_R(N) \leq m-1$. By the induction hypothesis, there exists an exact sequence

$$0 \to C_m \to \cdots \to C_t \to \cdots \to C_1 \to F_0 \to M \to 0$$

such that F_0 is $\mathcal{X}$-flat and C_t is Gorenstein $\mathcal{X}$-flat and C_i is $\mathcal{X}$-flat for $i \neq t$ and $1 \leq t \leq m$.

Now it remains to show (2) for the case $t = 0$. In the sequence (2.1), put $N = coker(G_2 \to G_1)$. One gets the exactness of $0 \to G_m \to \cdots \to G_2 \to G_1 \to N \to 0$. By the induction hypothesis, we have an exact sequence of R-modules $0 \to F_m \to \cdots \to F_3 \to F_2 \to G_1' \to N \to 0$ with G_1' Gorenstein $\mathcal{X}$-flat and all F_i $\mathcal{X}$-flat for $2 \leq i \leq m$. Set $K = coker(F_3 \to F_2)$. The exactness of $0 \to K \to G_1' \to G_0 \to M \to 0$ gives rise to an exact sequence $0 \to K \to F_1 \to G_0' \to M \to 0$ with F_1 $\mathcal{X}$-flat and G_0' Gorenstein $\mathcal{X}$-flat by Proposition 2.5. Therefore, we have the exact sequence

$$0 \to F_m \to \cdots \to F_2 \to F_1 \to G_0' \to M \to 0$$

where G_0' is Gorenstein $\mathcal{X}$-flat and all F_i are $\mathcal{X}$-flat for $1 \leq i \leq m$. This completes the proof. $\square$

3. Tate Homology of Modules of Finite Gorenstein $\mathcal{X}$-flat Dimension

We begin with the following definition

Definition 3.1. A totally acyclic complex of $\mathcal{X}$-flat R-modules is an exact complex of $\mathcal{X}$-flat R-modules such that $- \otimes_R G$ exacts it for any $\mathcal{X}$-injective right R-module G.

Definition 3.2. Let M be an R-module. A Tate $\mathcal{X}$-flat resolution of M is a totally acyclic complex T of $\mathcal{X}$-flat R-modules such that it is compatible with a $\mathcal{X}$-flat resolution of M for $i \gg 0$, that is, there is an integer $m \geq 0$ such that the truncated complex $T_{\geq m} \equiv \cdots \to T_{m+1} \to T_m \to 0$ and $F_{\geq m} \equiv \cdots \to F_{m+1} \to F_m \to 0$ are isomorphic, where $F \xrightarrow{\simeq} M$ is a $\mathcal{X}$-flat resolution of M.

Example 3.3.
1. If M is an R-module of finite $\mathcal{X}$-flat dimension, then the zero complex is a Tate $\mathcal{X}$-flat resolution of M.
2. If M is a Gorenstein $\mathcal{X}$-flat R-module with T a totally acyclic complex of $\mathcal{X}$-flat R-modules such that $M \cong \ker(T_{-1} \to T_{-2})$, then T is a Tate $\mathcal{X}$-flat resolution of M.

Remark 3.4. If M is an R-module admitting a Tate $\mathcal{X}$-flat resolution, then $G\mathcal{X}\text{-}fd_R(M) < \infty$.

We now give a partial converse of Remark 3.4 in the following. Let R be a right $G\mathcal{X}F$-closed ring and M an R-module of finite Gorenstein $\mathcal{X}$-flat dimension, say d. Consider a $\mathcal{X}$-flat resolution

$$\cdots \to F_d \to F_{d-1} \to \cdots \to F_0 \to M \to 0$$

of M, and let $G = \ker(F_{d-1} \to F_{d-2})$, then G is Gorenstein $\mathcal{X}$-flat by Theorem 2.7 since R is right $G\mathcal{X}F$-closed, and so there is an exact sequence

$$T \equiv \cdots \to T_{d+1} \to T_d \to T_{d-1} \to \cdots \to T_0 \to T_{-1} \to \cdots$$

of $\mathcal{X}$-flat R-modules such that $G = \ker(T_{d-1} \to T_{d-2})$ and $- \otimes_R G$ exacts the sequence for any $\mathcal{X}$-injective right R-module G. We always may choose T_i to be equal to F_i for $i \geq d$, then T a Tate $\mathcal{X}$-flat resolution of M. Thus, we have the next result by Remark 3.4 that will be used frequently in this paper.

Proposition 3.5. *Let R be a right $G\mathcal{X}F$-closed ring and M an R-module, then $G\mathcal{X}\text{-}fd_R(M) \leq d$ if and only if M admits a Tate $\mathcal{X}$-flat resolution that is compatible with a $\mathcal{X}$-flat resolution of M for $i \geq d$.*

Definition 3.6. Let M be an R-module admitting a Tate $\mathcal{X}$-flat resolution T. For a left R-module N, the Tate homology of M with coefficients in N is defined as

$$\widehat{\mathcal{X}tor}_i^R(M, N) := H_i(T \otimes_R N).$$

Proposition 3.7. *The above definition of Tate homology is independent of the choice of T.*

Proof. Let T and T' be two Tate $\mathcal{X}$-flat resolutions of M such that T is compatible with a $\mathcal{X}$-flat resolution of M for $i \geq s$ and T' is compatible with a $\mathcal{X}$-flat resolution of M for $i \geq t$. In the following, we show that $H_i(T \otimes_R N) \cong H_i(T' \otimes_R N)$ for any $i \in \mathbb{Z}$.

Set $m = max\{s, t\}$. If $i > m$, then we have $H_i(T \otimes_R N) \cong Tor_i^R(M, N) \cong H_i(T' \otimes_R N)$. Now we assume that $i \leq m$. Let

$$0 \to N \to E_0 \to C_0 \to 0$$

be an exact sequence of left R-modules with E_0 is injective. Since E_0 $\mathcal{X}$-injective, then the sequence

$$0 \to N \otimes_R T \to E_0 \otimes_R T \to C_0 \otimes_R T \to 0$$

of complexes is exact. We have $H_i(T \otimes_R N) \cong H_{i+1}(T \otimes_R C_0)$ since $E_0 \otimes_R T$ is exact. By continuing this process, considering the exact sequence

$$0 \to N_i \to E_{i+1} \to C_{i+1} \to 0$$

with E_{i+1} $\mathcal{X}$-injective, we have $H_i(T \otimes_R N) \cong H_{i+j}(T \otimes_R C_{j-1})$. But for j large enough, such that $i + j > m$, we have $H_{i+j}(T \otimes_R C_{j-1}) \cong Tor_{i+j}^R(M, C_{j-1})$, and so $H_i(T \otimes_R N) \cong Tor_{i+j}^R(M, C_{j-1})$. Using the same method, we get that $H_i(T' \otimes_R N) \cong Tor_{i+j}^R(M, C_{j-1})$. Thus $H_i(T \otimes_R N) \cong H_i(T' \otimes_R N)$ for any $i \in \mathbb{Z}$, as desired. $\square$

Now we give some vanishing results of Tate homology $\widehat{\mathcal{X}tor}$ that are useful for proving our main result.

Lemma 3.8. *If M is an R-module of finite $\mathcal{X}$-flat dimension, then $\widehat{\mathcal{X}tor}_i^R(M,-)=0$ for any $i\in\mathbb{Z}$.*

Proof. Since $\mathcal{X}$-$fd(M)<\infty$, we may choose an $\mathcal{X}$-flat resolution of M of finite length, and so the Tate $\mathcal{X}$-flat resolution of M can be chosen to be zero complex. Thus $\widehat{\mathcal{X}tor}_i^R(M,N)=0$ for any left R-module N and any $i\in\mathbb{Z}$. □

Lemma 3.9. *Let M be an R-module admitting a Tate $\mathcal{X}$-flat resolution, then there exists an exact sequence*

$$0\to M'\to F_0\to M\to 0$$

with F_0 $\mathcal{X}$-flat, such that $\widehat{\mathcal{X}tor}_i^R(M',N)\cong\widehat{\mathcal{X}tor}_{i+1}^R(M,N)$ for any left R-module N and $i\in\mathbb{Z}$.

Proof. Let T be a Tate $\mathcal{X}$-flat resolution of M such that it is compatible with a $\mathcal{X}$-flat resolution F of M for $i\gg 0$, and let $M'=\ker(F_0\to M)$, then we get an exact sequence

$$0\to M'\to F_0\to M\to 0$$

with F_0 $\mathcal{X}$-flat. Note that $\Sigma^{-1}T$ is a Tate $\mathcal{X}$-flat resolution of M', then

$$\widehat{\mathcal{X}tor}_i^R(M',N)=H_i(\Sigma^{-1}T\otimes_R N)\cong H_{i+1}(T\otimes_R N)=\widehat{\mathcal{X}tor}_{i+1}^R(M,N)$$

for any left R-module N and $i\in\mathbb{Z}$. □

The following result is analogous to [10, Proposition 3.13]

Proposition 3.10. *Let $0\to N'\to N\to N''\to 0$ be an exact sequence of left R-modules. If M is an R-module admitting a Tate $\mathcal{X}$-flat resolution, then there is a long exact sequence*

$$\cdots\to\widehat{\mathcal{X}tor}_i^R(M,N')\to\widehat{\mathcal{X}tor}_i^R(M,N)\to\widehat{\mathcal{X}tor}_i^R(M,N'')\to\widehat{\mathcal{X}tor}_{i-1}^R(M,N')\to\cdots.$$

Proof. Let T be a Tate $\mathcal{X}$-flat resolution of M, then the sequence

$$0\to T\otimes_R N'\to T\otimes_R N\to T\otimes_R N''\to 0$$

of complexes is exact, and so the sequence

$$\cdots\to H_i(N'\otimes_R T)\to H_i(N\otimes_R T)\to H_i(N''\otimes_R T)\to H_{i-1}(N'\otimes_R T)\to\cdots$$

is exact. This gives the desired exact sequence. □

Lemma 3.11. *Let M be a Gorenstein $\mathcal{X}$-flat R-module, then there exists an exact sequence*

$$0\to M\to F\to M'\to 0$$

with F $\mathcal{X}$-flat and M' Gorenstein $\mathcal{X}$-flat, such that $\widehat{\mathcal{X}tor}_i^R(M,N)\cong\widehat{\mathcal{X}tor}_{i+1}^R(M',N)$ for any left R-module N and $i\in\mathbb{Z}$.

Proof. Let $T \equiv \cdots \to T_1 \to T_0 \to T_{-1} \to \cdots$ be a totally acyclic complex of $\mathcal{X}$-flat R-modules such that $M \cong \ker(T_{-1} \to T_{-2})$, and let $M' = \ker(T_{-2} \to T_{-3})$, then the sequence $0 \to M \to T_{-1} \to M' \to 0$ is exact, and T_{-1} is $\mathcal{X}$-flat and M' is Gorenstein $\mathcal{X}$-flat. Note that T is a Tate $\mathcal{X}$-flat resolution of M and ΣT is a Tate $\mathcal{X}$-flat resolution of M', then

$$\widehat{\mathcal{X}tor}_i^R(M, N) = H_i(T \otimes_R N) \cong H_{i+1}(\Sigma T \otimes_R N) = \widehat{\mathcal{X}tor}_{i+1}^R(M', N)$$

for any left R-module N and $i \in \mathbb{Z}$. $\square$

Next we prove the main result of this section.

Theorem 3.12. *Let R be a right $G\mathcal{X}F$-closed ring and M an R-module admitting a Tate $\mathcal{X}$-flat resolution, and let N be a left R-module admitting a Tate $\mathcal{X}$-flat resolution T_N, then for any $i \in \mathbb{Z}$,*

$$\widehat{\mathcal{X}tor}_i^R(M, N) \cong H_i(M \otimes_R T_N).$$

Proof. Let T_M be a Tate $\mathcal{X}$-flat resolution of M such that T_M is compatible with an $\mathcal{X}$-flat resolution of M for any $j \geq s$, and let T_N be compatible with an $\mathcal{X}$-flat resolution of N for any $l \geq t$. Set $m = max\{s, t\}$, then for any $i > n$,

$$\widehat{\mathcal{X}tor}_i^R(M, N) = H_i(T_M \otimes_R N) \cong Tor_i^R(M, N) \cong H_i(M \otimes T_N).$$

Now assume that $i \leq m$. We notice that $G\mathcal{X}\text{-}fd_R(M) < \infty$ by Remark 3.4.

Let $G\mathcal{X}\text{-}fd_R(M) = 0$, that is, M is Gorenstein $\mathcal{X}$-flat. In this case, $m = max\{0, t\} = t$. Then there is an exact sequence

$$0 \to M \to F \to M' \to 0$$

with F $\mathcal{X}$-flat and M' Gorenstein $\mathcal{X}$-flat, such that $\widehat{\mathcal{X}tor}_i^R(M, N) \cong \widehat{\mathcal{X}tor}_{i+1}^R(M', N)$ by Lemma 3.11. By continuing this process, we have

$$\widehat{\mathcal{X}tor}_i^R(M, N) \cong \widehat{\mathcal{X}tor}_{i+1}^R(M'', N) \tag{3.1}$$

for some Gorenstein $\mathcal{X}$-flat R-module M''. Note that M'' admits a Tate $\mathcal{X}$-flat resolution that is compatible with an $\mathcal{X}$-flat resolution of M'' for any $j \geq 0$, then, by the previous proof, we get that $\widehat{\mathcal{X}tor}_i^R(M'', N) \cong H_i(M'' \otimes_R T_N)$ for any $i > max\{0, t\} = t$. In particular, we have

$$\widehat{\mathcal{X}tor}_{t+1}^R(M'', N) \cong H_{t+1}(M'' \otimes_R T_N). \tag{3.2}$$

On the other hand, note that the sequence

$$0 \to M \otimes_R T_N \to F \otimes_R T_N \to M' \otimes_R T_N \to 0$$

of complexes is exact and $F \otimes_R T_N$ is exact, so $H_i(M \otimes_R T_N) \cong H_{i+1}(M' \otimes_R T_N)$. By continuing this process, we can get that

$$H_i(M \otimes_R T_N) \cong H_{t+1}(M'' \otimes_R T_N) \cong \widehat{\mathcal{X}tor}_{t+1}^R(M'', N) \cong \widehat{\mathcal{X}tor}_i^R(M, N),$$

where the second isomorphism holds by (3.1) and the last follows from (3.2).
Now suppose inductively that $G\mathcal{X}\text{-}fd_R(M) = d > 0$. Then there is an exact sequence

$$0 \to M' \to F \to N \to 0$$

with F $\mathcal{X}$-flat, such that $\widehat{\mathcal{X}tor}_{i-1}^{R}(M', N) \cong \widehat{\mathcal{X}tor}_{i}^{R}(M, N)$ by Lemma 3.9. Consider the exact sequence

$$0 \to M' \otimes_R T_N \to F \otimes_R M' \to M \otimes_R M' \to 0.$$

Note that $F \otimes_R M'$ is exact, then

$$H_i(M \otimes_R T_N) \cong H_{i-1}(M' \otimes_R T_N) \cong \widehat{\mathcal{X}tor}_{i-1}^{R}(M', N) \cong \widehat{\mathcal{X}tor}_{i}^{R}(M, N),$$

where the second isomorphism follows from the induction hypothesis since $G\mathcal{X}$-$fd_R(M') = d$ - 1.

□

Corollary 3.13. *Let R be a right $G\mathcal{X}F$-closed ring, and let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules and each of M', M, and M'' admits a Tate $\mathcal{X}$-flat resolution. If N is a left R-module admitting a Tate $\mathcal{X}$-flat resolution, then there is a long exact sequence*

$$\cdots \to \widehat{\mathcal{X}tor}_{i}^{R}(M', N) \to \widehat{\mathcal{X}tor}_{i}^{R}(M, N) \to \widehat{\mathcal{X}tor}_{i}^{R}(M'', N) \to \widehat{\mathcal{X}tor}_{i-1}^{R}(M', N) \to \cdots.$$

Proof. Let T_N be a Tate $\mathcal{X}$-flat resolution of N, then the sequence

$$0 \to M' \otimes_R T_N \to M \otimes_R T_N \to M'' \otimes_R T_N \to 0$$

is exact, and so the sequence

$$\cdots \to H_i(M' \otimes_R T_N) \to H_i(M \otimes_R T_N) \to H_i(M'' \otimes_R T_N) \to H_{i-1}(M' \otimes_R T_N) \to \cdots$$

is exact. This gives the desired exact sequence by Theorem 3.12. □

Lemma 3.14. *Let N be a left R-module of finite $\mathcal{X}$-flat dimension d, and let T be an exact sequence of $\mathcal{X}$-flat R-modules, then $T \otimes_R N$ is exact.*

Proof. If $d = 0$, that is, N is $\mathcal{X}$-flat, then $T \otimes_R N$ is exact clearly. Now assume that $d > 0$, then there is an exact sequence

$$0 \to N' \to F \to N \to 0$$

with F $\mathcal{X}$-flat and $\mathcal{X}$-dim$(N') = d - 1$. Thus the sequence

$$0 \to T \otimes_R N' \to T \otimes_R F \to T \otimes_R N \to 0$$

of complexes is exact. Note that $N' \otimes_R T$ is exact by induction hypothesis, and $T \otimes_R F$ is also exact, then $T \otimes_R N$ is exact. □

Lemma 3.15. *Let N be a left R-module of finite $\mathcal{X}$-flat dimension and M an R-module admitting a Tate $\mathcal{X}$-flat resolution, then for any $i \in \mathbb{Z}$,*

$$\widehat{\mathcal{X}tor}_{i}^{R}(M, N) = 0$$

Proof. Let T be a Tate $\mathcal{X}$-flat resolution of M. Since $\mathcal{X}$-fd$(N) < \infty$, we get that $T \otimes_R N$ is exact by Lemma 3.14, and so $\widehat{\mathcal{X}tor}_{i}^{R}(M, N) = H_i(T \otimes_R N) = 0$ for any $i \in \mathbb{Z}$. □

Acknowledgment. The first author was partially supported by University Grants Commission, New Delhi, Major Research Project, Grant No.: 41-773/2012(SR). The second author was supported by UGC, New Delhi, Non SAP BSR, Grant No.:11 - 105/2008(BSR).

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