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ON BREAK DETECTION IN VOLATILITY FUNCTION OF A $\tau-$DEPENDENT PROCESS

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Abstract. This paper investigates change point detection within financial time series volatility using the NonParametric AutoRegressive Conditionally Heteroscedastic (NPARCH) process. It introduces a classical CUSUM type change point test following the test procedure by [19]. In the absence of change (Null Hypothesis), the test statistic converges to a well-established distribution, simplifying the determination of the critical test value. Conversely, under the Alternative Hypothesis, the probability that the test statistic tends to infinity is one, indicating asymptotic power one. Simulation results are presented to confirm the efficacy of the proposed methodology.

1. Introduction

Detecting breakpoints is a topic of interest in financial time series analysis. Since [21], this topic has garnered significant attention in the literature and evolved into a crucial area within statistics, as evident in works such as [11, 5, 3, 6] and the related references. Particularly for ARCH and GARCH models, many studies have been devoted to the parametric case. [14] introduced two categories of tests for identifying volatility changes in a GARCH process. These tests involve procedures using squared model residuals and the likelihood ratio. [17] worked on identifying changes in the parameters in GARCH(1,1) models through the residual CUSUM test. [3] proposed a sequential monitoring scheme for detecting changes in the parameters of a GARCH(p, q) sequence. [23] examined the challenge of testing for parameter changes in ARMA-GARCH models. Additionally, in 2020, [23] investigated a sequential procedure for detecting changes in the parameters of ARMA-GARCH models, building upon the test procedure proposed by [3]. [13] developed a Cramer-von Mises-type test for a change-point detection in residuals of an ARCH model. The alternative nonparametric methods also have been proposed to address this problem. [19] examined a nonparametric model and proposed a protocol for identifying the break in volatility function. [20] used the squared residuals in developing a test statistic for unknown abrupt change point detection in volatility of the exchange rate returns.

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The literature on change-points is vast. For more details in parametric or nonparametric approaches, one can, for example, see [8, 10, 1, 4, 12, 2] and the references therein. Recently, [16] have developed a new protocol for identifying a break in the conditional mean of a NonParametric AutoRegressive Conditionally Heteroscedastic (NPARCH) process.

In the present work, we propose to extend the results of [16] to the detection of a single break in the volatility function of a NPARCH process. To begin with, we establish the asymptotic normality of the test statistic in the absence of break. Next, we show that the estimator of the regression function remains convergent despite the presence of the volatility break in the process. We conclude the results with the convergence of our test procedure.

This paper is also an extension of the results of [19] to a $\tau-$mixing processes. The study of [19] was carried out in the $\alpha-$mixing case. [7] showed the difference between these two types of mixings. Our methods of presenting our results differ from theirs.

This paper is articulated as follows. Section 2 presents the problem. The solutions and their proofs are presented in Section 3. They are illustrated through a simulation described in Section 4. In the Appendix, we describe and demonstrate the secondary results.

2. Problem statement

In this paper, we will use the same notations as those of [16]. We observe a sample $\{(X_1, Y_1), \cdots, (X_T, Y_t)\}$ of $(X_t, Y_t)_t$, where $(X_t, Y_t)_t$ is defined in Equation (1) of [15]. The aim of our work is to set up a statistical test to determine a possible break in the volatility function $M^2(\cdot)$. We will assume that the function $f(\cdot)$ has no such break. The statistical test is as follows:
Null Hypothesis (no break in volatility):

$$H_0 : Var(X_t|Y_t = \cdot) = M^2(\cdot),\ t = 1, \cdots, T;$$

Alternative Hypothesis (break in volatility at $[\theta_0 T]$, $0 < \theta_0 < 1$):

$$H_1 : Var(X_t|Y_t = \cdot) = \begin{cases} M^2(\cdot), & \text{if } t \leqslant [\theta_0 T]; \\ \widetilde{M}^2(\cdot), & \text{if } t > [\theta_0 T]; \end{cases}$$

$\widetilde{M}^2(\cdot)$ being a measurable function different from $M^2(\cdot)$.
The statistic test (is performed by [19]):

$$U_T = \sup_{0 \leqslant \theta \leqslant 1} \sup_{z \in \mathbb{R}^d} |\widetilde{U}_T(\theta, z)|,$$

where, for any $(\theta, z) \in [0, 1] \times \mathbb{R}^d$,

$$\widetilde{U}_T(\theta, z) = \frac{1}{\sqrt{T}} \sum_{t=1}^{[\theta T]} \left((X_t - \widehat{f}_T(Y_t))^2 - \widehat{M}_T^2(Y_t) \right) \prod_{i=1}^{d} \mathbb{I}(|X_{t-i}| \leqslant c_T) \prod_{i=1}^{d} \mathbb{I}(X_{t-i} \leqslant z_i); \tag{2.1}$$

and

- $(c_i)_i$ is a sequence of nonnegative numbers such that $c_T/T \longrightarrow \infty$ as $T \longrightarrow \infty$;

- $\widehat{f}_T(\cdot)$ is defined in Equation (3) of [15], it is the estimator of the regression function $f(\cdot)$;
- $\widehat{M}^2_T(\cdot)$ is defined in Equation (4) of [15], it is the estimator of the volatility function $M^2(\cdot)$.

The decision rule : at α level, the Null Hypothesis will be rejected if the value of the test statistic U_T is greater than the $(1-\alpha)-$quantile of the limiting distribution of this statistic. Otherwise, we accept it.

The choice of the kernel is made from the family of kernels satisfying condition:
(C): The kernel $\mu : \mathbb{R} \longrightarrow [0, +\infty[$ has compact support: $[-\lambda_\mu, \lambda_\mu]$, $\lambda_\mu > 0$.

Remark 2.1. The condition ***(C)*** is less restrictive than Hypothesis (A_2) of [16].

3. Main results

3.1. Asymptotic distribution in the absence of a rupture.

Theorem 3.1. *If:*

- *the functions $f(\cdot)$ and $M(\cdot)$ satisfy Assumption (C_1) of [15];*
- *the kernels $K(\cdot)$ and $\kappa(\cdot)$ satisfy condition **(C)**;*
- *the bandwidths H_T and B_T satisfy Assumption (C_3) of [15] and Assumption (A_4) of [16];*

then, under H_0, for T large enough we have:

$$\widetilde{U}_T(\theta, y) \xrightarrow{d} \varsigma W(\eta(y) \wedge \theta), \tag{3.1}$$

where $\varsigma^2 = \mathbb{E}\left(\xi_0^4\right)$, $(\theta, y) \in [0,1] \times \mathbb{R}^d$, $\eta(y)$ and $W(\eta(y) \wedge \theta)$ are defined in Theorem 2 of [16].

Proof. Let $(\theta, y) \in [0,1] \times \mathbb{R}^d$, according to Lemma A.1, for T big enough,

$$\begin{aligned}
\widetilde{U}_T(y,\theta) &= \frac{1}{\sqrt{T}} \sum_{t=1}^{[\theta T]} \left(\left(X_t - \widehat{f}_T(Y_t) \right)^2 - \widehat{M}^2_T(Y_t) \right) \prod_{j=1}^{d} \mathbb{I}(X_{t-j} \leqslant y_j) + o_p(1) \\
&= \frac{1}{\sqrt{T}} \sum_{t=1}^{[\theta T] \wedge [\eta(y)T]} \left(\left(X_{\nu(t)} - \widehat{f}_T(Y_{\nu(t)}) \right)^2 - \widehat{M}^2_T(Y_{\nu(t)}) \right) + o_p(1) \\
&= \frac{1}{\sqrt{T}} \sum_{t=1}^{[(\theta \wedge \eta(y))T]} \left(\left(X_{\nu(t)} - \widehat{f}_T(Y_{\nu(t)}) \right)^2 - \widehat{M}^2_T(Y_{\nu(t)}) \right) + o_p(1) \\
&= \frac{1}{\sqrt{T}} \sum_{t=1}^{[(\theta \wedge \eta(y))T]} \left(\left(\xi^2_{\nu(t)} - 1 \right) M^2(Y_{\nu(t)}) + o(1) \right) + o_p(1).
\end{aligned} \tag{3.2}$$

The second equality is due to [16] (see pages 12 and 13 of [16]). The third is due to the asymptotic equivalence between $[\theta T] \wedge [\eta(y)T]$ and $[(\theta \wedge \eta(y))T]$. The fourth equality is due to Lemma 1 and Lemma 2 of [15] and to the definition of the process (see Equation (1) of [15]). The function $\nu(\cdot)$ is defined on page 9 of

[16].
Consequently, we obtain, for T large

$$\begin{aligned}\frac{1}{\sqrt{T}}\widetilde{U}_T(y,\theta) &= \frac{1}{T}\sum_{t=1}^{[(\theta\wedge\eta(y))T]}\Big(\left(\xi^2_{\nu(t)}-1\right)\left(M^2(Y_{\nu(t)})-1\right)\Big)\\ &+ \frac{1}{T}\sum_{t=1}^{[(\theta\wedge\eta(y))T]}\left(\xi^2_{\nu(t)}-1\right)+o_p(1). \qquad (3.3)\end{aligned}$$

We show (see the proof of Equation (23) of [15]):

- $\sup_t Var\Big(\left(\xi^2_{\nu(t)}-1\right)\left(M^2(Y_{\nu(t)})-1\right)\Big)<\infty$;
- for $i\neq j$,
$$Cov\Big(\left(\xi^2_{\nu(i)}-1\right)\left(M^2(Y_{\nu(i)})-1\right),\left(\xi^2_{\nu(j)}-1\right)\left(M^2(Y_{\nu(j)})-1\right)\Big)=0;$$
- $\displaystyle\frac{1}{[(\theta\wedge\eta(y))T]}\sum_{t=1}^{[(\theta\wedge\eta(y))T]}\mathbb{E}\Big(\left(\xi^2_{\nu(t)}-1\right)\left(M^2(Y_{\nu(t)})-1\right)\Big)=0.$

By [22], we have almost surely $(a.s)$ for T large

$$\frac{1}{[(\theta\wedge\eta(y))T]}\sum_{t=1}^{[(\theta\wedge\eta(y))T]}\left(\xi^2_{\nu(t)}-1\right)\left(M^2(Y_{\nu(t)})-1\right)=o(1). \qquad (3.4)$$

So, we have $a.s.$, for T big enough

$$\frac{[(\theta\wedge\eta(y))T]}{T}\frac{1}{[(\theta\wedge\eta(y))T]}\sum_{t=1}^{[(\theta\wedge\eta(y))T]}\left(\xi^2_{\nu(t)}-1\right)\left(M^2(Y_{\nu(t)})-1\right)=o(1).$$

Back to (3.3), we get $a.s.$, for T big enough

$$\frac{1}{\sqrt{T}}\widetilde{U}_T(y,\theta)=\frac{1}{T}\sum_{t=1}^{[(\theta\wedge\eta(y))T]}\left(\xi^2_{\nu(t)}-1\right)+o_p(1).$$

So the asymptotic laws of $\widetilde{U}_T(\theta,y)$ and of $\displaystyle\frac{1}{\sqrt{T}}\sum_{t=1}^{[(\theta\wedge\eta(y))T]}\left(\xi^2_{\nu(t)}-1\right)$ are the same for T large. By applying Donsker's principle to $\displaystyle\frac{1}{\sqrt{T}}\sum_{t=1}^{[(\theta\wedge\eta(y))T]}\left(\xi^2_{\nu(t)}-1\right)$, we obtain Theorem 3.1. □

Applying the Continuous Mapping Theorem (see [18]) to (3.1), we obtain the following theorem.

Theorem 3.2. *Under the assumptions of Theorem 3.1, for T large, we have*

$$U_T \xrightarrow{d} \varsigma\sup_{0\leqslant\theta\leqslant 1}\sup_{y\in\mathbb{R}^d}|W(\eta(y)\wedge\theta)|.$$

3.2. **Convergence of the test procedure.** In this section, we will assume that $d = 1$, i.e.

$$X_t = f(X_{t-1}) + M(X_{t-1})\xi_{t-1},\ t \in \mathbb{Z}. \tag{3.5}$$

Based on the sample $(X_0, \cdots, X_T)$, we study the stability of the volatility function of $(X_t)_{t\in\mathbb{Z}}$ defined in (3.5) by considering the Null Hypothesis

$$H_0 : Var\,(X_t/X_{t-1} = \cdot) = M_0^2(\cdot),\ t = 1, \cdots, T;$$

against the one change alternative

$$H_1 : Var\,(X_t/X_{t-1} = \cdot) = \begin{cases} M_0^2(\cdot), & t \leqslant [\theta_0 T] \\ M_1^2(\cdot), & t \geqslant [\theta_0 T] + 1 \end{cases},\ \theta_0 \in]0, 1[. \tag{3.6}$$

$M_0^2(\cdot)$ and $M_1^2(\cdot)$ are two different measurable functions.

Remark 3.3. If $f(\cdot)$, $M_0(\cdot)$ and $M_1(\cdot)$ each satisfy Assumption (C_1) of [15] then the processes $\left(X_{[\theta_0 T]-|t|}, t \in \mathbb{Z}\right)$ and $\left(X_{[\theta_0 T]+1-|t|}, t \in \mathbb{Z}\right)$ are strictly stationary and ergodic. Under these conditions, any measurable transformation of either of these processes is strictly stationary and ergodic.

The authors in [15] showed that the convergence of $\widehat{f}_T(\cdot)$. A legitimate question that could now arise is whether this estimator remains convergent after the break in volatility. The answer to such a concern is given in the following theorem.

Theorem 3.4. *If:*

- *$f(\cdot)$ satisfies Assumption (C_1) of [15];*
- *the kernel $K(\cdot)$ satisfies condition **(C)**;*
- *the bandwidth H_T satisfies Assumption (C_3) of [15] and Assumption (A_4) of [16];*

then, under H_1, for T big enough we have

$$\widehat{f}_T(y) = f(y) + o_p(1).$$

Proof. According to Equation (8) of [15] and Lemma 3 of [15], we have *a.s.*, for T big enough

$$
\begin{aligned}
\left|\widehat{f}_T(y) - f(y)\right| &\leqslant \left| \frac{\displaystyle\sum_{t=1}^{T} K_t(y) M(Y_t)\xi_t}{\displaystyle\sum_{t=1}^{T} K_t(y)} \right| + o(1) \\
&= \left| \frac{\displaystyle\frac{1}{T}\sum_{t=1}^{T} K_t(y) M(Y_t)\xi_t}{\displaystyle\frac{1}{T}\sum_{t=1}^{T} K_t(y)} \right| + o(1) \\
&\leqslant \frac{\left|\displaystyle\frac{1}{T}\sum_{t=1}^{[\theta_0 T]} K_t(y) M_0(X_{t-1})\xi_t + \frac{1}{T}\sum_{t=[\theta_0 T]+2}^{T} K_t(y) M_1(X_{t-1})\xi_t\right|}{\displaystyle\frac{1}{T}\sum_{t=1}^{[\theta_0 T]+1} K_t(y) + \frac{1}{T}\sum_{t=[\theta_0 T]+2}^{T} K_t(y)} \\
&\quad + \frac{\left|\displaystyle\frac{1}{T} K_{[\theta_0 T]+1}(y) M_1(X_{[\theta_0 T]})\xi_{[\theta_0 T]+1}\right|}{\displaystyle\frac{1}{T}\sum_{t=1}^{[\theta_0 T]+1} K_t(y) + \frac{1}{T}\sum_{t=[\theta_0 T]+2}^{T} K_t(y)} + o(1). \qquad (3.7)
\end{aligned}
$$

By Remark 3.3, the processes $\Big(K_t(y),\ 1 \leqslant t \leqslant [\theta_0 T]+1\Big)$ and $\Big(K_t(y),\ [\theta_0 T]+2 \leqslant t \leqslant T\Big)$ are strictly stationary and ergodic. So, we have *a.s.*, for large T

$$
\frac{1}{T}\sum_{t=1}^{[\theta_0 T]+1} K_t(y) + \frac{1}{T}\sum_{t=[\theta_0 T]+2}^{T} K_t(y) = \theta_0 \mathbb{E}\left(K_1(y)\right) + (1-\theta_0)\mathbb{E}\left(K_T(y)\right) + o(1). \qquad (3.8)
$$

By [22], for T big enough, we have *a.s.*,

$$
\frac{1}{T}\sum_{t=1}^{[\theta_0 T]+1} K_t(y) M_0(X_{t-1})\xi_t = o(1) \qquad (3.9)
$$

and

$$
\frac{1}{T}\sum_{t=[\theta_0 T]+2}^{T} K_t(y) M_1(X_{t-1})\xi_t = o(1). \qquad (3.10)
$$

(See the proof of Equation (23) of [15])
According to [15] (see pages 3311 and 3312 of [15]), we have

$$Var\left(\frac{1}{T}K_{[\theta_0 T]+1}(y)M_1(X_{[\theta_0 T]})\xi_{[\theta_0 T]+1}\right) = \frac{1}{T^2}\mathbb{E}\left(K^2_{[\theta_0 T]+1}(y)M_1^2(X_{[\theta_0 T]})\right).$$

According to Remark 3.3, for any T, we have

$$\mathbb{E}\left(K^2_{[\theta_0 T]+1}(y)M_1^2(X_{[\theta_0 T]})\right) < \infty.$$

So, for T big enough

$$Var\left(\frac{1}{T}K_{[\theta_0 T]+1}(y)M_1(X_{[\theta_0 T]})\xi_{[\theta_0 T]+1}\right) = o(1). \tag{3.11}$$

Gathering (3.7)-(3.11), we get Theorem 3.4. □

The consistency of the test is stated as follows:

Theorem 3.5. *If:*

- *the functions $f(\cdot)$, $M_0(\cdot)$ and $M_1(\cdot)$ satisfy Assumption (C_1) of [15];*
- *the kernels $K(\cdot)$ and $\kappa(\cdot)$ satisfy condition **(C)**;*
- *the bandwidths H_T and B_T satisfy Assumption (C_3) of [15] and Assumption (A_4) of [16];*

then, under H_1, for T big enough we have

$$\mathbb{P}\left(U_T \longrightarrow \infty\right) = 1.$$

Proof. Let $\theta \in [0,1]$ be such that $\theta_0 T + 1 \leqslant \theta T$, where θ_0 is defined in (3.6). Let $y \in \mathbb{R}$. From Theorem 3.4 and Lemma A.2, for T big enough, we have

$$\begin{aligned}
\frac{1}{\sqrt{T}}\widetilde{U}_T(y,\theta) &= \frac{1}{T}\sum_{t=1}^{[\theta T]}\Big(\left(X_t - f(X_{t-1})\right)^2 - \big(r(\theta_0, X_{t-1}) - f^2(X_{t-1})\big)\Big)\mathbb{I}(X_{t-1} \leqslant y) \\
&+ o_p(1) \\
&= \frac{1}{T}\sum_{t=1}^{[\theta T]}\Big(\left(X_t - f(X_{t-1})\right)^2 + f^2(X_{t-1})\Big)\mathbb{I}(X_{t-1} \leqslant y) \\
&- \frac{1}{T}\sum_{t=1}^{[\theta T]} r(\theta_0, X_{t-1})\mathbb{I}(X_{t-1} \leqslant y) + o_p(1).
\end{aligned}$$

From Lemma A.3 and Lemma Lemma A.4, we get, for T big enough

$$\begin{aligned}
\frac{1}{\sqrt{T}}\widetilde{U}_T(y,\theta) &= \theta_0\Bigg(\mathbb{E}\Big(\left(M_0^2(X_0) + f^2(X_0)\right)\mathbb{I}(X_0 \leqslant y)\Big) - r(\theta_0, X_0)\mathbb{P}(X_0 \leqslant y)\Bigg) \\
&+ (\theta - \theta_0)\,\mathbb{E}\Big(\left(M_1^2(X_{T-1}) + f^2(X_{T-1})\right)\mathbb{I}(X_{T-1} \leqslant y)\Big) \\
&- (\theta - \theta_0)\, r(\theta_0, X_{T-1})\mathbb{P}(X_{T-1} \leqslant y) + o_p(1).
\end{aligned}$$

Let's put

$$\begin{aligned} s(\theta,\theta_0,y,X_0,X_{T-1}) &= \theta_0\Bigg(\mathbb{E}\Bigg(\left(M_0^2(X_0)+f^2(X_0)\right)\mathbb{I}(X_0\leqslant y)\Bigg)-r(\theta_0,X_0)\mathbb{P}(X_0\leqslant y)\Bigg)\\ &+ (\theta-\theta_0)\,\mathbb{E}\Bigg(\left(M_1^2(X_{T-1})+f^2(X_{T-1})\right)\mathbb{I}(X_{T-1}\leqslant y)\Bigg)\\ &- (\theta-\theta_0)\,r(\theta_0,X_{T-1})\mathbb{P}(X_{T-1}\leqslant y). \end{aligned}$$

Since $s(\theta,\theta_0,y,X_0,X_{T-1})\in\mathbb{R}$, we get for T large

$$\frac{\left|\widetilde{U}_T(y,\theta)\right|}{\sqrt{T}|s(\theta,\theta_0,y,X_0,X_{T-1})|}=1+o_p(1).$$

This gives us Theorem 3.5. □

4. Simulation study

We will now assess our test procedure empirically. The data will be generated using the following equation:

$$X_t=\xi_t M\left(X_{t-1}\right),\ t=1,\cdots,T; \tag{4.1}$$

where $(\xi_t)_{1\leqslant t\leqslant T}$ is a sequence of independent and identically distributed random variables with the distribution $N(0,1)$; $X_0=0.5$ and:

$$M(X_{t-1})=\begin{cases} M_0(X_{t-1}), & t\leqslant[\theta_0 T]\\ 0.3M_0(X_{t-1}), & t\geqslant[\theta_0 T]+1; \end{cases}$$

where $\theta_0\in\{0.2,0.5,0.8\}$ and

$$M_0: x\longmapsto 0.7+0.1|x|,\ x\in\mathbb{R}.$$

We choose:

- Epanechnikov kernel;
- $h_T=2.34\sigma_T T^{-1/5}$ as the bandwidth (see [10]), σ_T denotes the standard deviation of the sample;
- The function $x\longmapsto x^2$ as Orlicz function. (For detailed information on Orlicz functions, see [9].)

The results were obtained using R software. All the results were obtained after 100 replications.

Figure 1 is a graphic illustration of a break in volatility. The breakpoint is indicated in the picture by a vertical line. We can see the difference between the amplitudes of the variations on either side of this vertical line.

The simulations are carried out as follows. We artificially generate the data (from Equation (4.1)). We propose two sets of data: data with no break and data with a (single) break in volatility. For each type of data, we have 100, 200 and 300 observations. Regardless of the type of data and population size, we determine the empirical value of the test statistic (see Table 1) and the proportion of rejection of the Null Hypothesis (see Table 2). From the 300 observations of sample with no breaks, we determine the empirical distribution function, which we will

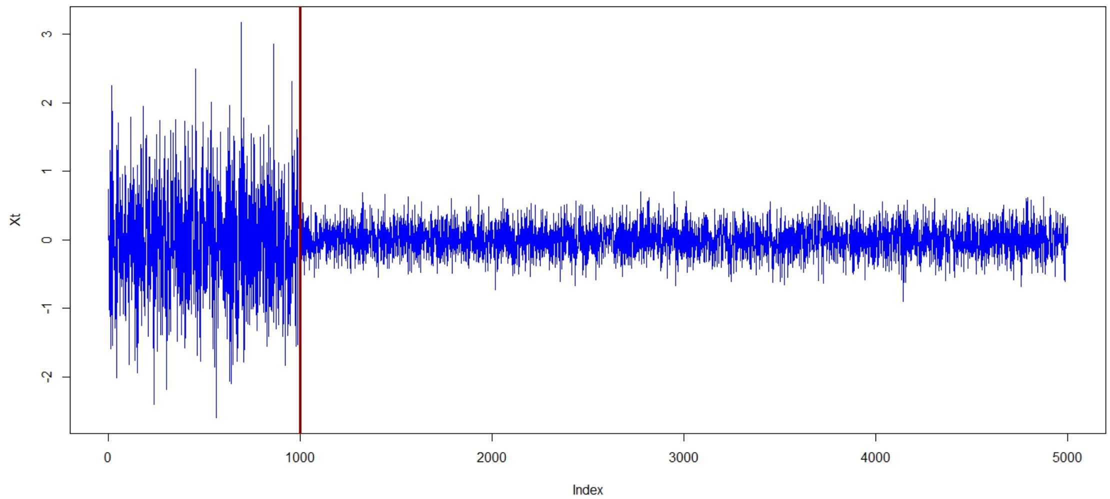

Figure 1. 5,000 observations of the process $(X_t,\ 1 \leqslant t \leqslant 5,000)$ defined in (4.1) with break in the volatility function at time 1,000.

call $\iota(\cdot)$, and some quantiles of $s_T \sup\limits_{0\leqslant\theta\leqslant 1} \sup\limits_{-1\leqslant y\leqslant 1} |W(\iota(y)\wedge\theta)|$, where s_T^2 denotes the fourth order empirical moment of white noise.

Table 1. Break detection test results of Model 4.1 at levels 10%, 5% and 1%

$Q(90\%)$	$Q(95\%)$	$Q(99\%)$	T	$\widehat{U_T}(0.2)$	$\widehat{U_T}(0.5)$	$\widehat{U_T}(0.8)$	$\overline{U_T}$
			100	4.7255	3.5081	1.9810	1.3153
3.1371	3.4926	3.7769	200	6.5403	4.1786	2.4131	1.8572
			300	6.6762	4.3492	3.9007	1.8888

In Table 1, $Q(1-\alpha)$ denotes the $(1-\alpha)-$quantile of $s_T \sup\limits_{0\leqslant\theta\leqslant 1} \sup\limits_{-1\leqslant y\leqslant 1} |W(\iota(y)\wedge\theta)|$ where $\alpha \in \{10\%, 5\%, 1\%\}$. The column $\widehat{U_T}(\theta_0)$ records the empirical values of the test statistic when the break occurs at time $[\theta_0 T]$, $\theta_0 \in \{0.2, 0.5, 0.8\}$. The column $\overline{U_T}$ contains the empirical values of the test statistic when there are no breaks in the sample.
We can see that all the numbers in the last column are smaller than the numbers in the first 3 columns. Also, almost all numbers in the $\widehat{U_T}(\theta_0)$ columns are all larger than the numbers in the first 3 columns (especially for large samples). These simulations therefore corroborate the decision rule (see page 3).

We then carried out tests on samples with a break (at time $[\theta_0 T]$, $\theta_0 \in \{0.2, 0.5, 0.8\}$) and on others with none. While varying the size of the samples, we noted the

Table 2. Proportion of rejections of the Null Hypothesis during detection tests carried out on the Model 4.1 at level α, $\alpha \in \{1\%, 5\%, 10\%\}$.

α	T \ θ_0	0.2	0.5	0.8	1
1%	100	1.00	0.02	0.00	0.00
	200	1.00	0.93	0.14	0.00
	300	1.00	1.00	0.37	0.00
	500	1.00	1.00	0.65	0.00
5%	100	1.00	0.38	0.00	0.00
	200	1.00	1.00	0.21	0.01
	300	1.00	1.00	0.38	0.03
	500	1.00	1.00	0.72	0.04
10%	100	1.00	0.72	0.00	0.00
	200	1.00	1.00	0.24	0.01
	300	1.00	1.00	0.53	0.06
	500	1.00	1.00	0.87	0.07

proportions of rejection of the Null Hypothesis in these two categories of samples. The results of this study are shown in Table 2. In this table, the column α designates the test level. The column $\theta_0 = 0.2$ designates the proportion of rejection of the Null Hypothesis after a test carried out on 100 samples of size T each presenting a break at time $[0.2T]$. The same applies to the columns $\theta_0 = 0.5$ and $\theta_0 = 0.8$. The column $\theta_0 = 1$ records the proportion of rejection of the Null Hypothesis after a test on 100 samples of size T with no break. In the Table 2, there are two cases for samples with a break. The Null Hypothesis is totally rejected for an early break. For late breaks, the percentage of rejection of the Null Hypothesis tends towards 100% as the sample size increases. In the absence of a break, this percentage would appear to increase towards the level of the test as the sample size increases.

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Appendix A. Secondary results

Lemma A.1. *Under the assumptions of Theorem 3.1, for T large, we have*

$$\mathbb{E}\left(\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{[\theta T]}\left((X_t-\widehat{f}_T(Y_t))^2-\widehat{M}_T^2(Y_t)\right)\prod_{i=1}^{d}\mathbb{I}(|X_{t-i}|>c_T)\prod_{i=1}^{d}\mathbb{I}(X_{t-i}\leqslant z_i)\right|\right)=o(1),$$

for any $\theta\in[0,1]$ and $z\in\mathbb{R}^d$.

Proof. Let $\theta\in[0,1]$ and $z\in\mathbb{R}^d$, we have

$$\begin{aligned}
&\mathbb{E}\left(\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{[\theta T]}\left((X_t-\widehat{f}_T(Y_t))^2-\widehat{M}_T^2(Y_t)\right)\prod_{i=1}^{d}\mathbb{I}(|X_{t-i}|>c_T)\prod_{i=1}^{d}\mathbb{I}(X_{t-i}\leqslant z_i)\right|\right)\\
\leqslant\ &\frac{1}{\sqrt{T}}\sum_{t=1}^{[\theta T]}\mathbb{E}\left(\left|(X_t-\widehat{f}_T(Y_t))^2-\widehat{M}_T^2(Y_t)\right|\prod_{i=1}^{d}\mathbb{I}(|X_{t-i}|>c_T)\right). \qquad \text{(A.1)}
\end{aligned}$$

From Hölder inequality, we have

$$\begin{aligned}
&\mathbb{E}\left(\left|(X_t-\widehat{f}_T(Y_t))^2-\widehat{M}_T^2(Y_t)\right|\prod_{i=1}^{d}\mathbb{I}(|X_{t-i}|>c_T)\right)\\
\leqslant\ &\left(\mathbb{E}\left(\left|(X_t-\widehat{f}_T(Y_t))^2-\widehat{M}_T^2(Y_t)\right|^2\right)\right)^{1/2}\left(\mathbb{E}\left(\prod_{i=1}^{d}\mathbb{I}(|X_{t-i}|>c_T)\right)\right)^{1/2} \qquad \text{(A.2)}
\end{aligned}$$

The process $(X_t)_t$ being strictly stationary and ergodic then the process $\left(\left|(X_t-\widehat{f}_T(Y_t))^2-\widehat{M}_T^2(Y_t)\right|^2\right)_t$ is stationary. This is because it is the measurable transformation of a strictly stationary and ergodic process. Consequently, there exists $a_1\in\mathbb{R}$ such that, for any t

$$\mathbb{E}\left(\left|(X_t-\widehat{f}_T(Y_t))^2-\widehat{M}_T^2(Y_t)\right|^2\right)=a_1.$$

We have also

$$\begin{aligned}
\mathbb{E}\left(\prod_{i=1}^{d}\mathbb{I}(|X_{t-i}|>c_T)\right)&\leqslant\mathbb{E}\left(\mathbb{I}(\|Y_t\|>c_T)\right)\\
&=\mathbb{P}\left(\|Y_t\|>c_T\right)\\
&\leqslant\frac{1}{c_T}\mathbb{E}\left(\|Y_t\|\right).
\end{aligned}$$

The process $(X_t)_t$ being strictly stationary and ergodic then the process $(\|Y_t\|)_t$ is stationary. Consequently, there exists $a_2\in\mathbb{R}$ such that, for any t

$$\mathbb{E}\left(\|Y_t\|\right)=a_2.$$

Back (A.2), we have

$$\mathbb{E}\left(\left|(X_t-\widehat{f}_T(Y_t))^2-\widehat{M}_T^2(Y_t)\right|\prod_{i=1}^{d}\mathbb{I}(|X_{t-i}|>c_T)\right)\leqslant\frac{1}{\sqrt{c_T}}\sqrt{a_1a_2}.$$

Consequently (A.1) becomes, for T large enough

$$\begin{aligned}
&\mathbb{E}\left(\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{[\theta T]}\left((X_t-\widehat{f}_T(Y_t))^2-\widehat{M}_T^2(Y_t)\right)\prod_{i=1}^{d}\mathbb{I}(|X_{t-i}|>c_T)\prod_{i=1}^{d}\mathbb{I}(X_{t-i}\leqslant z_i)\right|\right)\\
\leqslant\;& \frac{\sqrt{a_1a_2}[T\theta]}{\sqrt{T}\sqrt{c_T}}\\
=\;& \frac{\sqrt{a_1a_2}\sqrt{T}\theta}{\sqrt{c_T}}(1+o(1))\\
=\;& o(1).
\end{aligned}$$

The first equality is due to the asymptotic equivalence between $[T\theta]$ and $T\theta$. The last line is due to the definition of the sequence $(c_i)_i$. □

Lemma A.2. *Under the assumptions of Theorem 3.5, for T large, we have*

$$\widehat{M}_T^2(y)=r(\theta_0,y)-f^2(y)+o_p(1),$$

where

$$r(\theta_0,y)=\frac{\theta_0\mathbb{E}\left(\kappa_1(y)X_1^2\right)+(1-\theta_0)\mathbb{E}\left(\kappa_T(y)X_T^2\right)}{\theta_0\mathbb{E}\left(\kappa_1(y)\right)+(1-\theta_0)\mathbb{E}\left(\kappa_T(y)\right)}. \tag{A.3}$$

Proof. By [15] (*cf.* page 3298 of [15]) and Theorem 3.4, we have for T big enough

$$\widehat{M}_T^2(y)=\frac{\frac{1}{T}\sum_{t=1}^{T}\kappa_t(y)X_t^2}{\frac{1}{T}\sum_{t=1}^{T}\kappa_t(y)}-f^2(y)+o_p(1). \tag{A.4}$$

We have

$$\frac{1}{T}\sum_{t=1}^{T}\kappa_t(y)X_t^2=\frac{1}{T}\sum_{t=1}^{[\theta_0T]}\kappa_t(y)X_t^2+\frac{1}{T}\kappa_{[\theta_0T]+1}(y)X_{[\theta_0T]+1}^2+\frac{1}{T}\sum_{t=[\theta_0T]+2}^{T}\kappa_t(y)X_t^2.$$

We have

$$\mathbb{E}\left(\left|\frac{1}{T}\kappa_{[\theta_0T]+1}(y)X_{[\theta_0T]+1}^2\right|\right)\leqslant\frac{1}{T}\mathbb{E}\left(\left|\kappa_{[\theta_0T]+1}(y)X_{[\theta_0T]+1}^2\right|\right).$$

By Remark 3.3, we have $\mathbb{E}\left(\left|\kappa_{[\theta_0T]+1}(y)X_{[\theta_0T]+1}^2\right|\right)<\infty$. Therefore, we have, for T big enough

$$\mathbb{E}\left(\left|\frac{1}{T}\kappa_{[\theta_0T]+1}(y)X_{[\theta_0T]+1}^2\right|\right)=o(1).$$

Consequently, for T big enough, we have

$$\frac{\frac{1}{T}\sum_{t=1}^{T}\kappa_t(y)X_t^2}{\frac{1}{T}\sum_{t=1}^{T}\kappa_t(y)} = \frac{\frac{1}{T}\sum_{t=1}^{[\theta_0 T]}\kappa_t(y)X_t^2 + \frac{1}{T}\sum_{t=[\theta_0 T]+2}^{T}\kappa_t(y)X_t^2}{\frac{1}{T}\sum_{t=1}^{[\theta_0 T]+1}\kappa_t(y) + \frac{1}{T}\sum_{t=[\theta_0 T]+2}^{T}\kappa_t(y)} + o_{p(1)}.$$

Back to (A.4), we get Lemma A.2 using the same reasoning as in (3.8). □

Lemma A.3. *Under the assumptions of Theorem 3.5, for T large, we have a.s.*

$$\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{[\theta T]}\Big((X_t - f(X_{t-1}))^2 + f^2(X_{t-1})\Big)\mathbb{I}(X_{t-1}\leqslant y)\\
=\ & \theta_0\mathbb{E}\Big(\big(M_0^2(X_0) + f^2(X_0)\big)\,\mathbb{I}(X_0\leqslant y)\Big)\\
+\ & (\theta-\theta_0)\,\mathbb{E}\Big(\big(M_1^2(X_{T-1}) + f^2(X_{T-1})\big)\,\mathbb{I}(X_{T-1}\leqslant y)\Big) + o(1).
\end{aligned}$$

Proof. By Remark 3.3, we get *a.s.*, for T big enough

$$\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{[\theta_0 T]}\Big((X_t - f(X_{t-1}))^2 + f^2(X_{t-1})\Big)\mathbb{I}(X_{t-1}\leqslant y)\\
+\ & \frac{1}{T}\sum_{t=[\theta_0 T]+1}^{[\theta T]}\Big((X_t - f(X_{t-1}))^2 + f^2(X_{t-1})\Big)\mathbb{I}(X_{t-1}\leqslant y)\\
=\ & \theta_0\mathbb{E}\Big(\big((X_1 - f(X_0))^2 + f^2(X_0)\big)\,\mathbb{I}(X_0\leqslant y)\Big)\\
+\ & (\theta-\theta_0)\,\mathbb{E}\Big(\big((X_T - f(X_{T-1}))^2 + f^2(X_{T-1})\mathbb{I}(X_{T-1}\leqslant y)\big)\Big) + o(1)\\
=\ & \theta_0\mathbb{E}\Big(\big((\xi_0 M_0(X_0))^2 + f^2(X_0)\big)\,\mathbb{I}(X_0\leqslant y)\Big)\\
+\ & (\theta-\theta_0)\,\mathbb{E}\Big((\xi_{T-1}M_1(X_{T-1}))^2 + f^2(X_{T-1})\mathbb{I}(X_{T-1}\leqslant y)\Big) + o(1).
\end{aligned}$$

Since $M_1(X_0, f^2(X_0)$ and $\mathbb{I}(X_0\leqslant y)$ are measurable functions with respect to X_0 and ξ_0 is independent of X_0 then

$$\begin{aligned}
\mathbb{E}\Big(\big((\xi_0 M_0(X_0))^2 + f^2(X_0)\big)\,\mathbb{I}(X_0\leqslant y)\Big) &= \mathbb{E}\Big(\xi_0^2\Big)\mathbb{E}\Big(\big(M_0^2(X_0) + f^2(X_0)\big)\,\mathbb{I}(X_0\leqslant y)\Big)\\
&= \mathbb{E}\Big(\big(M_0^2(X_0) + f^2(X_0)\big)\,\mathbb{I}(X_0\leqslant y)\Big).
\end{aligned}$$

Similarly, we show

$$\mathbb{E}\Big((\xi_{T-1}M_1(X_{T-1}))^2 + f^2(X_{T-1})\mathbb{I}(X_{T-1}\leqslant y)\Big) = \mathbb{E}\Big(\big(M_1^2(X_{T-1}) + f^2(X_{T-1})\big)\,\mathbb{I}(X_{T-1}\leqslant y)\Big).$$

This completes the proof of Lemma A.3. □

Lemma A.4. *Under the assumptions of Theorem 3.5, for T large, we have*

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{[\theta T]} r(\theta_0, X_{t-1})\mathbb{I}(X_{t-1} \leqslant y) &= \theta_0 r(\theta_0, X_0)\mathbb{P}(X_0 \leqslant y)\\ &+ (\theta-\theta_0) r(\theta_0, X_{T-1})\mathbb{P}(X_{T-1} \leqslant y) + o(1),\end{aligned}$$

where $r(\theta_0, \cdot)$ is defined in (A.3).

Proof. We have $a.s.$, for T large

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{[\theta T]} r(\theta_0, X_{t-1})\mathbb{I}(X_{t-1} \leqslant y) &= \frac{1}{T}\bigg(\sum_{t=1}^{[\theta_0 T]} r(\theta_0, X_{t-1})\mathbb{I}(X_{t-1} \leqslant y)\\ &+ \sum_{t=[\theta_0 T]+1}^{[\theta T]} r(\theta_0, X_{t-1})\mathbb{I}(X_{t-1} \leqslant y)\bigg)\\ &= r(\theta_0, X_0)\frac{1}{T}\sum_{t=1}^{[\theta_0 T]}\mathbb{I}(X_{t-1} \leqslant y)\\ &+ r(\theta_0, X_T)\frac{1}{T}\sum_{t=[\theta_0 T]+1}^{[\theta T]}\mathbb{I}(X_{t-1} \leqslant y)\\ &= \theta_0 r(\theta_0, X_0)\mathbb{E}\Big(\mathbb{I}(X_0 \leqslant y)\Big)\\ &+ (\theta-\theta_0) r(\theta_0, X_{T-1})\mathbb{E}\Big(\mathbb{I}(X_{T-1} \leqslant y)\Big) + o(1)\\ &= \theta_0 r(\theta_0, X_0)\mathbb{P}(X_0 \leqslant y)\\ &+ (\theta-\theta_0) r(\theta_0, X_{T-1})\mathbb{P}(X_{T-1} \leqslant y) + o(1).\end{aligned}$$

The second equality is due to the Remark 3.3 □

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