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STOCHASTIC FRACTIONAL INTEGRODIFFERENTIAL EQUATIONS WITH JUMPS: APPLICATION TO AN AVERAGING PRINCIPLE

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Abstract. In this work, we deal with a stochastic fractional integrodifferential equations associated to a Poisson random measure. We first prove existence and uniqueness of solution in the case of Lipschitz coefficients but also in the non Lipschitz case. In the second part, we show an averaging principle in the sense of Khasminskii approach for a class of this equation with non Lipschitz coefficients and weak averaging conditions.

1. Introduction

Over the past twenty years, with the advent of fractional differential equations as a field of research, more and more researchers have become interested in the study of fractional calculus. Since then, different concepts of derivatives and fractional integrals have been introduced, such as the integrals and derivatives of Riemann-Liouville, Caputo (see [4], [9], [10]). With the development of fractional calculus, fractional integrodifferential equations appear in many domains such as electromagnetic waves and dynamic population system. It should be noted also that in the literature we find a lot of results dealing with existence and uniqueness of fractional integrodifferential equations ([2], [3], [1]).

Moreover, recall that various dynamic processes in engineering and science are subject to random perturbations of both external and internal environmental natures ([15], [13], [11], [22]). So to make these models more precise, Pedjeu and Ladde [11] develop the concept of dynamic processes operating under a set of linearly independent time scales and provide existence and uniqueness of solution of stochastic fractional differential equations by using the classical Picard-Lindelöf successive approximation scheme. They traited the Riemann-Liouville type fractional integral. In this spirit Umamaheswari et al. [17] established an existence and uniqueness of solution of stochastic fractional integrodifferential equations under the Lipschitz condition. These results extend the work of Pedjeu and Ladde [11] to integrodifferential equations. In this context Faye et al. [6] investigate the non Lipschitz case and provide a numerical solution of a family of

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stochastic fractional integrodifferential equations by using Euler Maruyama approximation. Also Diouf et al. [5] show an existence and uniqueness result of this class of equations by exploring a smoothness method.

Furthermore, it should be noted that asymptotic methods play an important role in investigating nonlinear dynamical systems in particular the averaging methods provide a powerful tool for simplifying stochastic dynamical systems and give approximate solutions to stochastic differential equations. After the averaging principle for deterministic differential equations, many authors contribute to developping these results for stochastic differential equations (see among others [20], [7], [21], [18], [12]). Xu et al. [19] present averaging principle for fractional stochastic differential equation with fractional order $0 < \alpha < 1$ and find the classical Khasminskii approach by showing that the mild solution of two equations before and after averaging are equivalence in the sense of mean square. In addition Guo et al. [8], under new averaging conditions, investigate averaging principle for Caputo fractional differential equations with compensated Poisson random measure in the Lipschitz case. They also show that the solution to a stochastic fractional differential system can be approximated by a corresponding averaging equation in the sense of mean square.

Inspired by these dynamics, we associate to the brownian motion a Lévy process of Poisson random measure type to take into account the discontinuity in certain random phenomena. Our present paper investigate stochastic fractional integrodifferential equations associated to a Poisson random measure (SFIDEJ). First we establish an existence and uniqueness result in the case of lipschitzian coefficients by using the classical Picard-Lindelöf iteration method. Secondly, we weaken the Lipschitz conditions by continuous coefficients and also prove an existence and uniqueness result for the solution. We used a smoothness technic based in Rong [14, Theorem 170]. All these results generalise those of Umamaheswari et al.[17] to the non lipschitz case with jump and also our previous work Faye et al.[6] and Diouf et al.[5] to integrodifferential with jump. Finally we investigate, under weak assumptions, an averaging principle by proving that a class of stochastic fractional integrodifferential equation with jump can be approximated by an averaged equation in the sense of mean square convergence. This extends the results of Xu et al.[19] and Guo et al.[8] to integrodifferential equation with non Lipschitz conditions.

The rest of the paper will be organized as follows: in section 2 we give some useful preliminaries notions. We show existence and uniqueness results in section 3. In section 4 we present an averaging principle for a family of stochastic fractional integrodifferential equations associated to a Poisson random measure.

2. Preliminaries

Definition 2.1. ($Riemann - Liouville\ fractional\ integral$).
The Riemann-Liouville fractional integral operator of order α of a function $f \in L^1(\mathbb{R}_+)$ is defined by

$$I_{0^+}^{(\alpha)} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \qquad \forall \alpha > 0,$$

where $\Gamma(.)$ is the Euler Gamma function defined by

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} \exp(-t) dt.$$

Definition 2.2. ($Riemann - Liouville\ fractional\ derivative$).
The Riemann-Liouville fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}^*$, is defined as

$$D_{0^+}^{(\alpha)} f(t) = \left(\frac{d}{dt}\right)^n I_{0^+}^{(n-\alpha)} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds,$$

where the function $f(t)$ has absolutely continuous derivatives upto order $(n-1)$.

Definition 2.3. (Multi-time scale Integral [11])
For $p \in \mathbb{N}$, $p > 1$, let $\{T_1, \cdots, T_p\}$ be a set of linearly independent time scales. Let $f : [a, b[\times \mathbb{R}^{p-1} \to \mathbb{R}^n$ be a continuous function defined by $f(t) = f(T_1(t), \cdots, T_p(t))$. The multi-time scale integral of the composite function f over an interval $[t_0, t] \subseteq]a, b[$ is defined as the sum of p integrals with respect to the time-scales $T_1, \cdots, T_p$. We denote it by If,

$$If(t) := \sum_{j=1}^{p} I_j f(t),$$

where the sense of the integral

$$I_j f(t) = \int_{t_0}^{t} f(s) dT_j(s)$$

depends on the time scale T_j, for each $j = 1, \cdots, p$.

Example 2.4. For $p = 4$, consider the linearly independent set consisting of time scales $T_1(t) := t$, $T_2(t) := B(t)$ where B is the standard Wiener process, $T_3(t) := \int_E N(t, de)$ where N be a Poisson random measure and $T_4(t) := t^\alpha$, $0 < \alpha < 1$ as defined before. In this case,

$$f(t) \equiv f(T_1(t), T_2(t), T_3(t), T_4(t)) \quad and \quad If(t) = I_1 f(t) + I_2 f(t) + I_3 f(t) + T_4(t),$$

where

$$I_1 f(t) = \int_{t_0}^{t} f(s) ds, \qquad I_2 f(t) = \int_{t_0}^{t} f(s) dB(s), \qquad I_3 f(t) = \int_{t_0}^{t}\int_E f(s) N(ds, de),$$

$$I_4 f(t) = \int_{t_0}^{t} f(s)(ds)^\alpha.$$

Under the set of time scales in example 2.4, we consider the following stochastic fractional differential equation: for all $t \in [0, T]$,

$$dx_t = b_1(t, x_t) dt + b_2(t, x_t)(dt)^\alpha + \sigma_1(t, x_t) dW_t + \int_E \sigma_2(t, x_t, e) N(dt, de). \tag{2.1}$$

Remark 2.5. 1) If $b_2 = 0$, then the equation (2.1) reduced to known SDE with jump

$$dx_t = b_1(t,x_t)dt + \sigma_1(t,x_t)dW_t + \int_E \sigma_2(t,x_t,e)N(dt,de),$$

whose fundamental properties and applications have been well studied for more than half-century.

2) If $\sigma_1 = \sigma_2 = 0$, we have the following generalized version of the classical deterministic fractional differential equation

$$dx(t) = b_1(t,x_t)dt + b_2(t,x_t)(dt)^\alpha.$$

3) If $b_2 = \sigma_1 = \sigma_2 = 0$, then equation (2.1) is the deterministic differential equation

$$dx(t) = b_1(t,x_t)dt.$$

3. Stochastic fractional integrodifferential equation

Consider $(\Omega,\mathcal{F},\mathbb{P})$ a complete probability space. For a fix real $T > 0$, we assume given two mutually independent processes:

- a $m-$dimensional standard Brownian motion $W = (W_t)_{0\le t\le T}$,
- a Poisson random measure N defined on $\mathbb{R}_+ \times E$.

The space E is equipped with its Borel field $\mathcal{B}(E)$ and a σ-finite measure π. We denote the compensated Poisson random measure associated to $N(dt,de)$ by $\widetilde{N}(dt,de) = N(dt,de) - \pi(de)dt$. Let $J = [0,T]$. Assume that $b_1, b_2 : J\times\mathbb{R}^p\times\mathbb{R}^p \to \mathbb{R}^p$, $\sigma_1 : J\times\mathbb{R}^p\times\mathbb{R}^p \to \mathbb{R}^{p\times m}$, $\sigma_2 : J\times\mathbb{R}^p\times\mathbb{R}^p\times E \to \mathbb{R}^p$ are $\mathcal{F}_t-$ adapted and measurable functions, where $\mathcal{F}_t$ is the $\sigma-$ field generated by W and $\widetilde{N}$ up to time t, defined by

$$\mathcal{F}_t = \sigma\left(W_s, \widetilde{N}(]0,s],U), \forall U \in \mathcal{B}(E), s \le t\right).$$

Given a real $\alpha \in]1/2,1[$ and X_0 a $\mathbb{R}^p$ value random variable independent of $\mathcal{F}_t$. In the following, we consider the stochastic fractional integrodifferential equation: for all $t \in J$,

$$\begin{aligned} X_t = X_0 \; &+ \int_0^t b_1\left(s, X_{s-}, \int_0^s f_1(s,u,X_u)\right)du)ds + \int_0^t b_2\left(s, X_{s-}, \int_0^s f_2(s,u,X_u)du\right)(ds)^\alpha \\ &+ \int_0^t \sigma_1\left(s, X_{s-}, \int_0^s g_1(s,u,X_u)du\right)dW_s \\ &+ \int_0^t\int_E \sigma_2\left(s, X_{s-}, \int_0^s g_2(s,u,X_u)du, e\right)\widetilde{N}(ds,de), \end{aligned} \tag{3.1}$$

where $f_i, g_i : J\times J\times\mathbb{R}^p \to \mathbb{R}^p$, $i = 1,2$ and $X_{t-} = \lim_{s\nearrow t} X_s$.
By a solution of equation(3.1), we mean a right continuous with left limits (r.c.l.l.) and $(\mathcal{F}_t)-$adapted, $\mathbb{R}^p$ valued process X_t satisfying equation(3.1) almost surely for $t \in [0,T]$.

3.1. **Lipschitz case.** In the following we assume that $b_i,\ \sigma_i,\ f_i,\ g_i,\ i=1,2$ verify the assumptions(**H**):
Let

$$F_i=\int_0^s f_i(s,u,X_u)du,\qquad G_i=\int_0^s g_i(s,u,X_u)du,$$
$$\widetilde{F}_i=\int_0^s f_i(s,u,\widetilde{X}_u)du,\qquad \widetilde{G}_i=\int_0^s g_i(s,u,\widetilde{X}_u)du.$$

(**H.1**) Linear growth condition
There exist two constantes $C>0$ and $c>0$ such that:

$$\begin{cases} \mid b_1\left(t,X,F_1\right)\mid^2+\mid b_2\left(t,X,F_2\right)\mid^2+\mid \sigma_1\left(t,X,G_1\right)\mid^2+\displaystyle\int_E \mid \sigma_2\left(t,X,G_2,e\right)\mid^2\pi(de)\\ \leq C^2\left(1+\mid X\mid^2+\mid F_1\mid^2+\mid F_2\mid^2+\mid G_1\mid^2+\mid G_2\mid^2\right),\\ \\ \mid F_1\mid^2+\mid G_1\mid^2+\mid F_2\mid^2+\mid G_2\mid^2\leq c^2(1+\mid X\mid^2). \end{cases}$$

(**H.2**) The Lipschitz condition:
There exist constantes $L>0$ and $l>0$ such that:

$$\begin{cases} \mid b_1(t,X,F_1)-b_1(t,\widetilde{X},\widetilde{F}_1)\mid^2+\mid b_2(t,X,F_2)-b_2(t,\widetilde{X},\widetilde{F}_2)\mid^2\\ +\mid \sigma_1(t,X,G_1)-\sigma_1(t,\widetilde{X},\widetilde{G}_1)\mid^2+\displaystyle\int_E \mid \sigma_2(t,X,G_2,e)-\sigma_2(t,\widetilde{X},\widetilde{G}_2,e)\mid^2\pi(de)\\ \leq L^2\left(\mid X-\widetilde{X}\mid^2+\mid F_1-\widetilde{F}_1\mid^2+\mid F_2-\widetilde{F}_2\mid^2+\mid G_1-\widetilde{G}_1\mid^2+\mid G_2-\widetilde{G}_2\mid^2\right),\\ \\ \mid F_1-\widetilde{F}_1\mid^2+\mid G_1-\widetilde{G}_1\mid^2+\mid F_2-\widetilde{F}_2\mid^2+\mid G_2-\widetilde{G}_2\mid^2\leq \ell^2\mid X-\widetilde{X}\mid^2. \end{cases}$$

Theorem 3.1. *Assume that $\mathbb{E}\left(\mid X_0\mid^2\right)<+\infty$ and condition (**H**) is in force, then the SFIDEJ (3.1) has a unique solution such that $\sup_{0\leq t\leq T}\mathbb{E}\left(|X_t|^2\right)<+\infty$.*

Proof. **Existence**:
Let us define $X_t^0=X_0$ and $X_t^k=X^k(t,\omega)$ inductively as follows:

$$\begin{aligned} X_t^{k+1} &= X_0+\int_0^t b_1(s,X_{s-}^k,\int_0^s f_1(s,u,X_u^k)du)ds+\int_0^t (t-s)^{\alpha-1}b_2(s,X_{s-}^k,\int_0^s f_2(s,u,X_u^k)du)ds\\ &+ \int_0^t \sigma_1(s,X_{s-}^k,\int_0^s g_1(s,u,X_u^k)du)dW_s\\ &+ \int_0^t\int_E \sigma_2(s,X_{s-}^k,\int_0^s g_2(s,u,X_u^k)du,e)\widetilde{N}(ds,de). \end{aligned} \tag{3.2}$$

So, with the help of the Cauchy-Schwarz inequality and applying the algebraic inequality

$(a+b+c+d+e)^2 \leq 5\left(a^2+b^2+c^2+d^2+e^2\right)$, we have

$$\begin{aligned} \mid X_t^{k+1} \mid^2 &\leq 5\Big(\mid X_0 \mid^2 + T\int_0^t \left| b_1\left(s, X_{s-}^k, \int_0^s f_1(s,u,X_u^k)du\right)\right|^2 ds \\ &+ \alpha^2 \frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \left| b_2\left(s, X_{s-}^k, \int_0^s f_2(s,u,X_u^k)du\right)\right|^2 ds \\ &+ \left| \int_0^t \sigma_1\left(s, X_{s-}^k, \int_0^s g_1(s,u,X_u^k)du\right) dW_s\right|^2 \\ &+ \left| \int_0^t \int_E \sigma_2\left(s, X_{s-}^k, \int_0^s g_2(s,u,X_u^k)du, e\right) \widetilde{N}(ds,de)\right|^2\Big), \end{aligned}$$

by typing the mathematical expectation and using the Itô isometry, we obtain

$$\begin{aligned} \mathbb{E} \mid X_t^{k+1} \mid^2 &\leq 5\Big(\mathbb{E} \mid X_0 \mid^2 + T\int_0^t \mathbb{E}\left| b_1\left(s, X_s^k, \int_0^s f_1(s,u,X_u^k)du\right)\right|^2 ds \\ &+ \alpha^2 \frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\left| b_2\left(s, X_s^k, \int_0^s f_2(s,u,X_u^k)du\right)\right|^2 ds \\ &+ \mathbb{E}\int_0^t \left| \sigma_1\left(s, X_s^k, \int_0^s g_1(s,u,X_u^k)du\right)\right|^2 ds \\ &+ \mathbb{E}\int_0^t \int_E \sigma_2\left| \left(s, X_s^k, \int_0^s g_2(s,u,X_u^k)du, e\right)\right|^2 \pi(de)ds\Big) \end{aligned}$$

and by $(\mathbf{H.1})$ we have, for $k = 0, 1, 2, \cdots$,

$$\begin{aligned} \mathbb{E} \mid X_t^{k+1} \mid^2 &\leq 5\Big(\mathbb{E} \mid X_0 \mid^2 + TC^2(1+c^2)\int_0^t \mathbb{E}(1+\mid X_s^k \mid^2)ds \\ &+ \alpha^2 \frac{T^{2\alpha-1}}{2\alpha-1}K^2(1+c^2)\int_0^t \mathbb{E}(1+\mid X_s^k \mid^2)ds \\ &+ C(1+c^2)\int_0^t \mathbb{E}(1+\mid X_s^k \mid^2)ds + K^2(1+c^2)\int_0^t \mathbb{E}(1+\mid X_s^k \mid^2)ds\Big) \\ &\leq 5\mathbb{E} \mid X_0 \mid^2 + 5C^2(1+c^2)(2+T+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1})\int_0^t \mathbb{E}(1+\mid X_s^k \mid^2)ds. \end{aligned}$$

Since $\mathbb{E}\left(\mid X_t^0 \mid^2\right) = \mathbb{E}\left(\mid X_0 \mid^2\right) < +\infty$, by induction, we have

$$\sup_{0\leq t\leq T} \mathbb{E}\left(|X_t^k|^2\right) < +\infty, \quad \text{for} \quad k = 1, 2, 3, \cdots.$$

Applying the Schwarz-inequality and Itô isometry, we obtain

$$\begin{aligned} \mathbb{E} \mid X_t^{k+1} - X_t^k \mid^2 &\leq 4T\int_0^t \mathbb{E}\left|\Delta b_1^k(s)\right|^2 ds + 4\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\left|\Delta b_2^k(s)\right|^2 ds \\ &+ 4\mathbb{E}\int_0^t \left|\Delta\sigma_1^k(s)\right|^2 ds + 4\mathbb{E}\int_0^t \int_E \left|\Delta\sigma_2^k(s,e)\right|^2 \pi(de)ds, \end{aligned}$$

where

$$\Delta b_i^k(s) = b_i(s, X_s^k, \int_0^s f_1(s,u,X_u^k)du) - b_i(s, X_s^{k-1}, \int_0^s f_1(s,u,X_u^{k-1})du),\ i=1,2,$$

$$\Delta\sigma_1^k(s) = \sigma_1(s, X_s^k, \int_0^s g_1(s,u,X_u^k)du) - \sigma_1(s, X_s^{k-1}, \int_0^s g_1(s,u,X_u^{k-1})du),$$

$$\Delta\sigma_2^k(s,e) = \sigma_2(s, X_s^k, \int_0^s g_2(s,u,X_u^k)du, e) - \sigma_2(s, X_s^{k-1}, \int_0^s g_2(s,u,X_u^{k-1})du, e).$$

The Lipschitz condition (**H.2**) give

$$\mathbb{E} \mid X_t^{k+1} - X_t^k \mid^2 \leq 4L^2(1+\ell^2)(2+T+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1}) \int_0^t \mathbb{E} \mid X_s^k - X_s^{k-1} \mid^2 ds. \tag{3.3}$$

From equation (3.2), by applying again the Schwarz inequality, the Itô isometry together with the linear growth conditions for $k=1$ we get

$$\begin{aligned}
\mathbb{E} \mid X_t^1 - X_t^0 \mid^2 &\leq 4T\int_0^t \mathbb{E}\left|b_1\left(s, X_0, \int_0^s f_1(s,u,X_0)du\right)\right|^2 ds \\
&+ 4\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\left|b_2\left(s, X_0, \int_0^s f_2(s,u,X_0)du\right)\right|^2 ds \\
&+ 4\mathbb{E}\int_0^t \left|\sigma_1\left(s, X_0, \int_0^s g_1(s,u,X_0)du\right)\right|^2 ds \\
&+ 4\mathbb{E}\int_0^t \int_E \sigma_2\left|\left(s, X_0, \int_0^s g_2(s,u,X_0)du, e\right)\right|^2 \pi(de)ds \\
&\leq 4C^2(1+c^2)(2+T+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1})\int_0^t \mathbb{E}(1+\mid X_0\mid^2)ds \\
&\leq 4C^2(1+c^2)(2+T+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1})(1+\mathbb{E}\mid X_0\mid^2)t.
\end{aligned} \tag{3.4}$$

Now for $k=1$, replacing $\mathbb{E} \mid X_t^1 - X_t^0 \mid^2$ in the inequality (3.3) with the value on the right hand side of inequality (3.4) and integrating, we obtain

$$\begin{aligned}
\mathbb{E} \mid X_t^2 - X_t^1 \mid^2 &\leq 4L^2(1+\ell^2)(2+T+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1})\int_0^t \mathbb{E}\mid X_s^1 - X_s^0\mid^2 ds \\
&\leq L^2(1+\ell^2)C^2(1+c^2)(1+\mathbb{E}\mid X_0\mid^2)4^2(2+T+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1})^2\int_0^t sds \\
&\leq K(1+\mathbb{E}\mid X_0\mid^2)\left[4^2(2+T+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1})^2\right]\frac{t^2}{2!},
\end{aligned}$$

where $K = L^2(1+\ell^2)C^2(1+c^2)$ is a constant that changes from one line to another.
For $k=2$, proceeding as before, we have

$$\mathbb{E} \mid X_t^3 - X_t^2 \mid^2 \leq K(1+\mathbb{E}\mid X_0\mid^2)\left[4(2+T+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1})\right]^3\frac{t^3}{3!}.$$

Thus, by the principle of mathematical induction, we have

$$\begin{aligned}
\mathbb{E} \mid X_t^{k+1} - X_t^k \mid^2 &\leq K(1+\mathbb{E}\mid X_0\mid^2)\left[4(2+T+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1})\right]^{k+1}\frac{t^{k+1}}{(k+1)!} \\
&\leq \frac{BM^{k+1}t^{k+1}}{(k+1)!},\ k=0,1,2,\cdots,
\end{aligned}$$

where $B = K(1+\mathbb{E} \mid X_0 \mid^2)$ and $M = 4(2+T+\alpha^2 \frac{T^{2\alpha-1}}{2\alpha-1})$. Thus

$$\sup_{0\leq t\leq T} \mathbb{E} \mid X_t^{k+1} - X_t^k \mid^2 \leq \frac{BM^{k+1}T^{k+1}}{(k+1)!}, \ k = 0, 1, 2, \cdots .$$

This implies the mean-square convergence of the successive approximations uniformly on J. That is,

$$\begin{aligned} \|X_t^m - X_t^n\|_{L^2} &\leq \sum_{k=n}^{m-1} \|X_t^{k+1} - X_t^k\|_{L^2} \\ &\leq \sum_{k=n}^{m-1} \int_0^T \frac{BM^{k+1}t^{k+1}}{(k+1)!} dt \\ &\leq \sum_{k=n}^{m-1} \frac{BM^{k+1}T^{k+2}}{(k+2)!} \to 0, \quad as\ m, n \to +\infty. \end{aligned}$$

Then, by applying the Chebyshev's inequality and summing up the resultant inequalities, we have

$$\sum_{k=1}^{+\infty} \mathbb{P}\left(\sup_{0\leq t\leq T} \mid X_t^{k+1} - X_t^k \mid^2 > \frac{1}{k^2} \right) \leq \sum_{k=1}^{+\infty} \frac{BM^{k+1}T^{k+2}k^4}{(k+2)!},$$

where the series on the right side converges by ratio test. Hence the series on the left side also converges, so by the Borel-Cantelli lemma, we conclude that $\sup_{0\leq t\leq T} \mid X_t^{k+1} - X_t^k \mid^2$ converges to 0, almost surely, that is, the successive approximations X_t^k converge, almost surely, uniformly on J to a limit X_t defined by

$$\lim_{n\to+\infty} \left(X_t^0 + \sum_{k=1}^{n} \left(X_t^k - X_t^{k-1} \right) \right) = \lim_{n\to+\infty} X_t^n = X_t.$$

Since X_t is the limit of nonanticipating functions and the uniform limit of a sequence of r.c.l.l. functions, it is itself nonanticipating and r.c.l.l. process
From (3.2), we have

$$\begin{aligned} X_t = X_0 &+ \int_0^t b_1(s, X_{s-}, \int_0^s f_1(s,u,X_u)du)ds + \alpha \int_0^t (t-s)^{\alpha-1} b_2(s, X_{s-}, \int_0^s f_2(s,u,X_u)du)ds \\ &+ \int_0^t \sigma_1(s, X_{s-}, \int_0^s g_1(s,u,X_u)du)dW_s \\ &+ \int_0^t \int_E \sigma_2(s, X_{s-}, \int_0^s g_2(s,u,X_u)du, e)\widetilde{N}(ds,de) \end{aligned} \tag{3.5}$$

for all $t \in J$. This completes the proof of the existence of solutions.
Uniqueness:
Let $X_t(\omega)$ and $Y_t(\omega)$ be solution processes through the initial data $(0, X_0)$ and $(0, Y_0)$ respectively, that is, $X(0,\omega) = X_0(\omega)$ and $Y(0,\omega) = Y_0(\omega)$, $\omega \in \Omega$. We

have, by virtue of the Schwarz inequality and the Itô isometry :

$$\begin{aligned}\mathbb{E}\mid X_t - Y_t\mid^2 \leq\ & 5\mathbb{E}\mid X_0 - Y_0\mid^2 + 5t\int_0^t \mathbb{E}\Big|b_1(s, X_s, \int_0^s f_1(s,u,X_u)du) - b_1(s, Y_s, \int_0^s f_1(s,u,Y_u)du)\Big|^2 ds \\ &+ 5\alpha^2 \frac{t^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\Big|b_2(s, X_s, \int_0^s f_2(s,u,X_u)du) - b_2(s, Y_s, \int_0^s f_2(s,u,Y_u)du)\Big|^2 ds \\ &+ 5\mathbb{E}\int_0^t \Big|\sigma_1(s, X_s, \int_0^s g_1(s,u,X_u)du) - \sigma_1(s, Y_s, \int_0^s g_1(s,u,Y_u)du)\Big|^2 ds \\ &+ 5\mathbb{E}\int_0^t\int_E \Big|\sigma_2(s, X_s, \int_0^s g_2(s,u,X_u)du, e) - \sigma_2(s, Y_s, \int_0^s g_2(s,u,Y_u)du, e)\Big|^2 \pi(de)ds.\end{aligned}$$

So by (**H.2**), we obtain

$$\mathbb{E}\mid X_t - Y_t\mid^2 \leq 5\mathbb{E}\mid X_0 - Y_0\mid^2 + 5L^2(1+\ell^2)(2 + T + \alpha^2\frac{T^{2\alpha-1}}{2\alpha-1})\int_0^t \mathbb{E}\mid X_s - Y_s\mid^2 ds.$$

We define $V_t = \mathbb{E}\mid X_t - Y_t\mid^2$. Then the function V satisfies

$$V_t \leq F + A\int_0^t V_s ds,$$

where $F = 5\mathbb{E}\mid X_0 - Y_0\mid^2$ and $A = 5L^2(1+\ell^2)(2 + T + \alpha^2\frac{T^{2\alpha-1}}{2\alpha-1})$. By applying the Gronwall inequality, we conclude that $V_t \leq F\exp(At)$. Now assume that $X_0 = Y_0$, then $F = 0$ and so $V_t = 0$ for all $t \geq 0$. That is, $\mathbb{E}\mid X_t - Y_t\mid^2 = 0$. Hence

$$\mathbb{P}\left\{|X_t - Y_t| = 0, \quad \forall t \in [0,T]\right\} = 1$$

and therefore, the solution of equation is unique.

□

3.2. **Non Lipschitz case.** We consider the following sets:

- $\mathcal{S}^2_{[0,T]}(\mathbb{R}^p)$ the spaces of $\mathcal{F}_t$-adapted r.c.l.l. processes

$$\psi : [0,T]\times\Omega \to \mathbb{R}^p, \quad \|\psi\|^2_{\mathcal{S}^2} = \mathbb{E}\left(\sup_{0\leqslant t\leqslant T}\mid\psi_t\mid^2\right) < \infty,$$

- $\mathcal{M}^2_{[0,T]}(\mathbb{R}^p)$ the spaces of $\mathcal{F}_t$-progressively measurable processes

$$\psi : [0,T]\times\Omega \to \mathbb{R}^p, \quad \|\psi\|^2_{\mathcal{M}^2} = \mathbb{E}\int_0^T \mid\psi_t\mid^2 dt < \infty.$$

In the following we assume that $b_j, \sigma_j,\ f_j, g_j,\ j = 1,2$ satisfy assumptions (**A**): let $F_j = \int_0^t f_j(t,s,X_s)ds, \quad G_j = \int_0^t g_j(t,s,X_s)ds, \quad j = 1,2.$

(A.1) Linear growth condition. There exist some constants $K > 0$ and $k_j > 0$, $j = 1,2$ such that
$|b_1(t,X_t,F_1)| + |b_2(t,X_t,F_2)| \leq K(1 + |X_t| + |F_1| + |F_2|)$,
$|\sigma_1(t,X_t,G_1)|^2 + \int_E|\sigma_2(t,X_t,G_2,e)|^2\pi(de) \leq K^2(1+|X_t|^2+|G_1|^2+|G_2|^2)$,
$|F_j| + |G_j| \leq k_j(1 + |X_t|)$.

(A.2) The functions $b_1(t,.,.),\ b_2(t,.,.),\ \sigma_1(t,.,.)$ are continuous in all variables,

$$\lim_{h\to 0}\int_E \mid \sigma_2(t,X_t+h,\int_0^t g_2(t,s,X_s+h)ds,e)-\sigma_2(t,X,\int_0^t g_2(t,s,X_s)ds,e)\mid^2\pi(de)=0.$$

(A.3) For each $N=1,2,\cdots$,

$$2\langle X_t-\widetilde{X}_t, b_1(t,X_t,F_1)-b_1(t,\widetilde{X}_t,\widetilde{F}_1)\rangle+\langle X_t-\widetilde{X}_t, b_2(t,X_t,F_2)-b_2(t,\widetilde{X}_t,\widetilde{F}_2)\rangle$$
$$+|\sigma_1(t,X_t,G_1)-\sigma_1(t,\widetilde{X}_t,\tilde{G}_1)|^2+\int_E|\sigma_2(t,X_t,G_2,e)-\sigma_2(t,\widetilde{X}_t,\tilde{G}_2,e)|^2\pi(de)$$
$$\leq c(t)\rho(|X_t-\widetilde{X}_t|^2)$$

as $|X_t|\vee|\widetilde{X}_t|\leq N,\ t\in[0,T]$, where $\int_0^T c(t)dt<+\infty$ and $\rho(u)\geq 0$, as $u\geq 0$, is non-random, strictly increasing, continuous and concave such that $\int_{0^+}\frac{1}{\rho(u)}du=+\infty$.

Remark 3.2. Note that $\rho(.)$ is concave and $\rho(0)=0$, then there exists two positives constants μ and λ such that $\rho(u)\leq\mu+\lambda u$.

We give the main result in this section :

Theorem 3.3. *Assume that $\mathbb{E}(\mid X_0\mid^2)<+\infty$ and $(\mathbf{A})$ is satisfied, then the SFIDEJ (3.1) has a unique solution such that $(X_t)_{t\geq 0}\in\mathcal{S}^2$.*

Before proving this theorem we need the following two lemmas:

Lemma 3.4. *Suppose conditions $(\mathbf{A.1})$ and $(\mathbf{A.2})$ are verified, then there exist functions: $b_i^n,\sigma_i^n,\ i=1,2,\ n\in\mathbb{N}^*$ verifying the following assumptions:*

1.
$$|b_1^n(t,X_t,F_1)|+|b_2^n(t,X_t,F_2)|\leq C\,(1+|X_t|)\quad \text{as}\quad n\geq N_0,$$
$$|\sigma_1^n(t,X,G_1)|^2+\int_E\mid\sigma_2^n(t,X,G_2,e)\mid^2\pi(de)\leq C\left(1+|X_t|^2\right),$$
where $N_0>0$ and $C>0$ are constants.

2.
$$|b_1^n(t,X_t,F_1)-b_1(t,\widetilde{X}_t,\widetilde{F}_1)|+|b_2^n(t,X,F_2)-b_2(t,\widetilde{X}_t,\tilde{F}_2)|\leq C_n\mid X_t-\widetilde{X}_t\mid,$$
$$|\sigma_1^n(t,X_t,G_1)-\sigma_1^n(t,\widetilde{X}_t,\tilde{G}_1)|^2+\int_E|\sigma_2^n(t,X_t,G_2,e)-\sigma_2^n(t,\widetilde{X}_t,\tilde{G}_2,e)|^2\pi(de)$$
$$\leq C_n\mid X_t-\widetilde{X}_t\mid,$$
where $C_n\geq 0$ is a constant depending on n.

3. *For any $N>0$ and for each $t\geq 0$, $\omega\in\Omega$, as $n\longrightarrow+\infty$*
$$\sup_{|X_t|\leq N}|b_1^n(t,X_t,F_1)-b_1(t,X_t,F_1)|+\sup_{|X_t|\leq N}|b_2^n(t,X_t,F_1)-b_2(t,X_t,F_1)|\longrightarrow 0,$$
$$\sup_{|X_t|\leq N}|\sigma_1^n(t,X_t,G_1)-\sigma_1(t,X_t,G_1)\mid^2+\sup_{|X_t|\leq N}\int_E|\sigma_2^n(t,X_t,G_2,e)-\sigma_2(t,X_t,G_2,e)\mid^2\pi(de)$$
$$\longrightarrow 0.$$

Proof. Let $\bar{F}_j^n = \int_0^t f_j(t,s,X_s^n - \frac{\bar{x}}{n})ds$, $\bar{G}_j^n = \int_0^t g_j(t,s,X_s^n - \frac{\bar{x}}{n})ds$, $j = 1,2$. Define

$$b_1^n(t,X_t,F_1) = \int_{\mathbb{R}^p} b_1(t,X_t - \frac{\overline{x}}{n}, \overline{F}_1)J(\overline{x})d\overline{x}, \tag{3.6}$$

where for all $u \in \mathbb{R}^p$

$$J(u) = \begin{cases} c\exp\left(-(1-\mid u\mid^2)^{-1}\right), & \text{for} \quad \mid u \mid< 1, \\ 0, & \text{otherwise,} \end{cases}$$

and the constant c satisfies $\int_{\mathbb{R}^p} J(u)du = 1$.

1. By definition (3.6) of b_1^n we have

$$|b_1^n(t,X_t,F_1)| \leqslant \int_{\mathbb{R}^p} |b_1(t,X_t - \frac{\overline{x}}{n}, \overline{F}_1)|J(\overline{x})d\overline{x}$$

and using the assumption (**A.1**) we have

$$\begin{aligned} |b_1^n(t,X_t,F_1)| &\leqslant \int_{\mathbb{R}^p} K\left(1+|X_t - \frac{\overline{x}}{n}| + k(1+|X_t-\frac{\overline{x}}{n}|)\right)J(\overline{x})d\overline{x} \\ &\leqslant K(1+k)\left[\int_{\mathbb{R}^p}(1+|X_t|)J(\overline{x})d\overline{x} + \frac{1}{n}\int_{\mathbb{R}^p}\overline{x}J(\overline{x})d\overline{x}\right]. \end{aligned}$$

Let $N_0 = \int_{\mathbb{R}^d} \overline{x}J(\overline{x})d\overline{x}$, we obtain

$$\begin{aligned} |b_1^n(t,X_t,F_1)| &\leqslant K(1+k)\left[1+|X_t| + \frac{N_0}{n}\right] \\ &\leqslant 2K(1+k)(1+|X_t|), \quad for \quad \frac{N_0}{n} \leq 1. \end{aligned}$$

Then b^n satisfies the point *1.*

2.

$$\begin{aligned} |b_1^n(t,X_t,F_1) - b_1(t,\widetilde{X}_t,\widetilde{F}_1)| &= \left|\int_{\mathbb{R}^p} b_1(t,X_s-\frac{\overline{x}}{n},\int_0^s f_1(s,r,X_s^n - \frac{\bar{x}}{n})ds)ds)J(\overline{x})d\overline{x}\right. \\ &- \left.\int_{\mathbb{R}^p} b_1(t,\widetilde{X}_s - \frac{\overline{x}}{n},\int_0^s f_1(s,r,\widetilde{X}_s^n - \frac{\bar{x}}{n})ds)ds)J(\overline{x})d\overline{x}\right|, \end{aligned}$$

then by a change of variable, we have

$$\begin{aligned} |b_1^n(t,X_t,F_1) - b_1(t,\tilde{X}_t,\tilde{F}_1)| &\leqslant n^p\int_{\mathbb{R}^p}\left|b_1(s,\overline{x},\overline{F}_1)\right|\left|J\left(n(X_s-\overline{x})\right) - J\left(n(\tilde{X}_s-\overline{x})\right)\right|d\overline{x} \\ &\leqslant n^p\int_{\mathbb{R}^p} K(1+k)(1+\mid\overline{x}\mid)\left|J\left(n(X_s-\overline{x})\right) - J\left(n(\tilde{X}_s-\overline{x})\right)\right|d\overline{x} \\ &\leqslant C_n \mid X_t - \tilde{X}_t\mid \end{aligned}$$

because

$$n^pK(1+\sum_{i=1}^2 k_i)\int_{\mathbb{R}^p}\int_0^1 (1+\mid\overline{x}\mid)grad\left[J\left(n(X_s-\overline{x}+\theta(\tilde{X}_s-\overline{x}))\right)\right]d\theta d\overline{x} \leq C_n \mid X_t - \tilde{X}_t\mid.$$

3. By Heine-Borel's finite covering theorem for any $N > 0$ and any given $\varepsilon > 0$ one can find a $\delta > 0$ may depend on t and ω such that as $\frac{1}{n} < \delta$, for all $\mid X_t \mid \leq N$

$$\left| b_1\left(t, X_t - \frac{1}{n}, \int_0^t f_1(t,s,X_s - \frac{1}{n})ds\right) - b_1\left(t, X_t, \int_0^t f_1(t,s,X_s)ds)\right)\right| < \varepsilon$$

because b_1 is continuous. Hence, as $n > \frac{1}{\delta}$,

$$\begin{aligned}\sup_{|X_t|\leq N} |b_1^n(t,X_t,F_1) - b_1(t,X_t,F_1)| &= \sup_{|X_t|\leq N} \Big|\int_{\mathbb{R}^p} b_1(t, X_t - \frac{\overline{x}}{n}, \overline{F}_1)J(\overline{x})d\overline{x} - b_1(t,X_t,F_1)\Big| \\ &\leq \int_{\mathbb{R}^p} \sup_{|X_t|\leq N} \Big|b_1(t, X_t - \frac{\overline{x}}{n}, \overline{F}_1) - b_1(t,X_t,F_1)\Big| J(\overline{x})d\overline{x} \\ &\leq \int_{|\overline{x}|<1} \sup_{|X_t|\leq N} \Big|b_1(t, X_t - \frac{\overline{x}}{n}, \overline{F}_1) - b_1(t,X_t,F_1)\Big| J(\overline{x})d\overline{x} \\ &\leq \varepsilon.\end{aligned}$$

Then, taking $\varepsilon \to 0$, we have

$$\sup_{|X_t|\leq N} \left| b_1^n(t,X_t,F_1) - b_1(t,X_t,F_1)\right| \longrightarrow 0.$$

Thus point *3.* is true for b_1^n.
By defining b_2^n, σ_1^n and σ_2^n in the same way as b_1^n in (3.6), we show that b_2^n, σ_1^n and σ_2^n satisfy this lemma.

□

Lemma 3.5. *Let b_1^n, b_2^n, σ_1^n and σ_2^n defined by lemma 3.4. Then there exists an unique solution $(X_t^n; 0 \leq t \leq T)$ satisfying the following SDE:*

$$\begin{aligned} X_t^n = X_0 &+ \int_0^t b_1^n(s, X_{s-}^n, \int_0^s f_1(s,u,X_u^n)du)ds + \int_0^t (t-s)^{\alpha-1} b_2^n(s, X_{s-}^n, \int_0^s f_2(s,u,X_u^n)du)ds \\ &+ \int_0^t \sigma_1^n(s, X_{s-}^n, \int_0^s g_1(s,u,X_u^n)du)dW_s \\ &+ \int_0^t \int_E \sigma_2^n(s, X_{s-}^n, \int_0^s g_2(s,u,X_u^n)du, e)\widetilde{N}(ds,de) \end{aligned} \tag{3.7}$$

such that $\mathbb{E}\left(\sup_{0\leq t\leq T} |X_t^n|^2\right) < K(\alpha,T)$.

Proof. By theorem 3.1, this equation (3.7) admit a pathwise unique strong solution $(X_t^n)_{0\leq t\leq T}$ satisfying $\sup_{0\leq t\leq T} \mathbb{E}\left(|X_t^n|^2\right) < +\infty$.
First, prove that $\mathbb{E}(\sup_{0\leq t\leq T} |X_t^n|^2) \leq K(\alpha,T)$, where $K(\alpha,T)$ is a constant depending on α and T.
By the Cauchy-Schwarz inequality and applying the algebraic inequality

$$(a+b+c+d+f)^2 \leq 5\left(a^2+b^2+c^2+d^2+f^2\right),$$

we have

$$\begin{aligned} \mid X_t^n \mid^2 &\leq 5\Big(\mid X_0 \mid^2 + T\int_0^t \left| b_1\left(s, X_{s-}^n, \int_0^s f_1(s,u,X_u^n)du\right)\right|^2 ds \\ &+ \alpha^2 \frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \left| b_2\left(s, X_{s-}^n, \int_0^s f_2(s,u,X_u^n)du\right)\right|^2 ds \\ &+ \left|\int_0^t \sigma_1\left(s, X_{s-}^n, \int_0^s g_1(s,u,X_u^n)du\right) dW_s\right|^2 \\ &+ \left|\int_0^t\int_E \sigma_2\left(s, X_{s-}^n, \int_0^s g_2(s,u,X_u^n)du, e\right)\widetilde{N}(ds,de)\right|^2\Big), \end{aligned}$$

hence

$$\begin{aligned} \sup_{0\leq t\leq T} \mid X_t^n \mid^2 &\leq 5\Big(\mid X_0 \mid^2 + T\int_0^T \left| b_1\left(s, X_{s-}^n, \int_0^s f_1(s,u,X_u^n)du\right)\right|^2 ds \\ &+ \alpha^2 \frac{T^{2\alpha-1}}{2\alpha-1}\int_0^T \left| b_2\left(s, X_{s-}^n, \int_0^s f_2(s,u,X_u^n)du\right)\right|^2 ds \\ &+ \sup_{0\leq t\leq T}\left|\int_0^t \sigma_1\left(s, X_{s-}^n, \int_0^s g_1(s,u,X_u^n)du\right) dW_s\right|^2 \\ &+ \sup_{0\leq t\leq T}\left|\int_0^t\int_E \sigma_2\left(s, X_{s-}^n, \int_0^s g_2(s,u,X_u^n)du, e\right)\widetilde{N}(ds,de)\right|^2\Big). \end{aligned}$$

The mathematical expectation with the help of lemma 3.4 allow us to have,

$$\mathbb{E}\left(\sup_{0\leq t\leq T} \mid X_t^n \mid^2\right) \leq 4\Big(\mathbb{E}\left(\mid X_0 \mid^2\right) + (T+1+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1})(1+k)\int_0^T \left(1+\mathbb{E}\left(\mid X_s^n \mid^2\right)ds\right)$$

therefore, since $\mathbb{E}\left(|X_t^n|^2\right) < +\infty$, we find a constant $M > 0$ such that

$$\mathbb{E}\left(\sup_{0\leq t\leq T} \mid X_t^n \mid^2\right) \leq K(\alpha, T),$$

where

$$K(\alpha,T) = 4\Big(\mathbb{E}\left(\mid X_0 \mid^2\right) + (T+1+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1})(1+k)(1+M)T\Big).$$

□

The proof of theorem 3.3 can be given now:

Proof. **Existence**: Let equation (3.7).
Define $F_j^n = \int_0^t f_j(t,s,X_s^n)ds, \quad G_j^n = \int_0^t g_j(t,s,X_s^n)ds$ and
$\bar{F}_j^n = \int_0^t f_j(t,s,X_s^n - \frac{\bar{x}}{n})ds, \quad \bar{G}_j^n = \int_0^t g_j(t,s,X_s^n - \frac{\bar{x}}{n})ds$, for $j = 1, 2$.

Thank to Itô's formula, we get

$$\begin{aligned}\mathbb{E}\left(|X_t^m - X_t^n|^2\right) &= 2\mathbb{E}\int_0^t (X_s^m - X_s^n)(b_1^m(s, X_s^m, F_1^m) - b_1^n(s, X_s^n, F_1^n))ds \\ &+ 2\alpha\mathbb{E}\int_0^t (t-s)^{\alpha-1}(X_s^m - X_s^n)(b_2^m(s, X_s^m, F_2^m) - b_2^n(s, X_s^n, F_2^n))ds \\ &+ \mathbb{E}\int_0^t |\sigma_1^m(s, X_s^m, G_1^m) - \sigma_1^n(s, X_s^n, G_1^n)|^2 ds \\ &+ \mathbb{E}\int_0^t \int_E |\sigma_2^m(s, X_s^m, G_2^m, e) - \sigma_2^n(s, X_s^n, G_2^n, e)|^2 \pi(de)ds,\end{aligned}$$

using the definition of the sequences b_i^n, σ_i^n, $i = 1, 2$ and the notations

$$\Delta b_i^{m,n}(s) = b_i(s, X_s^m - \frac{\bar{x}}{m}, \bar{F}_i^m) - b_i(s, X_s^n - \frac{\bar{x}}{n}, \bar{F}_i^n),$$

$$\Delta\sigma_1^{m,n}(s) = \sigma_1(s, X_s^m - \frac{\bar{x}}{m}, \bar{G}_1^m) - \sigma_1(s, X_s^n - \frac{\bar{x}}{n}, \bar{G}_1^n)$$

$$\Delta\sigma_2^{m,n}(s, e) = \sigma_2(s, X_s^m - \frac{\bar{x}}{m}, \bar{G}_2^m, e) - \sigma_2(s, X_s^n - \frac{\bar{x}}{n}, \bar{G}_2^n, e),$$

we get

$$\begin{aligned}\mathbb{E}\left(|X_t^m - X_t^n|^2\right) &= \mathbb{E}\int_0^t \int_{\mathbb{R}^p} 2(X_s^m - X_s^n)\Delta b_1^{m,n}(s)J(\bar{x})d\bar{x}ds \\ &+ 2\alpha\mathbb{E}\int_0^t (t-s)^{\alpha-1}\int_{\mathbb{R}^p}(X_s^m - X_s^n)\Delta b_2^{m,n}(s)J(\bar{x})d\bar{x}ds \\ &+ \mathbb{E}\int_0^t \int_{\mathbb{R}^p} |\Delta\sigma_1^{m,n}(s)|^2 J(\bar{x})d\bar{x}ds \\ &+ \mathbb{E}\int_0^t \int_{\mathbb{R}^p}\int_E |\Delta\sigma_2^{m,n}(s, e)|^2\pi(de)J(\bar{x})d\bar{x}ds\end{aligned}$$

and by asumption **(A.3)**, we obtain

$$\begin{aligned}\mathbb{E}\left(|X_t^m - X_t^n|^2\right) \le\ & 2\mathbb{E}\int_0^t\int_{\mathbb{R}^p} c(s)\rho(|X_s^m - X_s^n - (m^{-1} - n^{-1})\bar{x}|^2)J(\bar{x})d\bar{x}ds \\ &+2\alpha\mathbb{E}\int_0^t (t-s)^{\alpha-1}\int_{\mathbb{R}^p} c(s)\rho(|X_s^m - X_s^n - (m^{-1} - n^{-1})\bar{x}|^2)J(\bar{x})d\bar{x}ds \\ &+ 2\alpha\mathbb{E}\int_0^t (t-s)^{\alpha-1}|m^{-1} - n^{-1}|\int_{\mathbb{R}^p} |\bar{x}|J(\bar{x})d\bar{x}ds \\ &+ 2\mathbb{E}\int_0^t \int_{\mathbb{R}^p} |(m^{-1} - n^{-1})\bar{x}|J(\bar{x})d\bar{x}ds,\end{aligned}$$

hence as $0 \le t \le T$,

$$\begin{aligned}\mathbb{E}\left(|X_t^m - X_t^n|^2\right) &\le C(T, \alpha)\int_0^t c(s)\int_{\mathbb{R}^p} \rho(\mathbb{E}|X_s^m - X_s^n - (m^{-1} - n^{-1})\bar{x}|^2)J(\bar{x})d\bar{x}ds \\ &+ C(T, \alpha, N)(m^{-1} + n^{-1}). \qquad (3.8)\end{aligned}$$

based on lemma 3.5, it holds, for all n, $\mathbb{E}(\sup_{t\leq T}|X_t^n|^2)\leq K(\alpha,T)$.
Then Fatou's lemma gives us:

$$\limsup_{m,n\to+\infty}\mathbb{E}\left(|X_t^m-X_t^n|^2\right)\leq C(T,\alpha)\int_0^t c(s)\rho_1(\limsup_{m,n\to+\infty}\mathbb{E}|X_s^m-X_s^n|^2)ds,$$

where $\rho_1(u)=\rho(u)+u$. Therefore

$$\limsup_{m,n\to+\infty}\mathbb{E}\left(|X_t^m-X_t^n|^2\right)=0.$$

We also deduce from (3.8) that:

$$\limsup_{m,n\to+\infty}\mathbb{E}\int_0^T|X_t^m-X_t^n|^2dt=0.$$

So there exist $(X_t)_{t\geq 0}\in\mathcal{M}^2_{[0,T]}(\mathbb{R}^p)$ verifying:

$$\lim_{n\to+\infty}\mathbb{E}\int_0^T|X_t^n-X_t|^2dt=0$$

and then for each $t\geq 0$, $\lim_{n\to+\infty}\mathbb{E}|X_t^n-X_t|^2=0$.
Therefore X_t^n converge to X_t in probability for each t, and it is possible to select a subsequence (n_k) denoted again (n) such that $\mathbb{P}-$ a.s as $n\longrightarrow+\infty$, $X_t^n\longrightarrow X_t^0$, $\forall t=r_k,\quad k=1,2,\cdots$; where $(r_k)_{k\geq 1}\subset[0,T]$ represents all rational numbers in $[0,T]$. Consequently, by Fatou's Lemma, we obtain

$$\mathbb{E}\left(\sup_{t\leq T}|X_t^0|^2\right)\leq\mathbb{E}\left(\sup_k\lim_{n\longrightarrow+\infty}|X_{r_k}^n|^2\right)\leq\liminf_{n\longrightarrow+\infty}\mathbb{E}\left(\sup_{t\leq T}|X_t^n|^2\right)\leq K(T,\alpha).\quad(3.9)$$

Also X_t is r.c.l.l. as the uniform limit of a sequence of r.c.l.l. functions.

For $0\leq t\leq T$, let us now prove that, when n goes to infinite,

$$\int_0^t\int_E\sigma_2^n\left(s,X_{s-}^n,G_2^n,e\right)\widetilde{N}(ds,de)\longrightarrow\int_0^t\int_E\sigma_2\left(s,X_{s-},G_2,e\right)\widetilde{N}(ds,de),\quad\mathbb{P}-a.s.$$

For all $n\in\mathbb{N}$, assume that $\sup_{t\leq T}\mid X_t^n\mid\leq k_0$ and $\sup_{t\leq T}\mid X_t\mid\leq k_0$. However, as $t\in[0,T]$, for $\varepsilon>0$, let

$$p_n=\mathbb{P}\left(\left|\int_0^t\int_E\sigma_2^n\left(s,X_{s-}^n,G_2^n,e\right)\widetilde{N}(ds,de)-\int_0^t\int_E\sigma_2\left(s,X_{s-},G_2,e\right)\widetilde{N}(ds,de)\right|>\varepsilon\right).$$

and

$$G_2^X=\int_0^s g_2(s,r,X)dr.$$

We can write

$$\begin{aligned}p_n\ &\leq\ \frac{1}{\varepsilon}\mathbb{E}\int_0^T\int_E\left|\sigma_2^n\left(s,X_s^n,G_2^n,e\right)-\sigma_2\left(s,X_s^n,G_2^n,e\right)\right|^2\pi(de)ds\\&+\ \frac{1}{\varepsilon}\mathbb{E}\int_0^T\int_E\left|\sigma_2\left(s,X_s^n,G_2^n,e\right)-\sigma_2\left(s,X_s,G_2,e\right)\right|^2\pi(de)ds,\end{aligned}$$

considering the supremum for all $| X |\leq k_0$, we get

$$\begin{aligned} p_n &\leq \frac{1}{\varepsilon}\int_0^T \sup_{|X|}\int_E \left|\sigma_2^n\left(s,X,G_2^X,e\right)-\sigma_2\left(s,X,G_2^X,e\right)\right|^2\pi(de)ds \\ &+ \frac{1}{\varepsilon}\int_0^T \lim_{h\to 0}\sup_{|X|}\int_E \left|\sigma_2\left(s,X+h,G_2^{X+h},e\right)-\sigma_2\left(s,X_s,G_2^X,e\right)\right|^2\pi(de)ds \\ &:= I_1^n + I_2^h. \end{aligned}$$

By lemma 3.4, we get

$$\sup_{|X|}\int_E \left|\sigma_2^n\left(s,X,G_2^X,e\right)-\sigma_2\left(s,X,G_2^X,e\right)\right|^2\pi(de)\to 0.$$

We can apply Lebesgue's dominated convergence theorem with the help of linear growth assumption on σ_2 and σ_2^n, to get

$$I_1^n \longrightarrow 0, \quad \text{as} \quad n\to+\infty$$

Now since

$$\lim_{h\to 0}\sup_{|X|}\int_E \left|\sigma_2\left(s,X+h,G_2^{X+h},e\right)-\sigma_2\left(s,X_s,G_2^X,e\right)\right|^2\pi(de)=0,$$

there exists a small $\delta>0$ verifying

$$\sup_{|X|\leq k_0,|h|\leq\delta}\int_E \left|\sigma_2\left(s,X+h,G_2^{X+h},e\right)-\sigma_2\left(s,X_s,G_2^X,e\right)\right|^2\pi(de)<\epsilon,$$

so, we have, for this δ

$$I_2^h=0.$$

Consequently $p_n\to 0$ and then

$$\int_0^t\!\!\int_E \sigma_2^n\left(s,X_{s-}^n,G_2^n,e\right)\widetilde{N}(ds,de) \;\longrightarrow\; \int_0^t\!\!\int_E \sigma_2\left(s,X_{s-},G_2,e\right)\widetilde{N}(ds,de), \quad \mathbb{P}-a.s.$$

Using similar calculations, we can prove that, as $n\to\infty$,

$$\begin{aligned} \int_0^t b_1^n(s,X_s^n,F_1^n)ds &\longrightarrow \int_0^t b_1(s,X_s,F_1)ds, \quad \mathbb{P}-a.s, \\ \int_0^t (t-s)^{\alpha-1}b_2^n(s,X_s^n,F_2^n)ds &\longrightarrow \int_0^t (t-s)^{\alpha-1}b_2(s,X_s,F_2)ds, \quad \mathbb{P}-a.s, \\ \int_0^t \sigma_1^n(s,X_s^n,G_1^n)dW_s &\longrightarrow \int_0^t \sigma_1(s,X_s,G_1)dW_s, \quad \mathbb{P}-a.s. \end{aligned}$$

Then $(X_t)_{0\leq t\leq T}$ is a solution of SFIDEJ (3.1).
Uniqueness: Consider $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be solutions processes through the

initial value $X_0 = Y_0$. We have

$$\begin{aligned} X_t - Y_t &= \int_0^t b_1(s, X_s, F_1^X)ds - \int_0^t b_1(s, Y_s, F_1^Y)ds \\ &+ \alpha \int_0^t (t-s)^{\alpha-1} b_2(s, X_s, F_2^X)ds - \alpha \int_0^t (t-s)^{\alpha-1} b_2(s, Y_s, F_2^Y)ds \\ &+ \int_0^t \sigma_1(s, X_s, G_1^X)dW_s - \int_0^t \sigma_1(s, Y_s, G_1^Y)dW_s \\ &+ \int_0^t \int_E \sigma_2(s, X_{s-}, G_1^X, e)\widetilde{N}(ds, de) - \int_0^t \int_E \sigma_2^n(s, X_{s-}, G_2^Y, e)\widetilde{N}(ds, de) \\ &:= \int_0^t \Delta^{X,Y} b_1(s)ds + \alpha \int_0^t (t-s)^{\alpha-1} \Delta^{X,Y} b_2(s)ds \\ &+ \int_0^t \Delta^{X,Y} \sigma_1(s)dW_s + \int_0^t \int_E \Delta^{X,Y} \sigma_2(s,e)\widetilde{N}(ds, de). \end{aligned}$$

Using Itô formula, we obtain

$$\begin{aligned} \mathbb{E}\left(|X_t - Y_t|^2\right) &= 2\mathbb{E}\int_0^t (X_s - Y_s)|\Delta^{X,Y} b_1(s)|ds \\ &+ 2\alpha\mathbb{E}\int_0^t (t-s)^{\alpha-1}(X_s - Y_s)|\Delta^{X,Y} b_2(s)|ds \\ &+ \mathbb{E}\int_0^t |\Delta^{X,Y}\sigma_1(s)|^2 ds + \mathbb{E}\int_0^t \int_E |\Delta^{X,Y}\sigma_2(s,e)|^2 \pi(e)ds \end{aligned}$$

and by assumption **(A.3)**, we have

$$\begin{aligned} \mathbb{E}\left(|X_t - Y_t|^2\right) &\leq \int_0^t \rho\left(\mathbb{E}|X_s - Y_s|^2\right) ds + 2\alpha \int_0^t (t-s)^{\alpha-1} \rho\left(\mathbb{E}|X_s - Y_s|^2\right) ds \\ &+ \int_0^t \rho\left(\mathbb{E}|X_s - Y_s|^2\right) ds + \int_0^t \rho\left(\mathbb{E}|X_s - Y_s|^2\right) ds \\ &\leq \int_0^t (3 + 2(t-s)^{\alpha-1})\rho\left(\mathbb{E}|X_s - Y_s|^2\right) ds. \end{aligned}$$

Bihary's inequality gives

$$\mathbb{E}\left(|X_t - Y_t|^2\right) = 0.$$

Hence X is a modification of Y and then uniqueness is satisfied.

□

4. An averaging principle

The purpose of this section is to perform an averaging calculation for a stochastic integrodifferential equation in $\mathbb{R}^p$:

$$\begin{aligned} X_t^\varepsilon &= X_0 + \frac{\varepsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} b\Big(s, X_{s-}^\varepsilon, \int_0^s f_1(s,u,X_u^\varepsilon)du\Big)ds \\ &+ \frac{\sqrt{\varepsilon}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\sigma_1\Big(s, X_{s-}^\varepsilon, \int_0^s g_1(s,u,X_u^\varepsilon\Big)dW_s \qquad (4.1) \\ &+ \frac{\sqrt{\varepsilon}}{\Gamma(\alpha)} \int_0^t \int_E (t-s)^{\alpha-1}\sigma_2\Big(s, X_{s-}^\varepsilon, \int_0^s g_2(s,u,X_u^\varepsilon)du, e\Big)\widetilde{N}(ds,de), \end{aligned}$$

where X_0 is a random vector satisfying $\mathbb{E}|X_0|^2 < +\infty$ and ε is a positive small parameter belonging to $]0, \varepsilon_0]$ with ε_0 a fixed number.
We make the following assumptions
$(\mathbf{A.4})$: There exist measurable functions

$$\bar{b} : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p,\ \overline{\sigma}_1 : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p \times \mathbb{R}^m \quad and \quad \overline{\sigma}_2 : \mathbb{R}^p \times \mathbb{R}^p \times E \to \mathbb{R}^p$$

such that for all $t > 0$

$$\frac{1}{t^{2\alpha-1}}\int_0^t (t-s)^{2(\alpha-1)}|b(s,x,F_1(x)) - \bar{b}(x,F_1(x))|^2 ds \le \phi_1(t)(1+|x|^2+|F_1(x)|^2),$$

$$\frac{1}{t^{2\alpha-1}}\int_0^t (t-s)^{2(\alpha-1)}|\sigma_1(s,x,G_1(x)) - \overline{\sigma}_1(x,G_1(x))|^2 ds \le \phi_2(t)(1+|x|^2+|G_1(x)|^2),$$

$$\frac{1}{t^{2\alpha-1}}\int_0^t (t-s)^{2(\alpha-1)}\int_E|\sigma_2(s,x,G_2(x),e) - \overline{\sigma}_2(x,G_2(x),e)|^2 \pi(de) ds \le \phi_3(t)(1+|x|^2+|G_2(x)|^2),$$

where $\phi_i(t)$ are positive bounded function with

$$\lim_{t\to+\infty} \phi_i(t) = 0 \quad \text{for } i = 1,2,3 \ \text{ and } \ |F_1(x)|^2 + |G_1(x)|^2 + |G_2(x)|^2 \le k^2(1+|x|^2).$$

The solution of original equation (4.1) is well approximated, in the sense of mean square, by the solution of following equation

$$\begin{aligned} Z_t^\varepsilon &= X_0 + \frac{\varepsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\bar{b}(Z_{s-}^\varepsilon, F_1(Z_{s-}^\varepsilon))ds \\ &+ \frac{\sqrt{\varepsilon}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\overline{\sigma}_1(Z_{s-}^\varepsilon, G_1(Z_{s-}^\varepsilon))dW_s \\ &+ \frac{\sqrt{\varepsilon}}{\Gamma(\alpha)} \int_0^t \int_E (t-s)^{\alpha-1}\overline{\sigma}_2(Z_{s-}^\varepsilon, G_2(Z_{s-}^\varepsilon), e)\widetilde{N}(ds,de). \end{aligned}$$

Theorem 4.1. *Consider assumptions* $(\mathbf{A.1}) - (\mathbf{A.4})$ *are in force and* $\delta > 0$ *a arbitrarily small number. Then there exists* $L > 0$, $\varepsilon_1 \in]0, \varepsilon_0]$ *and* $\beta \in]0,1[$ *verifying for all* $\varepsilon \in]0, \varepsilon_1]$,

$$\sup_{0\le t\le L\varepsilon^{-\beta}} \mathbb{E}|X_t^\varepsilon - Z_t^\varepsilon|^2 \le \delta.$$

Proof. Let $T > 0$, for any $t \in [0,u] \subseteq [0,T]$ we have

$$\begin{aligned} X_t^\varepsilon - Z_t^\varepsilon &= \frac{\varepsilon}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[b(s, X_{s-}^\varepsilon, F_1(X_{s-}^\varepsilon)) - \overline{b}(Z_{s-}^\varepsilon, F_1(Z_{s-}^\varepsilon))\Big]ds \\ &+ \frac{\varepsilon}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[\sigma_1(s, X_{s-}^\varepsilon, G_1(X_{s-}^\varepsilon)) - \overline{\sigma}_1(Z_{s-}^\varepsilon, G_1(Z_{s-}^\varepsilon))\Big]dW_s \\ &+ \frac{\varepsilon}{\Gamma(\alpha)}\int_0^t\int_E (t-s)^{\alpha-1}\Big[\sigma_2(s, X_{s-}^\varepsilon, G_2(X_{s-}^\varepsilon), e) - \overline{\sigma}_2(Z_{s-}^\varepsilon, G_3(Z_{s-}^\varepsilon), e)\Big]\widetilde{N}(ds, de). \end{aligned}$$

By Itô formula, we have

$$\begin{aligned} \mathbb{E}|X_t^\varepsilon - Z_t^\varepsilon|^2 \leq\ & 2\frac{\varepsilon}{\Gamma(\alpha)}\mathbb{E}\int_0^t (t-s)^{\alpha-1}(X_s^\varepsilon - Z_s^\varepsilon)\Big[b(s, X_s^\varepsilon, F_1(X_s^\varepsilon)) - \overline{b}(Z_s^\varepsilon, F_1(Z_s^\varepsilon))\Big]ds \\ &+ \frac{\varepsilon}{\Gamma^2(\alpha)}\mathbb{E}\int_0^t (t-s)^{2(\alpha-1)}\Big|\sigma_1(s, X_s^\varepsilon, G_1(X_s^\varepsilon)) - \overline{\sigma}_1(Z_s^\varepsilon, G_1(Z_s^\varepsilon))\Big|^2 ds \\ &+ \frac{\varepsilon}{\Gamma^2(\alpha)}\mathbb{E}\int_0^t\int_E (t-s)^{2(\alpha-1)}\big|\sigma_2(s, X_s^\varepsilon, G_2(X_s^\varepsilon), e) - \overline{\sigma}_2(Z_s^\varepsilon, G_2(Z_s^\varepsilon), e)\big|^2\pi(de)ds \\ :=\ & J_1 + J_2 + J_3. \end{aligned}$$

For J_1, we get

$$\begin{aligned} J_1 &= 2\frac{\varepsilon}{\Gamma(\alpha)}\mathbb{E}\int_0^t (t-s)^{\alpha-1}(X_s^\varepsilon - Z_s^\varepsilon)\Big[b(s, X_s^\varepsilon, F_1(X_s^\varepsilon)) - \overline{b}(Z_s^\varepsilon, F_1(Z_s^\varepsilon))\Big]ds \\ &= 2\frac{\varepsilon}{\Gamma(\alpha)}\mathbb{E}\int_0^t (t-s)^{\alpha-1}(X_s^\varepsilon - Z_s^\varepsilon)\Big[b(s, X_s^\varepsilon, F_1(X_s^\varepsilon)) - b(s, Z_s^\varepsilon, F_1(Z_s^\varepsilon))\Big]ds \\ &+ 2\frac{\varepsilon}{\Gamma(\alpha)}\mathbb{E}\int_0^t (t-s)^{\alpha-1}(X_s^\varepsilon - Z_s^\varepsilon)\Big[b(s, Z_s^\varepsilon, F_1(Z_s^\varepsilon)) - \overline{b}(Z_s^\varepsilon, F_1(Z_s^\varepsilon))\Big]ds. \end{aligned}$$

Thanks to Cauchy-Schwarz inequality, hypotheses (**A.3**) and (**A.4**), we obtain

$$\begin{aligned} J_1 &\leq \frac{\varepsilon}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\rho(\mathbb{E}|X_s^\varepsilon - Z_s^\varepsilon|^2) \\ &+ \frac{\varepsilon}{\Gamma(\alpha)}\int_0^t \mathbb{E}|X_s^\varepsilon - Z_s^\varepsilon|^2 ds + \frac{\varepsilon}{\Gamma(\alpha)}t^{2\alpha-1}(1+k^2)\phi_1(t)(1+\mathbb{E}|Z_t^\varepsilon|^2). \end{aligned}$$

For J_2, we have

$$\begin{aligned} J_2 &= \frac{\varepsilon}{\Gamma^2(\alpha)}\mathbb{E}\int_0^t (t-s)^{2(\alpha-1)}\Big|\sigma_1(s, X_s^\varepsilon, G_1(X_s^\varepsilon)) - \overline{\sigma}_1(Z_s^\varepsilon, G_1(Z_s^\varepsilon))\Big|^2 ds \\ &= \frac{\varepsilon}{\Gamma^2(\alpha)}\mathbb{E}\int_0^t (t-s)^{2(\alpha-1)}\Big|\sigma_1(s, X_s^\varepsilon, G_1(X_s^\varepsilon)) - \sigma_1(s, Z_s^\varepsilon, G_1(Z_s^\varepsilon))\Big|^2 ds \\ &+ \frac{\varepsilon}{\Gamma^2(\alpha)}\mathbb{E}\int_0^t (t-s)^{2(\alpha-1)}\Big|\sigma_1(s, Z_s^\varepsilon, G_1(Z_s^\varepsilon)) - \overline{\sigma}_1(Z_s^\varepsilon, G_1(Z_s^\varepsilon))\Big|^2 ds. \end{aligned}$$

By using the assumptions (**A.3**) and (**A.4**), we have

$$J_2 \leq \frac{\varepsilon}{\Gamma^2(\alpha)}\int_0^t (t-s)^{\alpha-1}\rho(\mathbb{E}|X_s^\varepsilon - Z_s^\varepsilon|^2)ds + \frac{\varepsilon}{\Gamma(\alpha)}t^{2\alpha-1}(1+k^2)\phi_2(t)(1+\mathbb{E}|Z_t^\varepsilon|^2).$$

Similary, we get

$$J_3 \leq \frac{\varepsilon}{\Gamma^2(\alpha)} \int_0^t (t-s)^{\alpha-1} \rho(\mathbb{E}|X_s^\varepsilon - Z_s^\varepsilon|^2) ds + \frac{\varepsilon}{\Gamma(\alpha)} t^{2\alpha-1}(1+k^2)\phi_3(t)(1+\mathbb{E}|Z_t^\varepsilon|^2).$$

Finally, we obtain

$$\begin{aligned}
\mathbb{E}|X_t^\varepsilon - Z_t^\varepsilon|^2 \leq\ & \frac{\varepsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \rho(\mathbb{E}|X_s^\varepsilon - Z_s^\varepsilon|^2) ds \\
+\ & \frac{\varepsilon}{\Gamma(\alpha)} \int_0^t \mathbb{E}|X_s^\varepsilon - Z_s^\varepsilon|^2 ds + \frac{\varepsilon}{\Gamma(\alpha)} t^{2\alpha-1}(1+k^2)\phi_1(t)(1+\mathbb{E}|Z_t^\varepsilon|^2) \\
+ & \frac{\varepsilon}{\Gamma^2(\alpha)} \int_0^t (t-s)^{2(\alpha-1)} \rho(\mathbb{E}|X_s^\varepsilon - Z_s^\varepsilon|^2) ds + t^{2\alpha-1}(1+k^2)\phi_2(t)(1+\mathbb{E}|Z_t^\varepsilon|^2) \\
+ & \frac{\varepsilon}{\Gamma^2(\alpha)} \int_0^t (t-s)^{2(\alpha-1)} \rho(\mathbb{E}|X_s^\varepsilon - Z_s^\varepsilon|^2) ds + t^{2\alpha-1}(1+k^2)\phi_3(t)(1+\mathbb{E}|Z_t^\varepsilon|^2).
\end{aligned}$$

Noting that $\sup_{0\leq t\leq T} \mathbb{E}|Z_t^\varepsilon|^2$ is finite and $\phi_i(t), i = 1, 2, 3$ bounded, we have

$$\begin{aligned}
\mathbb{E}|X_t^\varepsilon - Z_t^\varepsilon|^2 \leq\ & \frac{\varepsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \rho(\mathbb{E}|X_s^\varepsilon - Z_s^\varepsilon|^2) ds \\
+\ & \frac{2\varepsilon}{\Gamma^2(\alpha)} \int_0^t (t-s)^{2(\alpha-1)} \rho(\mathbb{E}|X_s^\varepsilon - Z_s^\varepsilon|^2) ds \\
+\ & \frac{\varepsilon}{\Gamma(\alpha)} \int_0^t \mathbb{E}|X_s^\varepsilon - Z_s^\varepsilon|^2 ds + \frac{C_1\varepsilon}{\Gamma(\alpha)} t^{2\alpha-1} + \frac{C_2\varepsilon}{\Gamma^2(\alpha)} t^{2\alpha-1}.
\end{aligned}$$

Hence by Remark 3.2, we obtain

$$\begin{aligned}
\mathbb{E}|X_t^\varepsilon - Z_t^\varepsilon|^2 \leq\ & \frac{\varepsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (\mu + \lambda\mathbb{E}|X_s^\varepsilon - Z_s^\varepsilon|^2) ds \\
+\ & \frac{2\varepsilon}{\Gamma^2(\alpha)} \int_0^t (t-s)^{2(\alpha-1)} (\mu + \lambda\mathbb{E}|X_s^\varepsilon - Z_s^\varepsilon|^2) ds \\
+\ & \frac{\varepsilon}{\Gamma(\alpha)} \int_0^t \mathbb{E}|X_s^\varepsilon - Z_s^\varepsilon|^2 ds + \frac{C_1\varepsilon}{\Gamma(\alpha)} t^{2\alpha-1} + \frac{C_2\varepsilon}{\Gamma^2(\alpha)} t^{2\alpha-1}.
\end{aligned}$$

Then

$$\begin{aligned}
\mathbb{E}|X_t^\varepsilon - Z_t^\varepsilon|^2 \leq\ & \frac{\mu\varepsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + \frac{\lambda\varepsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathbb{E}|X_s^\varepsilon - Z_s^\varepsilon|^2 ds \\
+\ & \frac{2\mu\varepsilon}{\Gamma^2(\alpha)} \int_0^t (t-s)^{2(\alpha-1)} ds + \frac{2\lambda\varepsilon}{\Gamma^2(\alpha)} \int_0^t (t-s)^{2(\alpha-1)} \mathbb{E}|X_s^\varepsilon - Z_s^\varepsilon|^2 ds \\
+\ & \frac{\varepsilon}{\Gamma(\alpha)} \int_0^t \mathbb{E}|X_s^\varepsilon - Z_s^\varepsilon|^2 ds + \frac{C_1\varepsilon}{\Gamma(\alpha)} t^{2\alpha-1} + \frac{C_2\varepsilon}{\Gamma^2(\alpha)} t^{2\alpha-1} \\
\leq\ & \left(\frac{\mu\varepsilon}{2\Gamma(\alpha)} + \frac{2\mu\varepsilon}{\Gamma^2(\alpha)}\right) \int_0^t [(t-s)^{\alpha-1} + 1]^2 ds + \frac{C_1\varepsilon}{\Gamma(\alpha)} t^{2\alpha-1} + \frac{C_2\varepsilon}{\Gamma^2(\alpha)} t^{2\alpha-1} \\
+\ & \left(\frac{\lambda\varepsilon}{\Gamma(\alpha)} + \frac{2\lambda\varepsilon}{\Gamma^2(\alpha)} + \frac{\varepsilon}{\Gamma(\alpha)}\right) \int_0^t [(t-s)^{\alpha-1} + 1]^2 \mathbb{E}|X_s^\varepsilon - Z_s^\varepsilon|^2 ds.
\end{aligned}$$

By using Gronwall's lemma, we get

$$\sup_{0\le t\le u}\mathbb{E}|X_t^\varepsilon - Z_t^\varepsilon|^2 \le \left(\frac{C_1\varepsilon}{\Gamma(\alpha)}u^{2\alpha-1} + \frac{C_2\varepsilon}{\Gamma^2(\alpha)}u^{2\alpha-1} + (\frac{\mu\varepsilon}{2\Gamma(\alpha)} + \frac{2\mu\varepsilon}{\Gamma^2(\alpha)})(\frac{u^{2\alpha-1}}{2\alpha-1} + \frac{u^\alpha}{\alpha} + u)\right) \times \exp\left[(\frac{\mu\varepsilon}{2\Gamma(\alpha)} + \frac{2\mu\varepsilon}{\Gamma^2(\alpha)})(\frac{u^{2\alpha-1}}{2\alpha-1} + \frac{u^\alpha}{\alpha} + u)\right].$$

Consequently we can choose $\beta \in]0,1[$ and $L > 0$ satisfying for all $t \in [0, L\varepsilon^{-\beta}] \subseteq [0,T]$, we have

$$\sup_{0\le t\le L\varepsilon^{-\beta}}\mathbb{E}|X_t^\varepsilon - Z_t^\varepsilon|^2 \le C\varepsilon^{1-\beta},$$

where

$$C = \left(L^{2\alpha-1}\varepsilon^{2\beta(1-\alpha)}(\frac{C_1}{\Gamma(\alpha)} + \frac{C_2}{\Gamma^2(\alpha)}) + (\frac{\mu}{2\Gamma(\alpha)} + \frac{2\mu}{\Gamma^2(\alpha)})(\frac{L^{2\alpha-1}\varepsilon^{2\beta(1-\alpha)}}{2\alpha-1} + \frac{L^\alpha\varepsilon^{-\alpha\beta+\beta}}{\alpha} + L)\right) \times \exp\left[(\frac{\mu}{2\Gamma(\alpha)} + \frac{2\mu}{\Gamma^2(\alpha)})(\frac{L^{2\alpha-1}\varepsilon^{-2\alpha\beta+\beta+1}}{2\alpha-1} + \frac{L^\alpha\varepsilon^{-\alpha\beta+1}}{\alpha} + L\varepsilon^{1-\beta})\right].$$

Then, for any given number $\delta > 0$, we can take $\varepsilon \in]0, \varepsilon_1]$, verifying for all $t \in [0, L\varepsilon^{-\beta}] \subseteq [0,T]$,

$$\sup_{0\le t\le L\varepsilon^{-\beta}}\mathbb{E}|X_t^\varepsilon - Z_t^\varepsilon|^2 \le \delta.$$

□

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