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TOPOLOGICAL DUALS AND KÖTHE TOEPLITZ DUALS OF SOME DOUBLE SEQUENCE SPACES DEFINED BY A MODULUS FUNCTION

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Abstract. In the present paper, topological duals and Köthe-Toeplitz Duals of some double sequence spaces defined by a modulus function are studied. Also, we investigate perfectness, normality and monotonicity of some vector valued modulus sequences.

1. Introduction

Modulus Function was introduced by Nakano [1], in 1953, and used to solve some structural problems of the scalar FK-spaces theory. Ruckle [3] used the modulus function to construct the space $\mathcal{L}(f)$ of complex sequences and solve some problems encountered in theoretical investigations of classical FK-spaces by $\mathcal{L}(f)$. Later Maddox [5] introduced the class of sequences which are strongly Cesaro summable with respect to a modulus function. Since then a lot of sequence spaces have been defined by a modulus function for some special aims. Yılmaz [2] defined the sequence space $\lambda(X_k, r, f, s)$ by a modulus function and constructed its FK-structure under some conditions. Further, Yılmaz gave generalized Köthe-Toroeplitz duals of some important vector-valued modulus sequence spaces, and by these results, Yılmaz [4] outlined the basics of operator matrix transformations of modulus sequence spaces in the sense of Maddox [5].

Gökhan and Çolak [6], [7] have proved that $\mathcal{M}_u(t)$ and $C_p(t)$, $C_{bp}(t)$ are complete paranormed spaces of double sequences and given the $\alpha-, \beta-, \gamma-$duals of the spaces $\mathcal{M}_u(t)$ and $C_{bp}(t)$. Zeltser [8] had essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [9] introduced the statistical convergent and strongly Cesaro summable double sequences. The authors in [10] and[11] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem. They also have introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequence $x = (x_{j,k})$ into one whose core is a subset of the $M-$core of x.

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Altay and Basar [12] have defined the spaces $BS, BS(t), CS_p, CS_{bp}, CS_r$ and BV of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), C_p, C_{bp}, C_r$ and $\mathcal{L}_u$, respectively. They also examined some properties of those sequence spaces and determined the $\alpha-$duals of the spaces BS, CS_{bp}, BV and the $\beta(\nu)-$duals of the spaces CS_{bp}, CS_r of double series. In this study ν is a kind of convergence for double sequence.

In this study for simplicity in notation, here and after we abbreviate the summations $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}$ and $\sum_{i=0}^{m}\sum_{j=0}^{n}$ by $\sum_{i,j}, \sum_{i,j=0}^{m,n}$ respectively.

We denote the set of all real or complex valued double sequences by Ω which is a vector space with coordinatewise addition and scalar multiplication. Any vector subspace of Ω is called as a double sequence space. A double sequence is bounded if

$$\|x\| = \sup_{m,n\geq 0} |x_{mn}| < \infty.$$

The space $\mathcal{M}_u$ of all bounded double sequences is defined by

$$\mathcal{M}_u = \left\{ x = (x_{mn}) \in \Omega : \|x\|_{\infty} = \sup_{m,n\in\mathbb{N}} |x_{mn}| < \infty \right\},$$

which is a Banach space with the norm given by

$$\|x\|_{\infty} = \sup_{m,n\in\mathbb{N}} |x_{mn}| < \infty,$$

where $\mathbb{N} = \{0, 1, 2, ...\}$. The space $\mathcal{L}_u$ is defined as

$$\mathcal{L}_u = \left\{ x = (x_{mn}) \in \Omega : \sum_{m,n} |x_{mn}| < \infty \right\}$$

and it is a Banach space with the norm

$$\|x\|_1 = \sum_{m,n} |x_{mn}|.$$

Let us consider the sequence $x = (x_{mn}) \in \Omega$. If for all $\varepsilon > 0$, there exist $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $l \in \mathbb{C}$ such that

$$|x_{mn} - l| < \varepsilon,$$

for all $m, n > n_0$, then it is said that the double sequence x is convergent in the Pringsheim's sense to the limit l.Then we write

$$p - \lim x_{mn} = l,$$

where $\mathbb{C}$ denotes the complex field. By C_p, we denote the space of all convergent double sequences in the Pringsheim's sense. It is well known that there are such sequences in the space C_p but not in the space $\mathcal{M}_u$. So we may mention the space C_{bp} of the double sequences which are both convergent in the Pringsheim's sense and bounded, i.e. $C_{bp} = C_p \cap \mathcal{M}_u$. Moricz [13] proved that C_{bp} is a Banach space with the norm $\|, \|_{\infty}$. A sequence in the space C_p is said to be regularly convergent, if it is a single convergent sequence with respect to each index .It is denoted by C_r.

Now, denote the space of all X-valued double sequences by $\Omega(X)$, where X is a Banach space. The topology of (X) is locally convex topology produced by the family of seminorms, $p_{ij}(x) = \|x_{ij}\|$. This topology is metrizable and total paranorm giving this metric is

$$g(x) = \sum_{i,j} \frac{1}{2^{i+j}} \frac{\|x_{ij}\|}{1+\|x_{ij}\|}.$$

A modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such that

(1) $f(x) = 0$ if and only if $x = 0$,

(2) $f(x+y) \leq f(x) + f(y)$, for all $x \geq 0, y \geq 0$,

(3) f is increasing,

(4) f is continuous from the right at 0. Since $\mid f(x) - f(y) \mid \leq f(\mid x-y \mid)$, it follows from condition (4) that f is continuous on $[0,\infty)$.

Let f be a modulus function and Ω is the space of all scalar double sequences. We present some double sequence spaces in the following:

$$\begin{aligned}
\mathcal{L}_u(f) &= \{x \in \Omega : \sum_{m,n} f(|x_{mn}|) < \infty\}, \\
\mathcal{L}_u(X) &= \{x = (x_{mn}) \in \Omega(X) : \sum_{m,n} \|x_{mn}\| < \infty\}, \\
\mathcal{M}_u(X) &= \{x = (x_{mn}) \in \Omega(X) : \sup_{m,n} |x_{mn}| < \infty\}, \\
\Phi(X) &= \{x = (x_{mn}) \in \Omega(X) : \exists k_0 \ni \text{ for } m \geq k_0 \text{ or } n \geq k_0 \quad x_{mn} = 0\}, \\
\mathcal{L}_u(X,f) &= \{x \in \Omega(X) : \sum_{m,n} f(\|x_{mn}\|) < \infty\},
\end{aligned}$$

where $\|,\|$ is the norm of X. Further X' denotes the continuous dual of X. α-dual and $\beta(\nu)$-dual of some double sequence space E are given by

$$\begin{aligned}
E^{\alpha} &= \{(a_{ij}) \in \Omega : \ \sum |a_{ij}x_{ij}| < \infty, \text{ for all } x \in E\}, \\
E^{\beta(\nu)} &= \{(a_{ij}) \in \Omega : \quad \nu - \sum a_{ij}x_{ij} \text{ exists for all } x \in E \}.
\end{aligned}$$

Now we will give some definitions for vector valued double sequence spaces. Let E be a non-empty subset of $\Omega(X)$. Define $\alpha-$dual of E by

$$E^{\alpha} = \{(\rho_{nk}) \in \Omega\left(X'\right) : \text{ for all } x \in E, \sum |\rho_{nk}(x_{nk})| < \infty\},$$

and $\beta(\nu)-$dual of E by

$$E^{\beta(\nu)} = \{(\rho_{nk}) \in \Omega\left(X'\right) : \text{for all } x \in E, \ \nu - \sum \rho_{nk}(x_{nk}) \, \text{exists}\}.$$

In this study "$\nu - \sum_{k,l} x_{kl}$ exists" means that $\nu - \lim_{m,n} \sum_{k=1}^{m} \sum_{l=1}^{n} x_{kl}$ exists.

Note that, always $E^{\alpha} \subset E^{\beta(\nu)}$ and if $E \subseteq F \subseteq \Omega(X)$, then $E^{\delta} \supseteq F^{\delta}$, where $\delta \in \{\alpha, \beta(\nu)\}$. Let $E^{\alpha\alpha} = (E^{\alpha})^{\alpha}$. Then E is called perfect whenever $E = E^{\alpha\alpha}$.

Also, $E^{\alpha} \subset \Omega\left(X'\right)$ and hence $E^{\alpha\alpha} \subset \Omega\left(X''\right)$. If X is a normed space, X' and X'' are Banach spaces and for each $x \in X$, the function given by

$$\begin{aligned} \widehat{x} &: X' \to \mathbb{C} \\ \widehat{x}(f) &= f(x) \end{aligned}$$

is linear and continuous. Then, the transformation defined by

$$\begin{aligned} k &: X \to X'' \\ x &\to k(x) = \widehat{x}, \end{aligned}$$

is called embedding of X into X''. If k is onto then X is called reflexive. A reflexive normed space is complete, so it is a Banach space. We know that $\lambda \subset \lambda''$, for some scalar sequence space λ. Similarly, for some $E \subset \Omega(X)$, $E \subset E^{\delta\delta}$, $(\delta = \alpha, \beta(\nu))$, providing that X is reflexive normed space. Further $E^{\delta} = E^{\delta\delta\delta}$, in this case.

Let $E \subset \Omega(X)$ be a double sequence space and $x = (x_{nk}) \in E$. E is called normal, if for each $(\rho_{nk}) \in \Omega(X')$,

$$\left|\rho_{nk}(y_{nk})\right| \leq \left|\rho_{nk}(x_{nk})\right|,$$

implies $y = (y_{nk}) \in E$.

If λ is a normal sequence space then $\lambda(X) \subset \Omega(X)$ is normal. E is called monotone, if for each $x = (x_{kl}) \in E$ and $y = (y_{ij}) \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}$,

$$x.y = (x_{kl}.y_{ij}) \in E.$$

Note that normality of E implies its monotonicity.

2. Dual Spaces

Lemma 2.1. [4]*Let A be a set and (X, P) be a locally convex space. Then for all $x \in l(A, X)$*

$$x = \sum_{a\in\Lambda} (I_a \circ x)(a),$$

where P is a family of seminorms on X and

$$\begin{aligned} I_a &: X \to l(A, X) \\ b &\neq a \Rightarrow y(b) = 0, y(a) = t \text{ and } I_a(t) = y. \end{aligned}$$

Theorem 2.2. *If X is a normed space and $f \in \mathcal{L}_u(X)$, then f is represented by*

$$f(x) = \sum_{i,j} \Psi_{ij}(x_{ij}), \qquad \Psi \in \mathcal{M}_u(X'),$$

and hence

$$\mathcal{L}_u(X)' = \mathcal{M}_u(X').$$

Proof. From Lemma 2.1, $(S_F(x) : F \in \mathcal{F})$ is the net of finite summations and it converges to x, where

$$x = \sum_{i,j\in\mathbb{N}\times\mathbb{N}} I_{ij}(x_{ij}) \in \mathcal{L}_u(X).$$

If $f \in \mathcal{L}_u(X)'$, then we get

$$f(x) = \sum_{i,j \in \mathbb{N}\times\mathbb{N}} (f \circ I_{ij})(x_{ij}),$$

by the continuity of f. Let us define

$$\begin{aligned} \Psi &: \mathbb{N}\times\mathbb{N} \to X' \\ \Psi_{ij} &= f \circ I_{ij}. \end{aligned}$$

Then, for each $(i,j) \in \mathbb{N}\times\mathbb{N}$,

$$\|\Psi(i,j)\| \leq \|f\| \|I_{ij}\| = \|f\| < \infty,$$

and so $\|\Psi\| = Sup\{\|\Psi(i,j)\| : (i,j) \in \mathbb{N}\times\mathbb{N}\} \leq \|f\|$. Hence we obtain $\Psi \in \mathcal{M}_u(X')$. Further,

$$\begin{aligned} |f(x)| &\leq \left\|\sum_{i,j} \Psi_{ij}(x_{ij})\right\|, \\ &\leq \sum_{i,j} \|\Psi_{ij}\| \|(x_{ij})\|, \\ &\leq \|\Psi\| . \|x\|_u, \end{aligned}$$

which gives $\|f\| \leq \|\Psi\|$. So, we get $\|\Psi\| = \|f\|$.

We can define an isometry T such that

$$\begin{aligned} T &: \mathcal{L}_u(X)' \to \mathcal{M}_u(X') \\ Tf &= \Psi, \end{aligned}$$

for each $\Psi \in \mathcal{M}_u(X')$ .Also consider a linear functional f on $\mathcal{L}_u(X)$ given by

$$f(x) = \sum_{i,j \in \mathbb{N}\times\mathbb{N}} \Psi_{ij}(x_{ij}).$$

In this case $Tf = \Psi$. Since $\Psi_{ij} = f \circ I_{ij}$ and T are continuous by the fact that

$$|f(x)| \leq \|\Psi\| . \|x\|_u,$$

then we obtain $Tf = \Psi$. Moreover, T is onto. Hence T is an equivalence, i.e.,T is a linear isometry from $\mathcal{L}_u(X)'$ onto $\mathcal{M}_u(X')$. □

Lemma 2.3. [4]*Let f be an unbounded modulus function which is not the identity on $[0,\infty)$. Then, for each $x \in [0,\infty)$, there exists a positive integer n such that*

$$f\left(\frac{x}{n}\right) > \frac{1}{n}$$

[4].

Theorem 2.4. *If f is an unbounded modulus function which is not the identity on $[0,\infty)$, then $\mathcal{L}_u(X,f)$ is not locally convex.*

Proof. Let us take $D = \{x : P(x) \leq 1\}$ in $\mathcal{L}_u(X, f)$, where

$$P(x) = \sum_i \sum_j f(\|x_{ij}\|).$$

and D is a neighborhood of zero in $\mathcal{L}_u(X, f)$, Since D contains the sphere

$$\{x : P(x) \leq 1\},$$

In order to show that D does not contain any convex neighborhood of zero, let N be a convex neighborhood of zero in $\mathcal{L}_u(X, f)$. Then, for some $\delta > 0$, N contains

$$\{x : P(x) \leq \delta\}.$$

Since f is unbounded, we can take an $\xi > 0$ such that $f(\xi) = \delta$. Now define

$$\begin{aligned} I_{ij} &: X \to \mathcal{L}_u(X, f) \\ I_{ij}(t) &= \begin{bmatrix} 0 & 0 & \ldots & 0 & \ldots \\ 0 & 0 & \ldots & 0 & \ldots \\ . & . & \ldots & . & \ldots \\ 0 & 0 & \ldots & t & \ldots \\ 0 & 0 & \ldots & 0 & \ldots \end{bmatrix}. \end{aligned}$$

If $U_{ij} \in S_X$, where $S_X = \{x \in X : \|x\| = 1\}$, then, $I_{ij}(\xi U_{ij}) \in N$, for each (i, j). Since

$$P(I_{ij}(\xi U_{ij})) = f(\|\xi U_{ij}\|) = f(\xi) = \delta,$$

and

$$I_{ij}(\xi U_{ij}) \in \{x : P(x) \leq \delta\}.$$

from Lemma 2.3, we can find $m, n \in \mathbb{Z}^+$ such that

$$f\left(\frac{\xi}{mn}\right) > \frac{1}{mn}.$$

Using the convexity of N, we have

$$x = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} I_{ij}(\xi U_{ij}) = \begin{bmatrix} \frac{\xi u_{11}}{mn} & \frac{\xi u_{12}}{mn} & \ldots & \frac{\xi u_{1n}}{mn} & 0 \\ . & . & \ldots & . & . \\ \frac{\xi u_{m1}}{mn} & \frac{\xi u_{m2}}{mn} & \ldots & \frac{\xi u_{mn}}{mn} & 0 \\ 0 & 0 & \ldots & 0 & . \\ 0 & 0 & \ldots & 0 & . \end{bmatrix}.$$

But, since we have

$$\begin{aligned} P(x) &= \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(\left\|\frac{\xi u_{ij}}{mn}\right\|\right), \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(\left\|\frac{\xi}{mn}\right\|\right) = m.n.f\left(\frac{\xi}{mn}\right) > \frac{mn}{mn} = 1, \end{aligned}$$

then we get $x \notin D$, i.e., $N \nsubseteq D$,which shows that D, the neighbourhood of zero, does not contain a convex neighbourhood. Thus, $\mathcal{L}_u(X, f)$ can not be locally convex. □

Theorem 2.5. *If X is a reflexive normed space and $E \subset \Omega(X)$ is perfect, then the space E is normal.*

Proof. Since E is perfect , $E = E^{\alpha\alpha}$. For $x = (x_{nk}) \in E$ and for each $(g_{nk}) \in \Omega(X')$, let us take

$$|g_{nk}(y_{nk})| \leq |g_{nk}(x_{nk})|. \tag{2.1}$$

By perfectness of E, if $x \in E$,then $x \in E^{\alpha\alpha} \subseteq \Omega(X'') = \Omega(X)$. For each $(g_{nk}) \in E^{\alpha}$, we get

$$\sum_{n,k} |g_{nk}(x_{nk})| < \infty. \tag{2.2}$$

From (2.1) and(2.2) we obtain

$$\sum_{n,k} |g_{nk}(y_{nk})| < \infty,$$

which shows $y = (y_{nk}) \in E^{\alpha\alpha} = E$. So, E is a normal space. □

Theorem 2.6. *Normality implies monotonicity for any double sequence space $E \subset \Omega(X)$.*

Proof. Let $z = (z_{nk})$ be a double sequence defined by $z_{nk} = x_{nk}.y_{nk}$, for $x = (x_{nk}) \in E$ and $y = (y_{nk}) \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}$. By definition of $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$, the finite terms which are different from each other in sequence y are repeated infinity times. Let $\mu_1, \mu_2, ..., \mu_n$ be such terms and say that $\mu = \max\{|\ \mu_i\ |: 1 \leq i \leq n\}$. It is clear that $|\ y_{nk}\ | \leq \mu$. Also, for each $(g_{nk}) \in \Omega(X')$,we have

$$|g_{nk}(z_{nk})| = |y_{nk} g_{nk}(x_{nk})| = |y_{nk}| \, |g_{nk}(x_{nk})| \leq \mu. \, |g_{nk}(y_{nk})|.$$

Thus, using the normality of E, we get $z \in E$,which implies that E is monotone. □

Theorem 2.7. *If $E \subset \Omega(X)$ is a monotone double sequence space, then we have $E^{\alpha} = E^{\beta(\nu)}$.*

Proof. If $(g_{nk}) \in E^{\beta(\nu)}$, then for each $(x_{nk}) \in E$,

$$\nu - \sum_{n,k} g_{nk}(x_{nk})$$

exists. Now, we define a double sequence (z_{nk}) by

$$z_{nk} = \begin{cases} \frac{|g_{nk}(x_{nk})|}{g_{nk}(x_{nk})} x_{nk}, & g_{nk}(x_{nk}) \neq 0, \\ 0, & g_{nk}(x_{nk}) = 0. \end{cases}$$

Since E is monotone, then we have $z = (z_{nk}) \in E$. Thus

$$\nu - \sum_{n,k} g_{nk}(z_{nk})$$

exists,

$$\begin{aligned} \nu - \sum_{n,k=1}^{\infty} g_{nk}\left(z_{nk}\right) &= \sum_{n,k=1}^{\infty} g_{nk}\left(\frac{\left|g_{nk}\left(x_{nk}\right)\right|}{g_{nk}\left(x_{nk}\right)} x_{nk}\right), \\ &= \sum_{n,k=1}^{\infty} \frac{\left|g_{nk}\left(x_{nk}\right)\right|}{g_{nk}\left(x_{nk}\right)} g_{nk(}x_{nk)}, \\ &= \sum_{n,k=1}^{\infty} \left|g_{nk}\left(x_{nk}\right)\right|. \end{aligned}$$

Thus $(g_{nk}) \in E^{\alpha}$. □

Corollary 2.8. *$\alpha-$duals and $\beta(\nu)$ duals of $\mathcal{L}_u(X)$, $\mathcal{M}_u(X)$, $\Phi(X)$ and $\Omega(X)$ are overlap.*

Theorem 2.9. *The space $\mathcal{L}_u(f)$ is normal.*

Proof. If $x = (x_{nk}) \in \mathcal{L}_u(f)$ then we have $\sum f(|x_{nk}|) < \infty$. Assume that $|y_{nk}| \leq |x_{nk}|$, then we write

$$f(|y_{nk}|) < f(|x_{nk}|),$$

which implies

$$\sum_{n,k} f(|y_{nk}|) < \sum_{n,k} f(|x_{nk}|) < \infty.$$

So we get

$$\sum_{n,k} f(|y_{nk}|) < \infty$$

which means that $y = (y_{nk}) \in \mathcal{L}_u(f)$. □

Corollary 2.10. *The space $\mathcal{L}_u(f)$ is monotone. Thus, $\mathcal{L}_u(f)^{\alpha} = \mathcal{L}_u(f)^{\beta(\nu)} = \mathcal{M}_u$.*

Corollary 2.11. *Since $\mathcal{L}_u(f)$ is normal, then $\mathcal{L}_u(X,f)$ becomes normal. Accordingly $\mathcal{L}_u(X,f)$ is monotone in the same time, α and $\beta(\nu)$ duals of it are overlap, i.e. $\mathcal{L}_u(X,f)^{\alpha} = \mathcal{L}_u(X,f)^{\beta(\nu)}$.*

Theorem 2.12. *If λ is a normal scalar double sequence space, then we have*

$$\lambda(X)^{\alpha} = \lambda^{\alpha}(X').$$

Proof. If $(f_{nk}) \in \lambda^{\alpha}(X')$, then $(\|f_{nk}\|) \in \lambda^{\alpha}$. In this case, for each $(u_{nk}) \in \lambda$. We have

$$\sum_{n,k} \|f_{nk}\| . |u_{nk}| < \infty.$$

On the other hand, for each $x=(x_{nk})\in\lambda(X), (\|x_{nk}\|)\in\lambda$, we obtain

$$\sum_{n,k}|f_{nk}(x_{nk})|\leq\sum_{n,k}\|f_{nk}\|\,\|x_{nk}\|<\infty,$$

that is, $(f_{nk})\in\lambda(X)^{\alpha}$.

Conversely, suppose that $(f_{nk})\in\lambda(X)^{\alpha}$. From definition of $(\|f_{nk}\|)$, for each k, there exists a $(y_{nk})\in X$, such that $\|f_{nk}\|\leq 2.\,|f_{nk}(y_{nk})|$ and $\|y_{nk}\|\leq 1$. For each $(u_{nk})\in\lambda$, define sequence (z_{nk}) by $z_{nk}=u_{nk}.y_{nk}$. From normality of λ, Since we have

$$\|z_{nk}\|=|u_{nk}|\,\|y_{nk}\|<|u_{nk}|$$

and $(\|z_{nk}\|)\in\lambda$, then we get $z\in\lambda(X)$. Thus we write

$$\begin{aligned}\sum_{n,k}\|f_{nk}\|.|u_{nk}| &\leq 2.\sum_{n,k}|u_{nk}|\,|f_{nk}(y_{nk})|\\ &= 2\sum_{n,k}|f_{nk}(u_{nk}.y_{nk})|\\ &= 2\sum_{n,k}|f_{nk}(z_{nk})|<\infty,\end{aligned}$$

which implies $(\|f_{nk}\|)\in\lambda^{\alpha}$ and $(f_{nk})\in\lambda^{\alpha}(X')$. □

Remark 2.13. In the scalar case, perfectness of a sequence space is needed normality; normality is needed monotonicity. This valid for vector valued sequence space when X is a reflexive normed space.

Corollary 2.14. *$\mathcal{L}_u(X)$, $\mathcal{M}_u(X)$, $\Phi(X)$ and $\Omega(X)$ are normal and monotone.*

Corollary 2.15. *Let X be a reflexive normed space. Then. we have*

$$\mathcal{L}_u(X)^{\alpha}=\mathcal{L}_u(X)^{\beta(\nu)}=\mathcal{M}_u(X').$$

Theorem 2.16. *If λ is a normal double sequence space, then, we have*

$$\lambda(X,f)^{\alpha}=\lambda(X,f)^{\beta(\nu)}=(\lambda(f))^{\alpha}(X').$$

Corollary 2.17. *In above Theorem, if we take $\lambda=\mathcal{L}_u$ then we have*

$$\mathcal{L}_u(X,f)^{\alpha}=\mathcal{L}_u(X,f)^{\beta(\nu)}=(\mathcal{L}_u(f))^{\alpha}(X')=\mathcal{M}_u(X').$$

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