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SINCLAIR'S THEOREM FOR SPECTRALLY BOUNDED JORDAN HIGHER DERIVATIONS

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ABSTRACT. In this article, we prove Sinclair's Theorem for spectrally bounded Jordan higher derivations on Banach algebras. Furthermore, it is proved that the image of a higher derivation , under certain conditions, is contained in the Jacobson radical.

1. Introduction and preliminaries

Let $\mathcal{A}$ be any complex Banach algebra. A linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a derivation if it satisfies the Leibnitz rule $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$. A linear mapping $D : \mathcal{A} \to \mathcal{A}$ is called a Jordan derivation if $D(a \cdot b) = D(a) \cdot b + a \cdot D(b)$ holds for all pairs $a, b \in \mathcal{A}$, where $a \cdot b$ denotes the Jordan product $ab+ba$. Obviously, every derivation is a Jordan derivation. As a generalization of a derivation, a linear mapping $d : \mathcal{A} \to \mathcal{A}$ is called a $\sigma-derivation$ if it satisfies the generalized Leibnitz rule $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$ for all $a, b \in \mathcal{A}$, where $\sigma : \mathcal{A} \to \mathcal{A}$ is a linear mapping (see [6]). The author ([7], [8]) generalized a σ-derivation as follows:

A linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a generalized $\sigma-derivation$ if there exists a $\sigma-derivation$ $d : \mathcal{A} \to \mathcal{A}$ such that $\delta(ab) = \delta(a)\sigma(b)+\sigma(a)d(b)$ for all $a, b \in \mathcal{A}$. If we define a sequence $\{d_n\}$ of linear mappings on $\mathcal{A}$ by $d_0 = I$ and $d_n = \frac{\delta^n}{n!}$, where I is the identity mapping and δ is a derivation on $\mathcal{A}$, then the Leibnitz rule ensures us that d_n's satisfy the condition

$$d_n(ab) = \sum_{k=0}^{n} d_k(a)d_{n-k}(b) \tag{1.1}$$

for each $a, b \in \mathcal{A}$ and each non-negative integer n. This motivates us to consider the sequences $\{d_n\}$ of linear mappings on an algebra $\mathcal{A}$ satisfying (1.1). Such a sequence is called a higher derivation. "Higher derivations were introduced by Hasse and Schmidt [9], and algebraists sometimes call them Hasse-Schmidt derivations". Though, if $\delta : \mathcal{A} \to \mathcal{A}$ is a derivation, then $d_n = \frac{\delta^n}{n!}$ is a higher derivation; this is not the only example of a higher derivation. This kind of higher derivation is called an ordinary higher derivation. A sequence $\{d_n\}$ of

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linear mappings on $\mathcal{A}$ is called a Jordan higher derivation if $d_n(a \cdot b) = \sum_{k=0}^{n} d_k(a) \cdot d_{n-k}(b)$ $(a, b \in \mathcal{A})$ such that $a \cdot b = ab + ba$. As an interesting result, Shafeng Xu and Zhankui Xiao [[17], Corollary 4.4] proved that every Jordan higher derivation on a 2-torsion free semiprime ring is a higher derivation.

"Note that higher derivations, as natural generalizations of derivations, are closely related to derivations". Mirzavaziri [10] characterized all higher derivations on an algebra $\mathcal{A}$ in terms of the derivations on $\mathcal{A}$. Indeed, he proved that each higher derivation is a combination of compositions of derivations. Furthermore, he studied the innerness of higher derivations [11].

Let us introduce a background of our investigation. In 1955 Singer and Wermer obtained a fundamental result which started investigation into the ranges of derivations on Banach algebras. The result states that every bounded derivation on a commutative Banach algebra maps into the Jacobson radical. In the same paper they conjectured that the assumption of boundedness is not necessary. In 1988 Thomas [15] proved this conjecture. Kil-Woung Jun and Young-whan Lee [16] proved that the image of a continuous strong higher derivation is contained in the Jacobson radical. Brešar and Mathieu [2] proved that every spectrally bounded derivation on a unital Banach algebra maps into the Jacobson radical. We recall that a linear mapping $T : \mathcal{A} \to \mathcal{B}$ is called spectrally bounded if there is a constant $M \geq 0$ such that $r(T(a)) \leq Mr(a)$ for all $a \in \mathcal{A}$, where $r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}$ denotes the spectral radius of a and $\mathcal{B}$ is also a complex Banach algebra. Moreover, Mathieu [13] proved the following result:

Let d be a derivation on a Banach algebra $\mathcal{A}$. Then the following three conditions are equivalent:

(i) $[a, d(a)] \in rad(\mathcal{A})$ for all $a \in \mathcal{A}$;

(ii) d is spectrally bounded;

(iii) $d(\mathcal{A}) \subseteq rad(\mathcal{A})$,

where $[a, b] = ab - ba$ and $rad(\mathcal{A})$ denotes the Jacobson radical of $\mathcal{A}$.

"Note that there exist unbounded spectrally bounded derivations, but that the separating space of each such derivation is contained in the radical. Moreover, non-zero inner derivations on semi-simple Banach algebras provide examples of bounded spectrally unbounded derivations."

In 1969 Sinclair [14] proved that If $D : \mathcal{A} \to \mathcal{A}$ is a continuous derivation and $\mathcal{P}$ is any primitive ideal of $\mathcal{A}$, then $D(\mathcal{P}) \subseteq \mathcal{P}$.

It is the main purpose of the present paper to extend Sinclair's Theorem for spectrally bounded Jordan higher derivations on Banach algebras. Furthermore, it is proved that the image of a strong higher derivation on a commutative Banach algebra is contained in the Jacobson radical.

2. Main results

Throughout this paper, $\mathcal{A}$ denotes a complex Banach algebra. A linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a derivation if $\delta(ab) = \delta(a)b + a\delta(b)$ holds for all pairs $a, b \in \mathcal{A}$. A sequence $\{d_n\}$ of linear mappings on $\mathcal{A}$ is called a *higher derivation* if $d_n(ab) = \sum_{k=0}^{n} d_k(a)d_{n-k}(b)$ for each $a, b \in \mathcal{A}$ and each non-negative integer n.

If $d_0 = I$, the identity mapping on $\mathcal{A}$, then the higher derivation $\{d_n\}$ is called *strong*. It is clear that d_1 is an ordinary derivation.

Definition 2.1. A (strong) higher derivation $\{d_n\}$ is called *spectrally bounded* (*spectrally infinitesimal*) if there is a sequence $\{M_n\}$ of non-negative numbers such that $r(d_n(a)) \leq M_n r(a)$ $(r(d_n(a)) = 0)$ for all $a \in \mathcal{A}$, $n \in \mathbb{N}\bigcup\{0\}$. The sequence $\{M_n\}$ is called the corresponding numerical sequence of $\{d_n\}$.

A (strong) higher derivation $\{d_n\}$ is called *uniformly spectrally bounded* if there is a constant $M \geq 0$ such that $r(d_n(a)) \leq Mr(a)$ for all $a \in \mathcal{A}$. A linear mapping $T : \mathcal{A} \to \mathcal{B}$ is called *spectral isometry* if $r(T(a)) = r(a)$ for each $a \in \mathcal{A}$, where $\mathcal{B}$ is also a complex Banach algebra. Observe that the canonical epimorphism $\psi : \mathcal{A} \to \frac{\mathcal{A}}{rad(\mathcal{A})}$ is a spectral isometry.

The next lemma has been proved in [[18], Lemma 2.5]

Lemma 2.2. *Every spectrally bounded Jordan derivation on a Banach algebra $\mathcal{A}$ leaves each primitive ideal of $\mathcal{A}$ invariant.*

Theorem 2.3. *[Higher Sinclair's Theorem] Every spectrally bounded Jordan strong higher derivation $\{d_n\}$ on a Banach algebra $\mathcal{A}$, leaves each primitive ideal of $\mathcal{A}$ invariant.*

Proof. Let $\{d_n\}$ be a spectrally bounded Jordan strong higher derivation associated with the corresponding numerical sequence $\{M_n\}$ and let $\mathcal{P}$ be a primitive ideal of $\mathcal{A}$. It follows from part (ii) of Proposition 24-12 of [1] that, $\mathcal{A}$ has an irreducible representation $\pi : \mathcal{A} \to L(\mathcal{X})$ on a Banach space $\mathcal{X}$ with kernel $\mathcal{P}$, where $L(\mathcal{X})$ is the algebra of all linear mappings on $\mathcal{X}$. It follows from Lemma 2.2 that, $d_1(\mathcal{P}) \subseteq \mathcal{P}$. Now, we are going to show that $d_2(\mathcal{P}) \subseteq \mathcal{P}$. According to the proof of Lemma 1 of [16], we have

$$\begin{aligned} d_k^n(a^n) = \sum_{\sum_{j=1}^n a_{i,j}=k,\ i=1,2,\dots,n} & d_{a_{n,1}}(d_{a_{n-1,1}}(\dots(d_{a_{2,1}}(d_{a_{1,1}}(a)))\dots)) \\ & .d_{a_{n,2}}(d_{a_{n-1,2}}(\dots(d_{a_{2,2}}(d_{a_{1,2}}(a)))\dots)) \\ & \dots d_{a_{n,n}}(d_{a_{n-1,n}}(\dots(d_{a_{2,n}}(d_{a_{1,n}}(a)))\dots)) \end{aligned}$$

where $a_{i,j} \geq 0$. For $n > 2$ and $0 \leq a_{i,j} (i, j = 1, 2, ..., n)$, let

$$\begin{aligned} & a_{1,1} + a_{1,2} + \dots + a_{1,n} = 2, \\ & a_{2,1} + a_{2,2} + \dots + a_{2,n} = 2, \\ & . \\ & . \\ & . \\ & a_{n,1} + a_{n,2} + \dots + a_{n,n} = 2. \end{aligned}$$

If $a_{i,j} = 1$ for some $i, j = 1, 2, ..., n$, then for all $n > 2$

$$\begin{aligned} z = & d_{a_{n,1}}(d_{a_{n-1,1}}(\dots(d_{a_{2,1}}(d_{a_{1,1}}(a)))\dots)) \\ & .d_{a_{n,2}}(d_{a_{n-1,2}}(\dots(d_{a_{2,2}}(d_{a_{1,2}}(a)))\dots)) \\ & \dots d_{a_{n,n}}(d_{a_{n-1,n}}(\dots(d_{a_{2,n}}(d_{a_{1,n}}(a)))\dots)) \end{aligned}$$

is an element of $\mathcal{P}$. If $a_{i,j} \neq 0$ for all $i, j = 1, 2, ..., n$, then for all $n > 2$ $z = (d_2(a))^n$. Thus we have $d_2^n(a^n) - n!(d_2(a))^n \in \mathcal{P}$ for all $a \in \mathcal{P}$ and $n \in \mathbb{N}$. Then

$$\begin{aligned} r(d_2(a) + \mathcal{P}) &= (r((d_2(a))^n + \mathcal{P}))^{\frac{1}{n}} \\ &= (n!)^{\frac{-1}{n}} (r(d_2^n(a^n) + \mathcal{P}))^{\frac{1}{n}} \\ &\leq (n!)^{\frac{-1}{n}} (r(d_2^n(a^n)))^{\frac{1}{n}} \\ &\leq (n!)^{\frac{-1}{n}} M_2 r(a), \quad (a \in \mathcal{P},\ n \in \mathbb{N}) \end{aligned}$$

whence $d_2(a) + \mathcal{P} \in Q(\frac{\mathcal{A}}{\mathcal{P}})$ for each $a \in \mathcal{P}$. Let $a \in \mathcal{P}$ and assume that $d_2(a) \notin \mathcal{P}$. Note that the normed division algebra $\mathcal{D} = \{T \in L(\mathcal{X}) : bT(x) = T(bx), b \in \mathcal{A}, x \in \mathcal{X}\}$ is $\mathbb{C}I$ (see p.128 of [1]). If $\pi(d_2(a)) \in \mathbb{C}I$, then $b\pi(d_2(a))(x) = \pi(d_2(a))(bx)$, and hence $bd_2(a)x = d_2(a)bx$ for all $b \in \mathcal{A}$, $x \in \mathcal{X}$. This means that $bd_2(a) - d_2(a)b \in \mathcal{P}$ for all $b \in \mathcal{A}$. So $r((b+\mathcal{P})(d_2(a)+\mathcal{P})) \leq r(b+\mathcal{P})r(d_2(a)+\mathcal{P}) = 0$ $(b \in \mathcal{A})$. By the quasinilpotency of $d_2(a) + \mathcal{P}$, we can find that $d_2(a) + \mathcal{P} \in rad(\frac{\mathcal{A}}{\mathcal{P}}) = \{0\}$. Thus it is obtained that $d_2(a) \in \mathcal{P}$. This contradiction will show that $\pi(d_2(a)) \notin \mathbb{C}I$. Then $\{I, \pi(d_2(a))\}$ is linearly independent. It follows from Lemma 2.2 of [2] that there is $x_0 \in \mathcal{X}$ such that $\{x_0, \pi(d_2(a))(x_0)\} = \{x_0, d_2(a)x_0\}$ is linearly independent. Thus, by the Jacobson density theorem, we choose an element c_0 in $\mathcal{A}$ such that $c_0x_0 = x_0$ and $c_0d_2(a)x_0 = x_0 - d_2(a)x_0$. Then

$$\begin{aligned} x_0 &= c_0d_2(a)x_0 + d_2(a)x_0 \\ &= c_0d_2(a)x_0 + d_2(a)c_0x_0 \\ &= (c_0d_2(a) + d_2(a)c_0)x_0. \end{aligned}$$

It means that $(d_2(a).c_0)x_0 = x_0$, where $a.b$ is introduced in introduction. We claim that $d_2(a).c_0 + \mathcal{P}$ is not quasinilpotent. Arguing by contradiction, suppose that $d_2(a).c_0 + \mathcal{P}$ is quasinilpotent. We know that $bd_2(a) - d_2(a)b \in \mathcal{P}$ $(b \in \mathcal{A}$, $a \in \mathcal{P})$ and it means that $d_2(a).c_0 + \mathcal{P} = d_2(a)c_0 + c_0d_2(a) + \mathcal{P} = 2c_0d_2(a) + \mathcal{P}$. Clearly, $\frac{\mathcal{A}d_2(a)}{\mathcal{P}}$ is a left ideal of $\frac{\mathcal{A}}{\mathcal{P}}$ for $a \in \mathcal{P}$. Furthermore, we have $r(bd_2(a)+\mathcal{P}) = r((b+\mathcal{P})(d_2(a)+\mathcal{P})) \leq r(b+\mathcal{P})r(d_2(a)+\mathcal{P}) = 0$, so $\frac{\mathcal{A}d_2(a)}{\mathcal{P}} \in Q(\frac{\mathcal{A}}{\mathcal{P}})$. Using the Proposition 2.2.3 of [4], we conclude that $\frac{\mathcal{A}d_2(a)}{\mathcal{P}} \subseteq rad(\frac{\mathcal{A}}{\mathcal{P}}) = \{0\}$, i.e. $bd_2(a) \in \mathcal{P}$ for all $b \in \mathcal{A}$. Thus $d_2(a).c_0 + \mathcal{P} = 2c_0d_2(a) + \mathcal{P} = \mathcal{P}$ and it results that $d_2(a).c_0 \in \mathcal{P} = ker(\pi)$. Therefore $\pi(d_2(a).c_0)(x) = 0$ for all $x \in \mathcal{X}$. Putting $x = x_0$, it is obtained that $0 = \pi(d_2(a).c_0)(x_0) = (d_2(a).c_0)x_0 = x_0$ and it is a contradiction. Consequently, $d_2(a) \cdot c_0 + \mathcal{P}$ is not quasinilpotent. However $d_2(a) \cdot b + \mathcal{P}$ is quasinilpotent since, $d_2(a) \cdot b + \mathcal{P} = d_2(a \cdot b) - a \cdot d_2(b) - d_1(a) \cdot d_1(b) + \mathcal{P} \in d_2(\mathcal{P}) + \mathcal{P} \subseteq Q(\frac{\mathcal{A}}{\mathcal{P}})$ for all $a \in \mathcal{P}, b \in \mathcal{A}$. This contradiction implies that $d_2(a) \in \mathcal{P}$ and since a is an arbitrary element of $\mathcal{P}$, $d_2(\mathcal{P}) \subseteq \mathcal{P}$. Of course, the arguments in the above proof can be used for proving $d_n(\mathcal{P}) \subseteq \mathcal{P}$ for all $n \geq 3$. □

Next, we will prove that the image of a strong higher derivation on a Banach algebra, under certain conditions, is contained in the Jacobson radical.
The following result is proved by Mathieu [13].

Theorem 2.4. *Let d be a derivation on a Banach algebra $\mathcal{A}$. Then the following three conditions are equivalent:*
(i) $[a, d(a)] \in rad(\mathcal{A})$ for all $a \in \mathcal{A}$;
(ii) d is spectrally bounded;
(iii) $d(\mathcal{A}) \subseteq rad(\mathcal{A})$,
where $[a, b] = ab - ba$ and $rad(\mathcal{A})$ denotes the Jacobson radical of $\mathcal{A}$.

Theorem 2.5. *[Thomas's Theorem, Theorem 5.2.48 of [3]] Let D be a derivation on a commutative Banach algebra $\mathcal{A}$. Then $D(\mathcal{A}) \subseteq rad(\mathcal{A})$.*

Theorem 2.6. *[[10], Theorem 2.3] Let $\{d_n\}$ be a higher derivation on an algebra $\mathcal{A}$ with $d_0 = I$. Then there is a sequence $\{\delta_n\}$ of derivations on $\mathcal{A}$ such that*

$$d_n = \sum_{i=1}^{n} \Big(\sum_{\sum_{j=1}^{i} r_j = n} \big(\prod_{j=1}^{i} \frac{1}{r_j + ... + r_i} \big) \delta_{r_1} ... \delta_{r_i} \Big).$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n$.

Theorem 2.7. *Let $\{d_n\}$ be a strong higher derivation on a commutative Banach algebra $\mathcal{A}$. Then $d_n(\mathcal{A}) \subseteq rad(\mathcal{A})$ for all $n \in \mathbb{N}$.*

Proof. This is an immediate conclusion from 2.5 and 2.6. $\square$

Theorem 2.8. *Suppose that $\{d_n\}$ is a spectrally bounded strong higher derivation with the corresponding sequence $\{\delta_n\}$ of derivations on $\mathcal{A}$, which introduced in Theorem 2.6. If $\delta_n(a)a = a\delta_n(a)$ for all $a \in \mathcal{A}$, $n \in \mathbb{N}$, then $d_n(\mathcal{A}) \subseteq rad(\mathcal{A})$ for all $n \in \mathbb{N}$.*

Proof. It follows from Theorem 2.3 and Theorem 2.6, that $\delta_n(\mathcal{P}) \subseteq \mathcal{P}$ for all $n \in \mathbb{N}$, where $\mathcal{P}$ is an arbitrary primitive ideal of $\mathcal{A}$. Then every derivation $\delta_n : \mathcal{A} \to \mathcal{A}$ "drops" to a derivation $\widehat{\delta_n} : \frac{\mathcal{A}}{\mathcal{P}} \to \frac{\mathcal{A}}{\mathcal{P}}$ by $\widehat{\delta_n}(a + \mathcal{P}) = \delta_n(a) + \mathcal{P}$ $(a \in \mathcal{A})$. By 5.2.28 (iii) of [3], $\widehat{\delta_n}$ is continuous, and so $\widehat{\delta_n}(a + P) \in Q(\frac{\mathcal{A}}{\mathcal{P}})$. By 1.5.29 (vii) of [3], $\delta_n(a) \in Q(\mathcal{A})$. Hence δ_n's are spectrally infinitesimal and so they are spectrally bounded for all $n \in \mathbb{N}$. It follows from Theorem 2.4 that $\delta_n(\mathcal{A}) \subseteq rad(\mathcal{A})$ for all $n \in \mathbb{N}$ and finally, Theorem 2.6 yields the desired result. $\square$

Theorem 2.9. *Let $\{d_n\}$ be a spectrally bounded strong higher derivation and let $\mathcal{P}$ be a primitive ideal of $\mathcal{A}$ such that $\frac{\mathcal{A}}{\mathcal{P}}$ is commutative. Then $d_n(\mathcal{A}) \subseteq \mathcal{P}$ for all $n \in \mathbb{N}$.*

Proof. By Theorem 2.3, $d_n(\mathcal{P}) \subseteq \mathcal{P}$ for all $n \in \mathbb{N}$, so such a higher derivation $\{d_n\}$ induces a higher derivation $\{D_n\}$ on a semi-simple Banach algebra $\frac{\mathcal{A}}{\mathcal{P}}$ by $D_n(a + \mathcal{P}) = d_n(a) + \mathcal{P}$ $(n \in \mathbb{N})$. By hypothesis, $\frac{\mathcal{A}}{\mathcal{P}}$ is commutative, so Theorem 2.7 implies that $D_n(\frac{\mathcal{A}}{\mathcal{P}}) \subseteq rad(\frac{\mathcal{A}}{\mathcal{P}}) = \{0\}$, and $d_n(\mathcal{A}) \subseteq \mathcal{P}$ $(n \in \mathbb{N})$ follows. $\square$

It is well-known that $Z(\mathcal{A})$, the center of $\mathcal{A}$, is a closed subalgebra of $\mathcal{A}$. We define $\frac{Z_{n+1}(\mathcal{A})}{Z_n(\mathcal{A})} = Z(\frac{\mathcal{A}}{Z_n(\mathcal{A})})$ for each positive integer n. It is clear that $Z_0(\mathcal{A}) = \{0\}$

if and only if $Z_1(\mathcal{A}) = Z(\mathcal{A})$. If $n = 2$, for example, we have

$$\begin{aligned}\frac{Z_2(\mathcal{A})}{Z(\mathcal{A})} &= Z(\frac{\mathcal{A}}{Z(\mathcal{A})})\\ &= \{a + Z(\mathcal{A}) \mid ab - ba \in Z(\mathcal{A}) \ \forall\ b \in \mathcal{A}\}\\ &= \{a + Z(\mathcal{A}) \mid (ab - ba)c = c(ab - ba) \ \forall\ b, c \in \mathcal{A}\}\\ &= \{a + Z(\mathcal{A}) \mid [[a, b], c] = 0 \ \forall\ b, c \in \mathcal{A}\}.\end{aligned}$$

Hence, $Z_2(\mathcal{A}) = \{a \in \mathcal{A} \mid [[a, b], c] = 0 \ \forall\ b, c \in \mathcal{A}\}$. Moreover, if $n = 3$ then we can find $Z_3(\mathcal{A}) = \{a \in \mathcal{A} \mid [[[a, b], c], d] = 0 \ \forall\ b, c, d \in \mathcal{A}\}$. Thus, by induction on n, we can conclude that

$$Z_n(\mathcal{A}) = \{a \in \mathcal{A} \mid \underbrace{[[[...[}_{n-times}\ a, a_1], a_2], a_3], ..., a_n] = 0 \ \forall\ a_1, a_2, a_3, ..., a_n \in \mathcal{A}\}.$$

Clearly, $\{0\} = Z_0(\mathcal{A}) \subseteq Z_1(\mathcal{A}) \subseteq Z_2(\mathcal{A}) \subseteq ... \subseteq Z_n(\mathcal{A}) \subseteq ... \subseteq \mathcal{A}$, and $Z_2(\mathcal{A}) = \mathcal{A}$ if and only if $Z(\frac{\mathcal{A}}{Z(\mathcal{A})}) = \frac{\mathcal{A}}{Z(\mathcal{A})}$. It means that $\frac{\mathcal{A}}{Z(\mathcal{A})}$ is commutative if and only if $Z_2(\mathcal{A}) = \mathcal{A}$.

Theorem 2.10. *Let $\{d_n\}$ be a strong higher derivation on a noncommutative Banach algebra $\mathcal{A}$. If $Z_2(\mathcal{A}) = \mathcal{A}$, then $d_n(\mathcal{A}) \subseteq rad(\mathcal{A})$ for all $n \in \mathbb{N}$.*

Proof. First of all, we define another product on $\mathcal{A}$ by the following form: $a \bullet b = \frac{ab+ba}{2}$ $(a, b \in \mathcal{A})$. Obviously, $a \bullet b = b \bullet a$, i.e. $(\mathcal{A}, \bullet)$ is commutative. Furthermore, $a \bullet (b + c) = a \bullet b + a \bullet c$ and $(a + b) \bullet c = a \bullet c + b \bullet c$ for all $a, b, c \in \mathcal{A}$. Since $Z_2(\mathcal{A}) = \mathcal{A}$, we have

$$\begin{aligned}a \bullet (b \bullet c) &= a \bullet (\frac{bc + cb}{2})\\ &= \frac{abc + acb + bca + cba}{4}\\ &= \frac{abc + cba + bac + cab}{4}\\ &= (a \bullet b) \bullet c\end{aligned}$$

for all $a, b, c \in \mathcal{A}$ and it means that $(\mathcal{A}, \bullet)$ is an associative algebra. Let a and b be two arbitrary elements of $\mathcal{A}$. Then $\|a \bullet b\| = \|\frac{ab+ba}{2}\| \le \|a\|\|b\|$ and consequently, $(\mathcal{A}, \bullet)$ is a commutative Banach algebra by the original norm on $\mathcal{A}$. We denote this algebra by $\widetilde{\mathcal{A}}$. Let $\delta : \mathcal{A} \to \mathcal{A}$ be a derivation. Then

$$\begin{aligned}\delta(a \bullet b) &= \delta(\frac{ab + ba}{2})\\ &= \frac{\delta(a)b + a\delta(b) + \delta(b)a + b\delta(a)}{2}\\ &= \frac{\delta(a)b + b\delta(a)}{2} + \frac{a\delta(b) + \delta(b)a}{2}\\ &= \delta(a) \bullet b + a \bullet \delta(b).\end{aligned}$$

In conclusion, $\delta : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}}$ is also a derivation. It follows from Theorem 2.5 that $\delta(\widetilde{\mathcal{A}}) \subseteq rad(\widetilde{\mathcal{A}}) = Q(\widetilde{\mathcal{A}}) = Q(\mathcal{A})$, i.e. $\delta(\mathcal{A}) \subseteq Q(\mathcal{A})$. Hence δ is spectrally bounded (indeed it is spectrally infinitesimal) and by using Theorem 2.4, we arrive at $\delta(\mathcal{A}) \subseteq rad(\mathcal{A})$. Finally, the proof is completed by Theorem 2.6.

$\square$

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References

1. F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer-Verlag, New York, 1973.
2. M. Brešar and M. Mathieu, *Derivations mapping into the radical III*, J. Func. Anal. **133** (1995) 21–29.
3. H. G. Dales, *Banach Algebras and Automatic Continuity*, London Math. Soc. Monographs, New Series, 24, Oxford University Press, New York, 2000.
4. H. G. Dales, Pietro Aiena, Jörg Eschmeier, Kjeld Laursen and George A. Willis, *Introduction to Banach Algebras, Operators and Harmonic Analysis*, Cambridge University Press, 2003.
5. M. Ferrero and C. Haetinger, *Higher derivations and a theorem by Herstein*, Quaestiones Mathematicae **25** (2) (2002) 249–257.
6. A. Hosseini, M. Hassani, A. Niknam and S. Hejazian, *Some results on σ-derivations*, Ann. Funct. Anal. **2** (2) (2011) 75-84.
7. A. Hosseini, M. Hassani and A. Niknam, *Generalized σ-derivation on Banach Algebras*, Bull. Iranian Math. Soc. **37** (2011) 87–100.
8. A. Hosseini, M. Hassani and A. Niknam, *Some results on generalized σ-derivations*, Journal of advanced research in pure mathematics. **5** (3) (2013) 61–72.
9. H. Hasse and F. K. Schmidt, *Noch eine Begrüdung der theorie der höheren Differential quotienten in einem algebraischen Funtionenkörper einer Unbestimmeten*. J. Reine Angew. Math. **177** (1937) 215–237.
10. M. Mirzavaziri, *Characterization of higher derivations on algebras*, Communications in algebra. **38** (3) (2010) 981–987.
11. M. Mirzavaziri, K. Naranjani and A. Niknam, *Innerness of higher derivations*, Banach J. Math. Anal, **4** (2) (2010) 121–128.
12. M. Mirzavaziri and E. O. Tehrani, *Generalized higher derivations are sequences of generalized derivations*, Journal of advanced research in pure mathematics. **3** (1) (2011) 75–86.
13. M. Mathieu, *Where to find the image of aderivation*, Functional analysis and operator theory. **38** (1994) 237–249.
14. A. M. Sinclair, *Continuous derivations on Banach algebras*. Proc. Amer. Math. Soc. **20** (1969) 166–170
15. M. P. Thomas, *The image of a derivation is contained in the radical*, Ann. of math. **128** (1988) 435-460.
16. K. Woung Jun and Y. Whan Lee, *The image of a continuous strong higher derivation is contained in the radical* , Bull. Korean Math. Soc. **33** (2) (1996) 229–233.
17. Shafeng Xu and Zhankui Xiao, *Jordan higher derivations revisited*, Gulf Journal of Mathematics. **2** (1) (2014) 11–21.
18. Yong-Soo Jung, Spectrally bounded Jordan derivations on Banach algebras, Bull. Korean Math. Soc. **36** (3) (1999) 477–481.

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