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HOLOMORPHY OF BASARAB LOOPS

GIDEON OKON EFFIONG[1*], TÈMÍTỌ́PẸ́ GBỌ́LÁHÀN JAIYÉỌLÁ[2,3], MARTIN CHUCKS OBI[4] AND LUKMAN SHINA AKINOLA[5]

This paper is dedicated to Professor A.R.T. Solarin at 70

Abstract. A loop $(Q,\cdot)$ is called a Basarab loop if the identities: $(x\cdot yx^\rho)\cdot(xz)=x\cdot yz$ and $(yx)\cdot(x^\lambda z\cdot x)=yz\cdot x$ hold. The holomorphy of a Basarab loop Q was investigated with respect to a group $A(Q)$ of automorphisms of the loop. Some necessary and sufficient conditions for an $A(Q)$-holomorph of a loop Q to be a left (right) Basarab loop or Basarab loop were established. Specifically, the $A(Q)$-holomorph of a loop Q was shown to be a left (right) Basarab loop if and only if Q is a left (right) Basarab loop and every element of $A(Q)$ is left (right) regular. The $A(Q)$-holomorph of a loop Q was shown to be a Basarab loop if and only if Q is a Basarab loop, every element of $A(Q)$ is both left and right nuclear and the $A(Q)$-generalized inner mappings of Q take some particular forms. These results were expressed in form of commutative diagrams. In any left (right) Basarab loop or Basarab loop Q, it was shown that the set of $\alpha\in A(Q)$ with four autotopic characterizations actually form normal subgroups of $A(Q)$.

1. Introduction and preliminaries

1.1. **Quasigroups and Loops.** Let G be a non-empty set. Define a binary operation $(\cdot)$ on G. If $x\cdot y\in G$ for all $x,y\in G$, then the pair $(G,\cdot)$ is called a *groupoid* or *Magma*.

If each of the equations:

$$a\cdot x=b \qquad \text{and} \qquad y\cdot a=b$$

has unique solutions in G for x and y respectively, then $(G,\cdot)$ is called a *quasigroup*.

If there exists a unique element $e\in G$ called the *identity element* such that for all $x\in G$, $x\cdot e=e\cdot x=x$, $(G,\cdot)$ is called a *loop*. We write xy instead of $x\cdot y$, and stipulate that $\cdot$ has lower priority than juxtaposition among factors to be multiplied. For instance, $x\cdot yz$ stands for $x(yz)$.

It can now be seen that a groupoid $(G,\cdot)$ is a quasigroup if its left and right translation mappings are bijections or permutations. In a loop $(G,\cdot)$ with identity

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* Corresponding author.
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element e, the *left inverse element* of $x \in G$ is the element $xJ_\lambda = x^\lambda \in G$ such that

$$x^\lambda \cdot x = e$$

while the *right inverse element* of $x \in G$ is the element $xJ_\rho = x^\rho \in G$ such that

$$x \cdot x^\rho = e.$$

If $x^\lambda = x^\rho$ for any $x \in G$, then we simply write $x^\lambda = x^\rho = x^{-1}$ or $J_\lambda = J_\rho = J$. Let x be a fixed element in a groupoid $(G, \cdot)$. The left and right translation maps of G, L_x and R_x respectively can be defined by

$$yL_x = x \cdot y \qquad \text{and} \qquad yR_x = y \cdot x.$$

It can now be seen that a groupoid $(G, \cdot)$ is a quasigroup if its left and right translation mappings are bijections or permutations. Since the left and right translation mappings of a quasigroup are bijective, then the inverse mappings L_x^{-1} and R_x^{-1} exist. Let

$$x\backslash y = yL_x^{-1} = xM_y \qquad \text{and} \qquad x/y = xR_y^{-1} = yM_x^{-1}$$

and note that

$$x\backslash y = z \Leftrightarrow x \cdot z = y \qquad \text{and} \qquad x/y = z \Leftrightarrow z \cdot y = x.$$

A loop $(G, \cdot)$ is said to have: (i) the left inverse property (LIP) if for all $x, y \in G$, $x^\lambda \cdot xy = y$; (ii) the right inverse property (RIP) if $x, y \in G$, $yx \cdot x^\rho = y$; and (iii) the inverse property (IP) if it has both left and right inverse properties.

Let a, b and c be three elements of a loop G. The loop associator of a, b and c is the unique element (a, b, c) of G which satisfies $(ab)c = \{a(bc)\}(a, b, c)$.

The right nucleus of G is defined by $N_\rho(G, \cdot) = \{a \in G \mid xy \cdot a = x \cdot ya \ \forall\ x, y \in G\}$. The left nucleus of G is defined by $N_\lambda(G, \cdot) = \{a \in G : ax \cdot y = a \cdot xy \ \forall\ x, y \in G\}$. The middle nucleus of G is defined by $N_\mu(G, \cdot) = \{a \in G : ya \cdot x = y \cdot ax \ \forall\ x, y \in G\}$. The nucleus of G is defined by $N(G, \cdot) = N_\lambda(G, \cdot) \cap N_\rho(G, \cdot) \cap N_\mu(G, \cdot)$. The centrum of G is defined by $C(G, \cdot) = \{a \in G : ax = xa \ \forall\ x \in G\}$. The center of G is defined by $Z(G, \cdot) = N(G, \cdot) \cap C(G, \cdot)$. $N_\rho(G, \cdot), N_\lambda(G, \cdot), N_\mu(G, \cdot), N(G, \cdot), Z(G, \cdot)$ are subgroups of $(G, \cdot)$.

. The group of all permutations on G is called the permutation group of G and denoted by $SYM(G)$. The group $\mathcal{M}(G, \cdot) = \left\langle \{R_x, L_x, \ : \ x \in G\} \right\rangle$ is called the multiplication group of $(G, \cdot)$ and $\mathcal{M}(G, \cdot) \le SYM(G)$.

If $e\alpha = e$ in a loop G such that $\alpha \in \mathcal{M}(G)$, then α is called an inner mapping and they form a group $Inn(G)$ called the inner mapping group. The right, left and middle inner mappings

$$R_{(x,y)} = R_x R_y R_{xy}^{-1} \ , L_{(x,y)} = L_x L_y L_{yx}^{-1} \text{ and } T_x = R_x L_x^{-1}$$

respectively generate the left inner mapping group $\mathrm{Inn}_\lambda(G)$, right inner mapping group $\mathrm{Inn}_\rho(G)$ and the middle inner mapping group $\mathrm{Inn}_\mu(G)$. If

$$\mathrm{Inn}_\lambda(G) \le AUM(G), \ \mathrm{Inn}_\rho(G) \le AUM(G), \ \text{and } \mathrm{Inn}_\mu(G) \le AUM(G),$$

then G is called a left A-loop(A$_\lambda$-loop), right A-loop(A$_\rho$-loop) and middle A-loop(A$_\mu$-loop) respectively. If $\mathrm{Inn}(G,\cdot) \le AUM(G,\cdot)$, then $(G,\cdot)$ is called an automorphic loop (A-loop)

For an overview of the theory of loops, readers may check [13, 15, 17, 20, 24, 33, 36, 37, 38].

The triple (A,B,C) of bijections of a loop $(G,\cdot)$ is called an autotopism if

$$xA \cdot yB = (x\cdot y)C \ \forall\ x,y \in G. \tag{1}$$

Such triples form a group $AUT(G,\cdot)$ called the autotopism group of $(G,\cdot)$. Furthermore, if $A = B = C$, then A is called an automorphism of $(G,\cdot)$. Such bijections form a group $AUM(G,\cdot)$ called the automorphism group of $(G,\cdot)$.

Let G and H be groups such that $\varphi : G \to H$ is an isomorphism. If $\varphi(g) = h$, then this would be expressed as $g \overset{\varphi}{\cong} h$.

Given any two sets X and Y. The statement '$f\ : X \to Y$ is defined as $f(x) = y,\ x \in X,\ y \in Y$'will at times be expressed as '$f\ : X \to Y\ \uparrow\ f(x) = y$'.

Definition 1.1. *Let $(G,\cdot)$ be a quasigroup. Then*

(1) *a bijection U is called autotopic if there exists $(U,V,W) \in AUT(G,\cdot)$; the set of all such mappings forms a group $\Sigma(G,\cdot)$.*
(2) *a bijection U is called ρ-regular if there exists $(I,U,U) \in AUT(G,\cdot)$; the set of all such mappings forms a group $\mathcal{P}(G,\cdot)$.*
(3) *a bijection U is called λ-regular if there exists $(U,I,U) \in AUT(G,\cdot)$; the set of all such mappings forms a group $\Lambda(G,\cdot) \le \Sigma(G,\cdot)$.*
(4) *a bijection U is called μ-regular if there exists a bijection U' such that $(U,U'^{-1},I) \in AUT(G,\cdot)$. U' is called the adjoint of U. The set of all μ-regular mappings forms a group $\Phi(G,\cdot) \le \Sigma(G,\cdot)$. The set of all adjoint mapping forms a group $\Psi(G,\cdot)$.*

Theorem 1.1. *(Isere et al. [23])*
Let $(G,\cdot)$ be a loop. Let

$$\omega : \mathcal{P}(G,\cdot) \to N_\rho(G,\cdot) \uparrow \omega(U) = eU, \vartheta : \Lambda(G,\cdot) \to N_\lambda(G,\cdot) \uparrow \vartheta(U) = eU,$$

$$\varphi : \Phi(G,\cdot) \to \Psi(G,\cdot) \uparrow \varphi(U) = U',\ \ \sigma : \Phi(G,\cdot) \to N_\mu(G,\cdot) \uparrow \sigma(U) = eU \ \ \text{and}$$

$$\beta : \Psi(G,\cdot) \to N_\mu(G,\cdot) \uparrow \beta(U') = eU'$$

Then $\mathcal{P}(G,\cdot) \overset{\omega}{\cong} N_\rho(G,\cdot),\ \Lambda(G,\cdot) \overset{\vartheta}{\cong} N_\lambda(G,\cdot),\ \Phi(G,\cdot) \overset{\varphi}{\cong} \Psi(G,\cdot),$

$$\Phi(G,\cdot) \overset{\sigma}{\cong} N_\mu(G,\cdot),\ \Psi(G,\cdot) \overset{\beta}{\cong} N_\mu(G,\cdot).$$

1.2. **Holomorphy of Quasigroups and Loops.**

Definition 1.2. *Let $(Q,\cdot)$ be a loop and $H(Q,\cdot) = A(Q) \times Q$ such that $A(Q) \le AUM(Q,\cdot)$. Define $(U,\ a) \circ (V,\ b) = (UV,\ aV \cdot b)$ for all $(U,a),\ (V,\ b) \in H$. Then the loop $H(Q,\cdot) = (H,\circ)$ is called $A(Q)$-holomorph of $(Q,\cdot)$ or simply the holomorph of $(Q,\cdot)$.*

For holomorphy theory of loops, readers may check [16, 5].

. Let $\alpha \in AUM(Q,\cdot)$. If $x^{\lambda} \cdot x\alpha \in N(Q,\cdot)$, then α is said to be left nuclear. If $x\alpha^{-1} \cdot x^{\rho} \in N(Q,\cdot)$, then α is said to be right nuclear. For any $x \in Q$ and $\alpha \in AUM(Q,\cdot)$, the generalized middle inner mapping

$$T_{(\cdot,\cdot)} \ : \ Q \times AUM(Q,\cdot) \to \mathcal{M}(Q) \text{ is defined as } T_{(x,\alpha)} = R_x L_{x\alpha}^{-1}.$$

Automorphism group of some types of rings and abelian group were of interest in Abdelwanis and Boua [2] and Abdelalim et al.[1] respectively.

Interestingly, Adeniran [3] and Robinson [34], Chiboka and Solarin [16], Bruck [12], Bruck and Paige [14], Robinson [35], Huthnance [21] and Adeniran [3] have respectively studied the holomorphs of Bol loops, conjugacy closed loops, inverse property loops, A-loops, extra loops, weak inverse property loops and Bruck loops. A set of results on the holomorph of some varieties of loops can be found in Jaiyéọlá [26, 27, 24]. Studies on the holomorph of generalized Bol loops can be found in Adeniran et al. [4] and Jaiyéọlá and Popoola [25]. Isere et al. [23] studied the holomorphy of Osborn loops and Jaiyéọlá et al. [28] studied the holomorphy of middle Bol loops. Ogunrinade et al. [31] studied the holomorphy of self distributive quasigroup while Ilojide et al. [22] studied the holomorphy of Fenyves BCI-algebras. Recently, Oyebo et al. [32] studied the holomorphy of (r,s,t)-inverse loops and their application to cryptography.

. In this paper, our objective is to explore the holomorphic structure of a left (right) Basarab loop and Basarab loop. Before this, we shall take some definitions, literature review on Basarab loops and state some existing results.

1.3. Basarab Loop.

Definition 1.3. *A loop $(Q,\cdot)$ is called a Basarab loop (or K-loop), if the identities:*

$$(x \cdot yx^{\rho}) \cdot xz = x \cdot yz, \quad yx \cdot (x^{\lambda}z \cdot x) = yz \cdot x \tag{2}$$

hold for all $x, y, z \in Q$.

A loop $(Q,\cdot)$ is called an automorphic inverse property Basarab loop (or IK-loop) means $(Q,\cdot)$ is a Basarab loop and the mapping J of $(Q,\cdot)$ is an automorphism of the loop $(Q,\cdot)$.

Example 1.1. *(Basarab [9])*

Let $\mathcal{F}$ be a field, $\mathcal{F}'$ be the set of non-zero elements of $\mathcal{F}$. Define on the set $\mathcal{Q} = \mathcal{F}' \times \mathcal{F}$ the operation $(\cdot)$ as follows:

$$(a,x) \cdot (b,y) = \big(a \cdot b,\ (a^{-1}-1)\cdot(b^{-1}-1) + b^{-1}x + y\big).$$

Then $(\mathcal{Q},\cdot)$ is a Basarab loop.

. Some early results on the class of loop called Basarab loop are seen in two prominent papers of Basarab [6, 7]. Basarab [9] used the result published by Belousov [10] to construct an example of a Basarab loop whose nucleus is an abelian group. In Basarab [8, 9], the author studied the relationship between a generalized Moufang loop, Osborn loop, VD-loop, and a Basarab loop, and a special type of a Basarab loop, known as IK-loop respectively. In this current study, an IK-loop is an automorphic inverse property Basarab loop.

Here are some existing results on Basarab loop.

Theorem 1.2. *(Basarab [9]) Let $(Q,\cdot)$ be a Basarab loop.*

1. *$N(Q,\cdot)$ contains the associator of any three elements of Q.*
2. *The quotient loop $Q/Z(Q,\cdot)$ is a group.*
3. *If $(Q,\cdot)$ is generated by one element, then it is solvable.*
4. *If $(Q,\cdot)$ has the automorphic inverse property, then it is nilpotent.*

Theorem 1.3. *(Basarab [7]) Let $(Q,\cdot)$ be a Basarab loop.*

1. *$N(Q,\cdot)$ is a nontrivial normal subloop.*
2. *The quotient loop $Q/N(Q,\cdot)$ is an abelian group.*
3. *If $N(Q,\cdot)$ has an odd order, then $(Q,\cdot)$ is solvable.*

Theorem 1.4. *(Basarab [8])*

1. *Any Basarab loop (any VD-loop) is a G-loop.*
2. *Any Basarab loop (VD-loop) is an Osborn loop.*
3. *A Basarab loop $(Q,\cdot)$ is a VD-loop if $x^2 \in N(Q,\cdot)$ for any $x \in Q$.*
4. *A VD-loop $(Q,\cdot)$ is a Basarab loop if $x^2 \in N(Q,\cdot)$ for any $x \in Q$.*

Here is a short review of most recent works on Basarab loops. Jaiyéọlá and Effiong [29] investigated Basarab loop and its variance with inverse properties. It was shown that a Basarab loop $(Q,\cdot)$ has the cross inverse property if and only if $(Q,\cdot)$ is an abelian group or all left (right) translations of $(Q,\cdot)$ are right (left) regular. In a Basarab loop, the following properties were found to be equivalent: flexibility property, right inverse property, left inverse property, inverse property, right alternative property, left alternative property and alternative property. A Basarab loop was shown to be a weak inverse property loop if it is flexible such that the middle inner mapping is contained in a permutation group. Effiong and Jaiyéọlá [18] characterized some subloops of a Basarab loop. Also, Jaiyéọlá and Effiong [30] studied total multiplication group of Basarab loop while Effiong et al. [19] studied the connections between Basarab loop and Buchsteiner loop through the examination of the relationship between Basarab loops and some other important classes of loops like: Osborn loop, conjugacy closed loop (CC-loop), Wilson loop, Moufang loop and extra loop. The authors established that Basarab loop is a class of loops which answers a cogent question raised by Goodaire and Robinson in 1990: 'Are there other equationally defined (and naturally characterized) classes of G-loops (like the classes; Wilson loops and extra loops) which are contained in the class of all conjugacy closed loops?'.

2. Main results

In this section, the holomorphy of a Basarab loop Q is investigated with respect to a group $A(Q)$ of automorphisms of the loop. Some necessary and sufficient conditions for an $A(Q)$-holomorph of a loop Q to be a left (right) Basarab loop or Basarab loop are established. The $A(Q)$-holomorph of a loop Q is shown to be a Basarab loop if and only if Q is a Basarab loop, every element of $A(Q)$ is both left and right nuclear and the $A(Q)$-generalized inner mappings of Q take some particular forms. These results are expressed in form of commutative diagrams. In any left (right) Basarab loop or Basarab loop Q, it is shown that the set of

$\alpha \in A(Q)$ with four autotopic characterizations actually form normal subgroups of $A(Q)$.

2.1. **Holomorphy of left Basarab loop.**

Theorem 2.1. *Let $(Q, \cdot)$ be a loop with $A(Q)$-holomorph $(H, \circ)$. Then, $(H, \circ)$ is a left Basarab loop if and only if*

$$(x\alpha \cdot yx^\rho) \cdot xz = x\alpha \cdot yz$$

for all $x, y, z \in Q$ and for all $\alpha \in A(Q)$.

Proof. Let $(Q, \cdot)$ be a loop and $A(Q) \leq AUM(Q, \cdot)$. Let $H = A(Q) \times Q$ such that $(\alpha, x) \circ (\beta, y) = (\alpha\beta, x\beta \cdot y)$ for all $(\alpha, x), (\beta, y) \in H$. Clearly,

$$(\alpha, x)^\rho = (\alpha^{-1}, x^\rho\alpha^{-1}) = (\alpha^{-1}, (x\alpha^{-1})^\rho). \quad \text{Therefore,}$$

$$\begin{gathered}
\{(\alpha, x) \circ ((\beta, y) \circ (\alpha, x)^\rho\} \circ ((\alpha, x) \circ (\gamma, z)) = (\alpha, x) \circ ((\beta, y) \circ (\gamma, z)) \\
\Leftrightarrow \{(\alpha, x) \circ ((\beta, y) \circ (\alpha^{-1}, x^\rho\alpha^{-1})\} \circ ((\alpha, x) \circ (\gamma, z)) = (\alpha, x) \circ ((\beta, y) \circ (\gamma, z)) \\
\Leftrightarrow \{(\alpha, x) \circ (\beta\alpha^{-1}, y\alpha^{-1} \cdot x^\rho\alpha^{-1})\} \circ (\alpha\gamma, x\gamma \cdot z) = (\alpha, x) \circ (\beta\gamma, y\gamma \cdot z) \\
\Leftrightarrow \big(\alpha(\beta\alpha^{-1}), x\beta\alpha^{-1} \cdot (y\alpha^{-1} \cdot x^\rho\alpha^{-1})\big) \circ (\alpha\gamma, x\gamma \cdot z) = (\alpha(\beta\gamma), x\beta\gamma \cdot (y\gamma \cdot z)) \\
\Leftrightarrow \big((\alpha(\beta\alpha^{-1}))\alpha\gamma, \big(x\beta\alpha^{-1} \cdot (y\alpha^{-1} \cdot x^\rho\alpha^{-1})\big)\, \alpha\gamma \cdot (x\alpha \cdot z)\big) = (\alpha\beta\gamma, x\beta\gamma \cdot (y\gamma \cdot z)) \\
\Leftrightarrow \big(\alpha\beta\gamma, (x\beta\alpha^{-1} \cdot (y\alpha^{-1} \cdot x^\rho\alpha^{-1}))\alpha\gamma \cdot (x\alpha \cdot z)\big) = (\alpha\beta\gamma, x\beta\gamma \cdot (y\gamma \cdot z)) \\
\Leftrightarrow \big((x\beta\alpha^{-1} \cdot (y\alpha^{-1} \cdot x^\rho\alpha^{-1}))\alpha\gamma\big) \cdot (x\alpha \cdot z) = x\beta\gamma \cdot (y\gamma \cdot z) \\
\Leftrightarrow \big((x\beta\alpha^{-1}\alpha\gamma) \cdot (y\alpha^{-1} \cdot x^\rho\alpha^{-1})\alpha\gamma\big) \cdot (x\gamma \cdot z) = x\beta\gamma \cdot (y\gamma \cdot z) \\
\Leftrightarrow \big((x\beta\alpha^{-1}) \cdot (y\alpha^{-1} \cdot x^\rho\alpha^{-1})\alpha\gamma\big) \cdot (x\alpha) \cdot z) = x\beta\gamma \cdot (y\gamma \cdot z) \\
\Leftrightarrow \big((x\beta\gamma) \cdot (y\alpha^{-1}\alpha\gamma \cdot x^\rho\alpha^{-1}\alpha\gamma)\big) \cdot (x\gamma \cdot z) = x\beta\gamma \cdot (y\gamma \cdot z) \\
\Leftrightarrow ((x\beta\gamma) \cdot (y\gamma \cdot x^\rho\gamma)) \cdot (x\gamma \cdot z) = x\beta\gamma \cdot (y\gamma \cdot z) \\
\Leftrightarrow ((x\beta\gamma) \cdot (y\gamma \cdot x^\rho\gamma))\, \gamma^{-1} \cdot (x\gamma \cdot z)\gamma^{-1} = (x\beta\gamma \cdot (y\gamma \cdot z))\gamma^{-1} \Leftrightarrow
\end{gathered}$$

$$(x\beta \cdot (y \cdot x^\rho)) \cdot (x \cdot z\gamma^{-1}) = x\beta \cdot (y \cdot z\gamma^{-1}) \tag{3}$$

Setting $\bar{z} = z\gamma^{-1}$ in (3) gives

$$(x\beta \cdot (y \cdot x^\rho)) \cdot (x \cdot \bar{z}) = x\beta \cdot (y \cdot \bar{z}) \tag{4}$$

In (4), setting $\bar{z} = z$ and $\beta = \alpha$ gives

$$(x\alpha \cdot yx^\rho) \cdot xz = x\alpha \cdot yz$$

for all $x, y, z \in Q$, and $\alpha \in A(Q)$. $\square$

Corollary 2.1. *Let $(Q, \cdot)$ be a loop with $A(Q)$-holomorph $(H, \circ)$. Then, $(H, \circ)$ is a left Basarab loop if and only if $(R_{x^\rho}L_{x\alpha}, L_x, L_{x\alpha}) \in AUT(Q, \cdot)$ for all $x \in Q$ and for all $\alpha \in A(Q)$.*

Proof. This follows from Theorem 2.1. $\square$

Lemma 2.1. *Let $(Q, \cdot)$ be a left Basarab loop with $A(Q)$-holomorph $(H, \circ)$. Then, $(H, \circ)$ is a left Basarab loop if and only if any of the following is true:*

1. *$(x\alpha \cdot (x\backslash y)) \cdot z = x\alpha \cdot (x\backslash(yz))$ for all $x, y, z \in Q$ and for all $\alpha \in A(Q)$.*
2. *$(L_x^{-1}L_{x\alpha}, I, L_x^{-1}L_{x\alpha}) \in AUT(Q, \cdot)$ for all $x \in Q$ and for all $\alpha \in A(Q)$.*
3. *$L_x^{-1}L_{x\alpha}$ is λ-regular for all $x \in Q$ and for all $\alpha \in A(Q)$.*

(4) $L_x^{-1}L_{x\alpha} \in \Lambda(Q,\cdot)$ *for all* $x \in Q$ *and for all* $\alpha \in A(Q)$.

Proof. Let $(Q,\cdot)$ be a left Basarab loop. Then the autotopism $T = (R_{x^\rho}L_x, L_x, L_x)$ holds. $(H,\circ)$ is a left Basarab loop if and only if $S = (R_{x^\rho}L_{x\alpha}, L_x, L_{x\alpha}) \in AUT(Q,\cdot)$ by Corollary 2.1

$$T = (R_{x^\rho}L_x, L_x, L_x) \in AUT(Q,\cdot) \Rightarrow T^{-1} = (L_x^{-1}R_{x^\rho}^{-1}, L_x^{-1}, L_x^{-1}) \in AUT(Q,\cdot).$$

$(H,\circ)$ is a left Basarab loop if and only if

$$\begin{aligned} T^{-1}S &= (L_x^{-1}R_{x^\rho}^{-1}, L_x^{-1}, L_x^{-1})(R_{x^\rho}L_{x\alpha}, L_x, L_{x\alpha}) \\ &= (L_x^{-1}R_{x^\rho}^{-1}R_{x^\rho}L_{x\alpha}, L_x^{-1}L_x, L_x^{-1}L_{x\alpha}) = (L_x^{-1}L_{x\alpha}, I, L_x^{-1}L_{x\alpha}) \in AUT(Q,\cdot). \end{aligned}$$

Also, for all $x, y, z \in Q$ and for all $\alpha \in A(Q)$,

$$\begin{aligned} (L_x^{-1}L_{x\alpha}, I, L_x^{-1}L_{x\alpha}) \in AUT(Q,\cdot) &\Leftrightarrow yL_x^{-1}L_{x\alpha} \cdot z = (yz)L_x^{-1}L_{x\alpha} \\ &\Leftrightarrow (x\alpha \cdot (x\backslash y)) \cdot z = x\alpha \cdot (x\backslash(yz)). \end{aligned}$$

The rest follows by Definition 1.1(3). □

Corollary 2.2. *Let* $(Q,\cdot)$ *be a loop with* $A(Q)$*-holomorph* $(H,\circ)$*. Then,* $(H,\circ)$ *is a left Basarab loop if and only if* $(Q,\cdot)$ *is a left Basarab loop and any of the following equivalent conditions holds:*

(1) $L_x^{-1}L_{x\alpha} \in \Lambda(Q,\cdot)$ *for all* $x \in Q$ *and for all* $\alpha \in A(Q)$.
(2) $x\alpha \cdot x^\rho \in N_\lambda(Q,\cdot)$ *for all* $x \in Q$ *and for all* $\alpha \in A(Q)$.
(3) $\alpha = J_\rho M_n^{-1}$ *for all* $\alpha \in A(Q)$ *and some* $n \in N_\lambda(Q,\cdot)$.
(4) $A(Q) \leq \Lambda(Q,\cdot)$.

Proof. Let $(H,\circ)$ be a left Basarab loop, then $\{I\} \times Q \leq (H,\circ)$ and $\{I\} \times Q \cong (Q,\cdot)$. Hence, $(Q,\cdot)$ is a left Basarab loop. Thus, by Lemma 2.1, $L_x^{-1}L_{x\alpha} \in \Lambda(Q,\cdot)$ for all $x \in Q$ and for all $\alpha \in A(Q)$.

Conversely, if $(Q,\cdot)$ is a left Basarab loop and 1. holds, then by Lemma 2.1, $(H,\circ)$ is a left Basarab loop.

1$\Leftrightarrow$ **2:** By Theorem 1.2, $L_x^{-1}L_{x\alpha} \overset{\vartheta}{\cong} eL_x^{-1}L_{x\alpha} \Leftrightarrow eL_x^{-1}L_{x\alpha} \in N_\lambda(Q,\cdot) \Leftrightarrow x\alpha \cdot x^\rho \in N_\lambda(Q,\cdot)$ for all $x \in Q$ and for all $\alpha \in A(Q)$.

2$\Leftrightarrow$ **3:** $x\alpha \cdot x^\rho \in N_\lambda(Q,\cdot) \Leftrightarrow x\alpha \cdot x^\rho = n \Leftrightarrow x\alpha = nR_{x^\rho}^{-1} = x^\rho M_n^{-1} \Leftrightarrow \alpha = J_\rho M_n^{-1}$ for all $\alpha \in A(Q)$ and some $n \in N_\lambda(Q,\cdot)$.

1$\Leftrightarrow$ **4:** For all $x \in Q$ and for all $\alpha \in A(Q)$, $L_x^{-1}L_{x\alpha} \in \Lambda(Q,\cdot) \Leftrightarrow L_{x\alpha} \in L_x\Lambda(Q,\cdot) \Leftrightarrow L_{x\alpha} = L_x\lambda$ for some $\lambda \in \Lambda(Q,\cdot) \Leftrightarrow (\alpha, I, \lambda) \in AUT(Q,\cdot) \Leftrightarrow \alpha \in \Lambda(Q,\cdot) \Leftrightarrow A(Q) \leq \Lambda(Q,\cdot)$.

□

Remark 2.1. *From Corollary 2.2, it can be deduced that the* $\Lambda(Q,\cdot)$*-holomorph* $H(Q,\cdot)$ *of a left Basarab loop* $(Q,\cdot)$ *is precisely a left Basarab loop.*

Corollary 2.3. *Let* $(Q,\cdot)$ *be a loop with* $A(Q)$*-holomorph* $(H,\circ)$*. Then,* $(H,\circ)$ *is a LIP left Basarab loop if and only if* $(Q,\cdot)$ *is a LIP left Basarab loop and any of the following equivalent conditions holds:*

(1) $JL_x^{-1}L_{x\alpha}J$ *is* μ*-regular with an adjoint* $(JL_x^{-1}L_{x\alpha}J)' = L_{x\alpha}^{-1}L_x$ *for all* $x \in Q$ *and for all* $\alpha \in A(Q)$.

(2) *$JL_x^{-1}L_{x\alpha}J \in \Phi(Q,\cdot)$ and $(JL_x^{-1}L_{x\alpha}J)' = L_{x\alpha}^{-1}L_x \in \Psi(Q,\cdot)$ for all $x \in Q$ and for all $\alpha \in A(Q)$.*
(3) *$x\alpha \cdot x^\rho,\ x \cdot x^\rho\alpha \in N_\mu(Q,\cdot)$ for all $x \in Q$ and for all $\alpha \in A(Q)$.*
(4) *$\alpha = J_\rho M_{n_1}^{-1} = M_{n_2}J_\lambda$ for all $\alpha \in A(Q)$ and some $n_1, n_2 \in N_\mu(Q,\cdot)$.*
(5) *$A(Q) \subseteq J\Phi(Q,\cdot)J$ and $(JA(Q)J)' \subseteq J\Phi(Q,\cdot)J \subseteq \Psi(Q,\cdot)$.*

Proof. It must first be noted that $(H,\circ)$ is a LIP loop if and only if $(Q,\cdot)$ is a LIP. Following Corollary 2.2, $(H,\circ)$ is a left Basarab loop if and only if $(Q,\cdot)$ is a left Basarab loop and $L_x^{-1}L_{x\alpha} \in \Lambda(Q,\cdot)$ for all $x \in Q$ and for all $\alpha \in A(Q)$.

In a LIPL, $(U,V,W) \in AUT(Q,\cdot) \Rightarrow (JUJ,W,U) \in AUT(Q,\cdot)$. So, for all $x \in Q$ and for all $\alpha \in A(Q)$:

$$L_x^{-1}L_{x\alpha} \in \Lambda(Q,\cdot) \Leftrightarrow$$
$$(L_x^{-1}L_{x\alpha}, I, L_x^{-1}L_{x\alpha}) \in AUT(Q,\cdot) \Leftrightarrow \big(JL_x^{-1}L_{x\alpha}J, L_x^{-1}L_{x\alpha}, I\big) \in AUT(Q,\cdot)$$

$\Leftrightarrow JL_x^{-1}L_{x\alpha}J$ is μ-regular with an adjoint $(JL_x^{-1}L_{x\alpha}J)' = L_{x\alpha}^{-1}L_x$.

1$\Leftrightarrow$ **2:** This is straightforward.

2$\Leftrightarrow$ **3:** By Theorem 1.2, $JL_x^{-1}L_{x\alpha}J \overset{\sigma}{\cong} eJL_x^{-1}L_{x\alpha}J \Leftrightarrow eJL_x^{-1}L_{x\alpha}J \in N_\mu(Q,\cdot) \Leftrightarrow x\alpha \cdot x^\rho \in N_\mu(Q,\cdot)$ for all $x \in Q$ and for all $\alpha \in A(Q)$.

Furthermore, $L_{x\alpha}^{-1}L_x \overset{\beta}{\cong} eL_{x\alpha}^{-1}L_x \Leftrightarrow eL_{x\alpha}^{-1}L_x \in N_\mu(Q,\cdot) \Leftrightarrow x \cdot x^\rho\alpha \in N_\mu(Q,\cdot)$ for all $x \in Q$ and for all $\alpha \in A(Q)$.

3$\Leftrightarrow$ **4:** $x\alpha \cdot x^\rho \in N_\mu(Q,\cdot) \Leftrightarrow x\alpha \cdot x^\rho = n_1 \Leftrightarrow x\alpha = n_1R_{x^\rho}^{-1} = x^\rho M_{n_1}^{-1} \Leftrightarrow \alpha = J_\rho M_{n_1}^{-1}$ for all $\alpha \in A(Q)$ and some $n_1 \in N_\mu(Q,\cdot)$.

Furthermore, $x \cdot x^\rho\alpha \in N_\mu(Q,\cdot) \Leftrightarrow x \cdot x^\rho\alpha = n_2 \Leftrightarrow x^\rho\alpha = n_2L_x^{-1} = xM_{n_2} \Leftrightarrow \alpha = M_{n_2}J_\lambda$ for all $\alpha \in A(Q)$ and some $n_2 \in N_\mu(Q,\cdot)$.

1$\Leftrightarrow$ **5:** For all $x \in Q$ and for all $\alpha \in A(Q)$, $JL_x^{-1}L_{x\alpha}J \in \Phi(Q,\cdot) \Leftrightarrow JL_x^{-1}L_{x\alpha}J = \phi$ for some $\phi \in \Phi(Q,\cdot) \Leftrightarrow L_x^{-1}L_{x\alpha} = J\phi J \Leftrightarrow L_{x\alpha} = L_xJ\phi J \Leftrightarrow (\alpha, I, J\phi J) \in AUT(Q,\cdot) \Leftrightarrow (J\alpha J, J\phi J, I) \in AUT(Q,\cdot) \Leftrightarrow J\alpha J \in \Phi(Q,\cdot)$ and $(J\alpha J)' = (J\psi J) = J\psi^{-1}J \in \Psi(Q,\cdot)$ for some $\psi \in \Psi(Q,\cdot) \Leftrightarrow A(Q) \subseteq J\Phi(Q,\cdot)J$ and $(JA(Q)J)' \subseteq J\Phi(Q,\cdot)J \subseteq \Psi(Q,\cdot)$.

Furthermore, for all $x \in Q$ and for all $\alpha \in A(Q)$, $(JL_x^{-1}L_{x\alpha}J)' = L_{x\alpha}^{-1}L_x \in \Psi(Q,\cdot) \Leftrightarrow L_x \in L_{x\alpha}\Psi(Q,\cdot) \Leftrightarrow L_x \in L_{x\alpha}\psi$ for some $\psi \in \Psi(Q,\cdot) \Leftrightarrow (\alpha, I, \psi^{-1}) \in AUT(Q,\cdot) \Leftrightarrow (J\alpha J, I, \psi^{-1}) \in AUT(Q,\cdot) \Leftrightarrow J\alpha J \in \Phi(Q,\cdot)$ and $(J\alpha J)' = \psi \in \Psi(Q,\cdot) \Leftrightarrow A(Q) \subseteq J\Phi(Q,\cdot)J$ and $(JA(Q)J)' \subseteq \Psi(Q,\cdot)$.

□

Corollary 2.4. *Let $(Q,\cdot)$ be a loop with $A(Q)$-holomorph $(H,\circ)$. If $(H,\circ)$ is a LIP left Basarab loop, then*

$$\begin{tikzcd} & x\alpha\cdot x^\rho,\ x\cdot x^\rho\alpha \\ JL_x^{-1}L_{x\alpha}J \arrow[ur, mapsto, "\sigma"] \arrow[r, mapsto, "\varphi", "\text{isomorphism}"'] & L_{x\alpha}^{-1}L_x \arrow[u, mapsto, "\beta"'] \end{tikzcd} \in \begin{tikzcd} & N_\mu(Q,\cdot) \\ \Phi(Q,\cdot) \arrow[ur, "\sigma"] \arrow[r, "\varphi", "\text{isomorphism}"'] & \Psi(Q,\cdot) \arrow[u, "\beta"', "\text{isomorphism}"] \end{tikzcd} \tag{5}$$

for all $x \in Q$, $\alpha \in A(Q)$. i.e. $\sigma = \varphi\beta$.

Proof. This follows from Corollary 2.3. □

Lemma 2.2. *Let $(Q,\cdot)$ be a loop with $A(Q)$-holomorph $(H,\circ)$. If $(H,\circ)$ is a left Basarab loop then the following are true for all $x,z\in Q$ and $\alpha\in A(Q)$:*

1. $(x\alpha\cdot x^\rho)\cdot z = x\alpha\cdot(x\backslash z)$;
2. $x\alpha\cdot x^\rho\in N_\lambda(Q,\cdot)$;
3. $(x\alpha\cdot x^\rho)\cdot xz = x\alpha\cdot z$;
4. $(x\alpha\cdot x^\rho)x = x\alpha$;
5. $(x\alpha\cdot x^\rho, x, z) = e$;
6. $L_{(x\alpha\cdot x^\rho)} = L_x^{-1}L_{x\alpha}$;
7. $L_{(x\alpha\cdot x^\rho)}\in\Lambda(Q,\cdot)$.

Proof. (1) By Lemma 2.1, $(H,\circ)$ is a left Basarab loop implies

$$(L_x^{-1}L_{x\alpha}, I, L_x^{-1}L_{x\alpha})\in AUT(Q,\cdot)\Rightarrow\ yL_x^{-1}L_{x\alpha}\cdot z = (yz)L_x^{-1}L_{x\alpha}\forall\ y,z\in Q.$$

By Corollary 2.2, $(H,\circ)$ is a left Basarab loop implies that $(Q,\cdot)$ a left Basarab loop. So, $R_{x^\rho} = L_xR_x^{-1}L_x^{-1}$ which implies $L_x^{-1} = R_xL_x^{-1}R_{x^\rho}$. Thus, for all $y,z\in Q$, $yR_xL_x^{-1}R_{x^\rho}L_{x\alpha}\cdot zI = (yz)L_x^{-1}L_{x\alpha}$ if and only if

$$yT_xR_{x^\rho}L_{x\alpha}\cdot zI = (yz)L_x^{-1}L_{x\alpha}, \forall\ x,y,z\in Q. \tag{6}$$

Setting $y=e$ in (6), then $eT_xR_{x^\rho}L_{x\alpha}\cdot zI = (ez)L_x^{-1}L_{x\alpha}\Rightarrow eR_{x^\rho}L_{x\alpha}\cdot zI = zL_x^{-1}L_{x\alpha}\Rightarrow(x\alpha\cdot x^\rho)\cdot z = x\alpha\cdot(x\backslash z)$.

(2) From 1, $(x\alpha\cdot x^\rho)\cdot z = x\alpha\cdot(x\backslash z)\Leftrightarrow zL_{(x\alpha\cdot x^\rho)} = zL_x^{-1}L_{x\alpha}\Leftrightarrow L_{(x\alpha\cdot x^\rho)} = L_x^{-1}L_{x\alpha}$. So,

$$\left(L_x^{-1}L_{x\alpha}, I, L_x^{-1}L_{x\alpha}\right) = (L_{(x\alpha\cdot x^\rho)}, I, L_{(x\alpha\cdot x^\rho)})\in AUT(Q,\cdot)\Rightarrow x\alpha\cdot x^\rho\in N_\lambda(Q,\cdot).$$

(3) In 1 above, $(x\alpha\cdot x^\rho)\cdot z = x\alpha\cdot(x\backslash z)$, setting $z=xz$ gives $(x\alpha\cdot x^\rho)\cdot xz = x\alpha\cdot z$.

(4) If $z=x$ in 3 above, then $(x\alpha\cdot x^\rho)x = x\alpha$.

(5) From 3 and 4 above , $(x\alpha\cdot x^\rho)\cdot xz = x\alpha\cdot z$ and $(x\alpha\cdot x^\rho)x\cdot z = x\alpha\cdot z$ respectively. Using these, we get $(x\alpha\cdot x^\rho)\cdot xz = x\alpha\cdot z = (x\alpha\cdot x^\rho)x\cdot z\Rightarrow(x\alpha\cdot x^\rho, x, z) = e$.

(6) This is gotten from the proof of 2.

(7) This is gotten from the proof of 2.

□

2.2. Holomorphy of right Basarab loop.

Theorem 2.2. *Let $(Q,\cdot)$ be a loop with $A(Q)$-holomorph $(H,\circ)$. Then, $(H,\circ)$ is a right Basarab loop if and only if for all $x,y,z\in Q$ and $\delta\in A(Q)$,*

$$(y\cdot x\delta)\cdot((x^\lambda\delta\cdot z)\cdot x) = yz\cdot x\quad or\quad(y\cdot x)\cdot((x^\lambda\cdot z)\cdot x\delta^{-1}) = (y\cdot z)\cdot x\delta^{-1}$$

Proof. Let $(Q,\cdot)$ be a loop and $A(Q)\le AUM(Q,\cdot)$. Let $H = A(Q)\times Q$ such that
$(\alpha,x)\circ(\beta,y) = (\alpha\beta, x\beta\cdot y)$, for all $(\alpha,x),(\beta,y)\in H$. Clearly,

$$(\beta,y)^\lambda = (\beta^{-1}, y^\lambda\beta^{-1}) = (\beta^{-1},(y\beta^{-1})^\lambda).\quad\text{Then,}$$

$$((\beta,y)\circ(\alpha,x))\cdot\{((\alpha,x)^\lambda\circ(\gamma,z))\circ(\alpha,x)\} = ((\beta,y)\circ(\gamma,z))\circ(\alpha,x)$$

$$\Leftrightarrow((\beta,y)\circ(\alpha,x))\circ\{((\alpha^{-1},x^\lambda\alpha^{-1})\circ(\gamma,z))\circ(\alpha,x)\} = ((\beta,y)\circ(\gamma,z))\circ(\alpha,x)$$

$$\Leftrightarrow(\beta\alpha, y\alpha\cdot x)\circ\{(\alpha^{-1}\gamma, x^\lambda\alpha^{-1}\gamma\cdot z)\circ(\alpha,x)\} = (\beta\gamma, y\gamma\cdot z)\circ(\alpha,x)$$

$$\Leftrightarrow (\beta\alpha, y\alpha\cdot x)\circ\{\alpha^{-1}\gamma\alpha, (x^\lambda\alpha^{-1}\gamma\cdot z)\alpha\cdot x\} = (\beta\gamma\alpha, (y\gamma\cdot z)\alpha\cdot x)$$
$$\Leftrightarrow (\beta\alpha\alpha^{-1}\gamma\alpha, (y\alpha\cdot x)\alpha^{-1}\gamma\alpha\cdot((x^\lambda\alpha^{-1}\gamma\cdot z)\alpha\cdot x)) = (\beta\gamma\alpha, (y\gamma\cdot z)\alpha\cdot x)$$
$$\Leftrightarrow (y\alpha\cdot x)\alpha^{-1}\gamma\alpha\cdot((x^\lambda\alpha^{-1}\gamma\cdot z)\alpha\cdot x) = (y\gamma\cdot z)\alpha\cdot x$$
$$\Leftrightarrow (y\alpha\alpha^{-1}\gamma\alpha\cdot x\alpha^{-1}\gamma\alpha)\cdot((x^\lambda\alpha^{-1}\gamma\alpha\cdot z\alpha)\cdot x) = (y\gamma\alpha\cdot z\alpha)\cdot x$$
$$\Leftrightarrow (y\gamma\alpha\cdot x\alpha^{-1}\gamma\alpha)\cdot((x^\lambda\alpha^{-1}\gamma\alpha\cdot z\alpha)\cdot x) = (y\gamma\alpha\cdot z\alpha)\cdot x.$$

Replace: $y\gamma\alpha$ with y; $z\alpha$ with z and $\alpha^{-1}\gamma\alpha$ with δ, then it follows that

$$(y\cdot x\delta)\cdot((x^\lambda\delta\cdot z)\cdot x) = (y\cdot z)\cdot x$$

□

Corollary 2.5. *Let $(Q,\cdot)$ be a loop with $A(Q)$-holomorph $(H,\circ)$. Then, $(H,\circ)$ is a right Basarab loop if and only if for all $x\in Q$ and $\delta\in A(Q)$,*

$$(R_{x\delta},\ L_{x^\lambda\delta}R_x,\ R_x)\in AUT(Q,\cdot) \quad \textit{or} \quad (R_x,\ L_{x^\lambda}R_{x\delta^{-1}}, R_{x\delta^{-1}})\in AUT(Q,\cdot)$$

Proof. The proof follows from Theorem 2.2. □

Lemma 2.3. *Let $(Q,\cdot)$ be a right Basarab loop with $A(Q)$-holomorph $(H,\circ)$. Then, $(H,\circ)$ is a right Basarab loop if and only if any of the following is true:*

1. *$y\cdot((z/x)\cdot x\delta^{-1}) = ((yz)/x)\cdot x\delta^{-1}$ holds for all $x\in Q$ and for all $\delta^{-1}\in A(Q)$.*
2. *$(I,\ R_x^{-1}R_{x\delta^{-1}}, R_x^{-1}R_{x\delta^{-1}})\in AUT(Q,\cdot)$ for all $x\in Q$ and for all $\delta^{-1}\in A(Q)$.*
3. *$R_x^{-1}R_{x\delta^{-1}}$ is ρ-regular for all $x\in Q$ and for all $\delta^{-1}\in A(Q)$.*
4. *$R_x^{-1}R_{x\delta^{-1}}\in\mathcal{P}(Q,\cdot)$ for all $x\in Q$ and for all $\delta^{-1}\in A(Q)$.*

Proof. Let $(Q,\cdot)$ be a right Basarab loop. Then the autotopism $T=(R_x, L_{x^\lambda}R_x, R_x)$ holds. By Corollary 2.5, the holomorph $(H,\circ)$ of $(Q,\cdot)$ is a right Basarab loop if and only if $S=(R_x, L_{x^\lambda}R_{x\delta^{-1}}, R_{x\delta^{-1}})$ is the autotopism of Q.

The autotopism $T=(R_x, L_{x^\lambda}R_x, R_x)\in AUT(Q)$ implies

$$T^{-1} = (R_x^{-1}, R_x^{-1}L_{x^\lambda}^{-1}, R_x^{-1})\in AUT(Q).$$

The holomorph $(H,\circ)$ of $(Q,\cdot)$ is a right Basarab loop if and only if

$$\begin{aligned} T^{-1}S &= (R_x^{-1}, R_x^{-1}L_{x^\lambda}^{-1}, R_x^{-1})(R_x, L_{x^\lambda}R_{x\delta^{-1}}, R_{x\delta^{-1}})\\ &= (R_x^{-1}R_x, R_x^{-1}L_{x^\lambda}^{-1}L_{x^\lambda}R_{x\delta^{-1}}, R_x^{-1}R_{x\delta^{-1}})\\ &= (I,\ R_x^{-1}R_{x\delta^{-1}}, R_x^{-1}R_{x\delta^{-1}})\in AUT(Q,\cdot)\end{aligned}$$

Also, for all $x,y,z\in Q$ and for all $\delta\in A(Q)$,

$$\big(I,\ R_x^{-1}R_{x\delta^{-1}}, R_x^{-1}R_{x\delta^{-1}}\big)\in AUT(Q,\cdot)\Leftrightarrow y\cdot zR_x^{-1}R_{x\delta^{-1}} = (yz)R_x^{-1}R_{x\delta^{-1}}\Leftrightarrow$$
$$y\cdot((z/x)\cdot x\delta^{-1}) = ((yz)/x)\cdot x\delta^{-1}.$$

The rest follows by Definition 1.1(2). □

Corollary 2.6. *Let $(Q,\cdot)$ be a loop with $A(Q)$-holomorph $(H,\circ)$. Then, $(H,\circ)$ is a right Basarab loop if and only if $(Q,\cdot)$ is a right Basarab loop and any of the following equivalent conditions holds:*

1. *$R_x^{-1}R_{x\delta}\in\mathcal{P}(Q,\cdot)$ for all $x\in Q$ and for all $\delta\in A(Q)$.*
2. *$x^\lambda\cdot x\delta\in N_\rho(Q,\cdot)$ for all $x\in Q$ and for all $\delta\in A(Q)$.*

(3) *$\delta = J_\lambda M_n$ for all $\delta \in A(Q)$ and some $n \in N_\rho(Q,\cdot)$.*
(4) *$A(Q) \leq \mathcal{P}(Q,\cdot)$.*

Proof. Let $(H,\circ)$ be a right Basarab loop, then $\{I\}\times Q \leq (H,\circ)$ and $\{I\}\times Q \cong (Q,\cdot)$. Hence, $(Q,\cdot)$ is a right Basarab loop. Thus, by Lemma 2.3, $R_x^{-1}R_{x\delta} \in \mathcal{P}(Q,\cdot)$ for all $x \in Q$ and for all $\delta \in A(Q)$.

Conversely, if $(Q,\cdot)$ is a right Basarab loop and 1. holds, then by Lemma 2.3, $(H,\circ)$ is a right Basarab loop.

1$\Leftrightarrow$ 2**:** By Theorem 1.2, $R_x^{-1}R_{x\delta} \overset{\psi}{\cong} eR_x^{-1}R_{x\delta} \Leftrightarrow eR_x^{-1}R_{x\delta} \in N_\rho(Q,\cdot) \Leftrightarrow x^\lambda \cdot x\delta \in N_\rho(Q,\cdot)$ for all $x \in Q$ and for all $\delta \in A(Q)$.

2$\Leftrightarrow$ 3**:** $x^\lambda \cdot x\delta \in N_\rho(Q,\cdot) \Leftrightarrow x^\lambda \cdot x\delta = n \Leftrightarrow x\delta = nL_{x^\lambda}^{-1} = x^\lambda M_n \Leftrightarrow \delta = J_\lambda M_n$ for all $\delta \in A(Q)$ and some $n \in N_\rho(Q,\cdot)$.

1$\Leftrightarrow$ 4**:** For all $x \in Q$ and for all $\delta \in A(Q)$, $R_x^{-1}R_{x\delta} \in \mathcal{P}(Q,\cdot) \Leftrightarrow R_{x\delta} \in R_x\mathcal{P}(Q,\cdot) \Leftrightarrow R_{x\delta} = R_x\rho$ for some $\rho \in \mathcal{P}(Q,\cdot) \Leftrightarrow (I,\delta,\rho) \in AUT(Q,\cdot) \Leftrightarrow \delta \in \mathcal{P}(Q,\cdot) \Leftrightarrow A(Q) \leq \mathcal{P}(Q,\cdot)$.

□

Remark 2.2. *From Corollary 2.6, it can be deduced that the $\mathcal{P}(Q,\cdot)$-holomorph $H(Q,\cdot)$ of a right Basarab loop $(Q,\cdot)$ is precisely a right Basarab loop.*

Corollary 2.7. *Let $(Q,\cdot)$ be a loop with $A(Q)$-holomorph $(H,\circ)$. Then, $(H,\circ)$ is a RIP right Basarab loop if and only if $(Q,\cdot)$ is a RIP right Basarab loop and any of the following equivalent conditions holds:*

(1) *$R_x^{-1}R_{x\delta}$ is μ-regular with an adjoint $(R_x^{-1}R_{x\delta})' = JR_{x\delta}^{-1}R_xJ$ for all $x \in Q$ and for all $\delta \in A(Q)$.*
(2) *$R_x^{-1}R_{x\delta} \in \Phi(Q,\cdot)$ and $(R_x^{-1}R_{x\delta})' = JR_{x\delta}^{-1}R_xJ \in \Psi(Q,\cdot)$ for all $x \in Q$ and for all $\delta \in A(Q)$.*
(3) *$x^\lambda \cdot x\delta,\ x^\lambda\delta \cdot x \in N_\mu(Q,\cdot)$ for all $x \in Q$ and for all $\delta \in A(Q)$.*
(4) *$\delta = J_\lambda M_{n_1} = \delta = M_{n_2}^{-1}J_\rho$ for all $\delta \in A(Q)$ and some $n_1, n_2 \in N_\mu(Q,\cdot)$.*
(5) *$A(Q) \subseteq J\Psi(Q,\cdot)J$, $J\Psi(Q,\cdot)J \subseteq \Phi(Q,\cdot)$ and $(J\Psi(Q,\cdot)J)' \subseteq JA(Q)J$.*

Proof. It should first be noted that $(H,\circ)$ is a RIP loop if and only if $(Q,\cdot)$ is a RIP. Following Corollary 2.6, $(H,\circ)$ is a right Basarab loop if and only if $(Q,\cdot)$ is a right Basarab loop and $R_x^{-1}R_{x\delta} \in \mathcal{P}(Q,\cdot)$ for all $x \in Q$ and for all $\delta \in A(Q)$.

In a RIPL, $(U,V,W) \in AUT(Q,\cdot) \Rightarrow (W,JVJ,U) \in AUT(Q,\cdot)$. So, for all $x \in Q$ and for all $\alpha \in A(Q)$:

$$R_x^{-1}R_{x\delta} \in \mathcal{P}(Q,\cdot) \Leftrightarrow (I, R_x^{-1}R_{x\delta}, R_x^{-1}R_{x\delta}) \in AUT(Q,\cdot) \Leftrightarrow$$

$$\big(R_x^{-1}R_{x\delta}, JR_x^{-1}R_{x\delta}J, I\big) \in AUT(Q,\cdot)$$

$\Leftrightarrow R_x^{-1}R_{x\delta}$ is μ-regular with an adjoint $(R_x^{-1}R_{x\delta})' = JR_{x\delta}^{-1}R_xJ$.

1$\Leftrightarrow$ 2**:** This is straightforward.

2$\Leftrightarrow$ 3**:** By Theorem 1.2, $R_x^{-1}R_{x\delta} \overset{\sigma}{\cong} eR_x^{-1}R_{x\delta} \Leftrightarrow eR_x^{-1}R_{x\delta} \in N_\mu(Q,\cdot) \Leftrightarrow x^\lambda \cdot x\delta \in N_\mu(Q,\cdot)$ for all $x \in Q$ and for all $\delta \in A(Q)$.

Furthermore, $JR_{x\delta}^{-1}R_xJ \overset{\beta}{\cong} eJR_{x\delta}^{-1}R_xJ \Leftrightarrow eJR_{x\delta}^{-1}R_xJ \in N_\mu(Q,\cdot) \Leftrightarrow x^\lambda\delta \cdot x \in N_\mu(Q,\cdot)$ for all $x \in Q$ and for all $\delta \in A(Q)$.

3$\Leftrightarrow$ **4:** $x^\lambda \cdot x\delta \in N_\mu(Q,\cdot) \Leftrightarrow x^\lambda \cdot x\delta = n_1 \Leftrightarrow x\delta = n_1 L_{x^\lambda}^{-1} = x^\lambda M_{n_1} \Leftrightarrow \delta = J_\lambda M_{n_1}$ for all $\delta \in A(Q)$ and some $n_1 \in N_\mu(Q,\cdot)$.

Furthermore, $x^\lambda\delta \cdot x \in N_\mu(Q,\cdot) \Leftrightarrow x^\lambda\delta \cdot x = n_2 \Leftrightarrow x^\lambda\delta = n_2 R_x^{-1} = xM_{n_2}^{-1} \Leftrightarrow \delta = M_{n_2}^{-1}J_\rho$ for all $\delta \in A(Q)$ and some $n_2 \in N_\mu(Q,\cdot)$.

1$\Leftrightarrow$ **5:** For all $x \in Q$ and for all $\delta \in A(Q)$, $R_x^{-1}R_{x\delta} \in \Phi(Q,\cdot) \Leftrightarrow R_x^{-1}R_{x\delta} = \phi$ for some $\phi \in \Phi(Q,\cdot) \Leftrightarrow R_{x\delta} = R_x\phi \Leftrightarrow (I,\delta,\phi) \in AUT(Q,\cdot) \Leftrightarrow (\phi, J\delta J, I) \in AUT(Q,\cdot) \Leftrightarrow \phi \in \Phi(Q,\cdot)$ and $\phi' = J\delta^{-1}J \in \Psi(Q,\cdot)$ for some $\phi \in \Phi(Q,\cdot) \Leftrightarrow A(Q) \subseteq J\Psi(Q,\cdot)J$.

Furthermore, for all $x \in Q$ and for all $\delta \in A(Q)$, $(R_x^{-1}R_{x\delta})' = JR_{x\delta}^{-1}R_xJ \in \Psi(Q,\cdot) \Leftrightarrow JR_{x\delta}^{-1}R_xJ = \psi$ for some $\psi \in \Psi(Q,\cdot) \Leftrightarrow R_{x\delta}^{-1}R_x = J\psi J$ for some $\psi \in \Psi(Q,\cdot) \Leftrightarrow R_x = R_{x\delta}^{-1}J\psi J$ for some $\psi \in \Psi(Q,\cdot) \Leftrightarrow (I,\delta,J\psi^{-1}J) \in AUT(Q,\cdot) \Leftrightarrow (J\psi^{-1}J, J\delta J, I) \in AUT(Q,\cdot) \Leftrightarrow J\psi^{-1}J \in \Phi(Q,\cdot)$ and $(J\psi^{-1}J)' = J\delta J \in \Psi(Q,\cdot) \Leftrightarrow J\Phi(Q,\cdot)J \subseteq \Psi(Q,\cdot)$ and $(J\Psi(Q,\cdot)J)' \subseteq JA(Q)J \subseteq \Psi(Q,\cdot)$.

□

Corollary 2.8. *Let $(Q,\cdot)$ be a loop with $A(Q)$-holomorph $(H,\circ)$. If $(H,\circ)$ is a RIP right Basarab loop, then*

$$\begin{tikzcd} & x^\lambda \cdot x\delta,\ x^\lambda\delta \cdot x \\ R_x^{-1}R_{x\delta} \arrow[ur, mapsto, "\sigma"] \arrow[r, mapsto, "\varphi", "isomorphism"'] & JR_{x\delta}^{-1}R_xJ \arrow[u, mapsto, "\beta"] \end{tikzcd} \in \begin{tikzcd} & N_\mu(Q,\cdot) \\ \Phi(Q,\cdot) \arrow[ur, "\sigma"] \arrow[r, "\varphi", "isomorphism"'] & \Psi(Q,\cdot) \arrow[u, "\beta", "isomorphism"'] \end{tikzcd} \tag{7}$$

for all $x \in Q$, $\delta \in A(Q)$. i.e. $\sigma = \varphi\beta$.

Proof. This follows from Corollary 2.7. □

Lemma 2.4. *Let $(Q,\cdot)$ be a loop with $A(Q)$-holomorph $(H,\circ)$. If $(H,\circ)$ is a right Basarab loop then for all $x,y \in Q$ and $\delta \in A(Q)$, the following are true:*

(1) $y \cdot (x^\lambda \cdot x\delta^{-1}) = (y/x) \cdot x\delta^{-1}$.
(2) $x^\lambda \cdot x\delta^{-1} \in N_\rho(Q,\cdot)$.
(3) $yx \cdot (x^\lambda \cdot x\delta^{-1}) = y \cdot x\delta^{-1}$.
(4) $x \cdot (x^\lambda \cdot x\delta^{-1}) = x\delta^{-1}$.
(5) $\big(y, x, x^\lambda \cdot x\gamma\big) = e$.
(6) $R_{x^\lambda \cdot x\delta^{-1}} = R_x^{-1}R_{x\delta^{-1}}$.
(7) $R_{x^\lambda \cdot x\delta^{-1}} \in \mathcal{P}(Q,\cdot)$.

Proof. $(H,\circ)$ is a right Basarab loop implies

$$(R_x, L_{x^\lambda}R_{x\delta^{-1}}, R_{x\delta^{-1}}) \in AUT(Q) \Rightarrow \forall\, y,z \in Q,\ yR_x \cdot zL_{x^\lambda}R_{x\delta^{-1}} = (yz)R_{x\delta^{-1}}.$$

By Corollary 2.6, $(Q,\cdot)$ is a right Basarab loop, hence, $L_{x^\lambda} = R_xL_x^{-1}R_x^{-1}$ which implies $R_x^{-1} = L_xR_x^{-1}L_{x^\lambda}$. Again by Lemma 2.3, $(H,\circ)$ is a right Basarab loop if and only if

$$(I,\ R_x^{-1}R_{x\delta^{-1}}, R_x^{-1}R_{x\delta^{-1}}) \in AUT(Q,\cdot).$$

(1) So, for all $x,y,z \in Q$, $yI \cdot zR_x^{-1}R_{x\delta^{-1}} = (yz)R_x^{-1}R_{x\delta^{-1}}$ implies for all $x,y,z \in Q$.

$$yI \cdot zL_xR_x^{-1}L_{x^\lambda}R_{x\delta^{-1}} = (yz)R_x^{-1}R_{x\delta^{-1}} \Rightarrow yI \cdot zT_x^{-1}L_{x^\lambda}R_{x\delta^{-1}} = (yz)R_x^{-1}R_{x\delta^{-1}}.$$

Thus, with $z = e$, we have

$$yI \cdot eT_x^{-1}L_{x^\lambda}R_{x\delta^{-1}} = (ye)R_x^{-1}R_{x\delta^{-1}} \Rightarrow yI \cdot eL_{x^\lambda}R_{x\delta^{-1}} = yR_x^{-1}R_{x\delta^{-1}}$$
$$\Rightarrow y \cdot (x^\lambda \cdot x\delta^{-1}) = (y/x) \cdot x\delta^{-1}.$$

(2) From 1, $yR_{x^\lambda \cdot x\delta^{-1}} = yR_x^{-1}R_{x\delta^{-1}} \Rightarrow R_{x^\lambda \cdot x\delta^{-1}} = R_x^{-1}R_{x\delta^{-1}}$. Thus,
$(I,\ R_x^{-1}R_{x\delta^{-1}}, R_x^{-1}R_{x\delta^{-1}}) = (I, R_{(x^\lambda \cdot \delta^{-1})}, R_{(x^\lambda \cdot \delta^{-1})}) \in AUT(Q) \Rightarrow x^\lambda \cdot x\delta^{-1} \in N_\rho(Q, \cdot)$.

(3) From 1, $y \cdot (x^\lambda \cdot x\delta^{-1}) = (y/x) \cdot x\delta^{-1}$. Do $y \mapsto yx$ to get
$$yx \cdot (x^\lambda \cdot x\delta^{-1}) = (yx/x) \cdot x\delta^{-1} \Rightarrow yx \cdot (x^\lambda \cdot x\delta^{-1}) = y \cdot x\delta^{-1}.$$

(4) Use $y = e$ in 3 to get $x \cdot (x^\lambda \cdot x\delta^{-1}) = x\delta^{-1}$.

(5) By 3 and 4, $yx \cdot (x^\lambda \cdot x\delta^{-1}) = y \cdot x\delta^{-1} = y \cdot x(x^\lambda \cdot x\delta^{-1})$ and this implies $\big(y, x, x^\lambda \cdot x\gamma\big) = e$.

(6) This is gotten from the proof of 2.

(7) This is gotten from the proof of 2.

□

2.3. **Holomorphy of Basarab loop.**

Corollary 2.9. *Let $(Q, \cdot)$ be a loop with $A(Q)$-holomorph $(H, \circ)$. $(H, \circ)$ is a Basarab loop if and only if any of the following is true:*

(1) $(R_{x^\rho}L_{x\alpha}, L_x, L_{x\alpha}), (R_{x\alpha}, L_{x^\lambda\alpha}R_x, R_x) \in AUT(Q, \cdot)$ *for all $x \in Q$ and $\alpha \in A(Q)$.*

(2) $(R_x, L_{x^\lambda}R_{x\alpha^{-1}}, R_{x\alpha^{-1}}), (R_{x^\rho\alpha^{-1}}L_x, L_{x\alpha^{-1}}, L_x)$ *for all $x \in Q$ and for all $\alpha \in A(Q)$.*

(3) $(x\alpha \cdot yx^\rho) \cdot xz = x\alpha \cdot yz$ *and* $(y \cdot x\alpha) \cdot ((x^\lambda\alpha \cdot z) \cdot x) = yz \cdot x$ *for all $x, y, z \in Q$ and for all $\alpha \in A(Q)$.*

Proof. The result follows from Corollaries 2.1, 2.5 and Theorems 2.1, 2.2. □

Lemma 2.5. *Let $(Q, \cdot)$ be a Basarab loop with $A(Q)$-holomorph $(H, \circ)$. Then, $(H, \circ)$ is a Basarab loop if and only if any of the following is true:*

(1) $(x\alpha \cdot (x\backslash y)) \cdot z = x\alpha \cdot (x\backslash(yz))$ *and* $y \cdot ((z/x) \cdot x\delta^{-1}) = ((yz)/x) \cdot x\delta^{-1}$ *holds for for all $x, y, z \in Q$ and for all $\alpha, \delta \in A(Q)$.*

(2) $(L_x^{-1}L_{x\alpha}, I, L_x^{-1}L_{x\alpha}), (I,\ R_x^{-1}R_{x\delta^{-1}}, R_x^{-1}R_{x\delta^{-1}}) \in AUT(Q, \cdot)$ *for all $x \in Q$ and for all $\alpha, \delta \in A(Q)$.*

(3) $L_x^{-1}L_{x\alpha}$ *is λ-regular and* $R_x^{-1}R_{x\delta^{-1}}$ *is ρ-regular for all $x \in Q$ and for all $\alpha, \delta \in A(Q)$.*

(4) $L_x^{-1}L_{x\alpha} \in \Lambda(Q, \cdot)$ *and* $R_x^{-1}R_{x\delta^{-1}} \in \mathcal{P}(Q, \cdot)$ *for all $x \in Q$ and for all $\alpha, \delta \in A(Q)$.*

Proof. This follows from Lemmas 2.1 and 2.3. □

Corollary 2.10. *Let $(Q, \cdot)$ be a loop with $A(Q)$-holomorph $(H, \circ)$. Then, $(H, \circ)$ is a Basarab loop if and only if $(Q, \cdot)$ is a Basarab loop and any of the following equivalent conditions holds:*

(1) $L_x^{-1}L_{x\alpha} \in \Lambda(Q, \cdot)$ *and* $R_x^{-1}R_{x\delta} \in \mathcal{P}(Q, \cdot)$ *for all $x \in Q$ and for all $\alpha, \delta \in A(Q)$.*

(2) $x\alpha \cdot x^{\rho} \in N_{\lambda}(Q,\cdot)$, $x^{\lambda} \cdot x\delta \in N_{\rho}(Q,\cdot)$ *for all* $x \in Q$ *and for all* $\alpha, \delta \in A(Q)$.
(3) $\alpha = J_{\rho}M_n^{-1}$ *and* $\delta = J_{\lambda}M_{n'}$ *for all* $\alpha \in A(Q)$ *and some* $n \in N_{\lambda}(Q,\cdot)$, $n' \in N_{\rho}(Q,\cdot)$.
(4) $A(Q) \leq \Lambda(Q,\cdot) \cap \mathcal{P}(Q,\cdot)$.

Proof. This follows by Corollaries 2.2, 2.6. □

Remark 2.3. *From Corollary 2.10, it can be deduced that the* $\Lambda(Q,\cdot) \cap \mathcal{P}(Q,\cdot)$*-holomorph* $H(Q,\cdot)$ *of a Basarab loop* $(Q,\cdot)$ *is precisely a Basarab loop.*

Lemma 2.6. *Let* $(Q,\cdot)$ *be a loop with* $A(Q)$*-holomorph* $(H,\circ)$.

(1) *Then,* $(H,\circ)$ *is a Basarab loop if and only if* Q *for all* $x, y \in Q$ *and* $\alpha \in A(Q)$*:*

$$(L_{x\alpha}R_x^{-1}, L_x, L_{x\alpha}),\ (R_{x\alpha}, R_xL_{x\alpha}^{-1}, R_x) \in AUT(Q,\cdot) \text{ and}$$
$$(x\alpha \cdot yx^{\rho})x = x\alpha \cdot y, \ \ x\alpha \cdot (x^{\lambda}\alpha \cdot y)x = yx.$$

(2) $(H,\circ)$ *is a Basarab loop if and only if for all* $x \in Q$ *and* $\delta \in A(Q)$*:*

$$(R_x, R_{x\delta^{-1}}L_x^{-1}, R_{x\delta^{-1}}),\ (L_xR_{x\delta^{-1}}^{-1}, L_{x\delta^{-1}}, L_x) \in AUT(Q,\cdot) \text{ and}$$
$$x(x^{\lambda}y \cdot x\delta^{-1}) = y \cdot x\delta^{-1}, \ \ x(y \cdot x^{\rho}\delta^{-1}) \cdot x\delta^{-1} = xy.$$

Proof. (1) Let $(R_{x^{\rho}}L_{x\alpha}, L_x, L_{x\alpha}) \in AUT(Q,\cdot)$. Then, for all $x, y, z \in Q$,

$$\begin{aligned} yR_{x^{\rho}}L_{x\alpha} \cdot zL_x = (yz)L_{x\alpha} &\Rightarrow (x\alpha \cdot yx^{\rho}) \cdot xz = x\alpha(yz) \ \ (\text{set } z = e) \\ &\Rightarrow (x\alpha \cdot yx^{\rho}) \cdot x = (x\alpha)y \Rightarrow yR_{x^{\rho}}L_{x\alpha}R_x = yL_{x\alpha} \\ &\Rightarrow R_{x^{\rho}}L_{x\alpha}R_x = L_{x\alpha} \Rightarrow R_{x^{\rho}}L_{x\alpha} = L_{x\alpha}R_x^{-1} \\ &\Rightarrow R_{x^{\rho}}L_{x\alpha}R_x = L_{x\alpha} \Rightarrow (x\alpha \cdot yx^{\rho})x = x\alpha \cdot y. \end{aligned}$$

Thus, $(Rx^{\rho}L_{x\alpha}, L_x, L_{x\alpha}) \in AUT(Q) \Rightarrow (L_{x\alpha}R_x^{-1}, L_x, L_{x\alpha}) \in AUT(Q)$. Also, let $(R_{x\alpha}, L_{x^{\lambda}\alpha}R_x, R_x) \in AUT(Q)$. Then for all $x, y, z \in Q$,

$$\begin{aligned} yR_{x\alpha} \cdot zL_{x^{\lambda}\alpha}R_x = (yz)R_x &\Rightarrow (y \cdot x\alpha) \cdot (x^{\lambda}\alpha)z \cdot x = yz \cdot x \ \ (\text{set } y = e) \\ &\Rightarrow x\alpha \cdot (x^{\lambda}\alpha \cdot z)x = zx \Rightarrow zL_{x^{\lambda}\alpha}R_xL_{x\alpha} = zR_x \\ &\Rightarrow L_{x^{\lambda}\alpha}R_xL_{x\alpha} = R_x \Rightarrow L_{x^{\lambda}\alpha}R_x = R_xL_{x\alpha}^{-1} \\ &\Rightarrow L_{x^{\lambda}\alpha}R_xL_{x\alpha} = R_x \Rightarrow x\alpha \cdot (x^{\lambda}\alpha \cdot y)x = yx. \end{aligned}$$

Thus, $(R_{x\alpha}, L_{x^{\lambda}\alpha}R_x, R_x) \in AUT(Q) \Rightarrow (R_{x\alpha}, R_xL_{x\alpha}^{-1}, R_x) \in AUT(Q)$.
The converse follows by doing the reverse in each case.

(2) This can be gotten from 1.

□

Theorem 2.3. *Let* $(Q,\cdot)$ *be a loop with* $A(Q)$*-holomorph* $(H,\circ)$.

(1) $(H,\circ)$ *is a Basarab loop if and only if*
   (a) $(Q,\cdot)$ *Basarab loop;*
   (b) $(L_{x\alpha}L_x^{-1}, I, L_{x\alpha}L_x^{-1}), (R_x^{-1}R_{x\alpha}, L_xL_{x\alpha}^{-1}, I) \in AUT(Q)$*; and*
   (c) $(x\alpha \cdot yx^{\rho})x = x\alpha \cdot y$, $x\alpha \cdot (x^{\lambda}\alpha \cdot y)x = yx$.
   for all $x, y, z \in Q$ *and* $\alpha \in A(Q)$.
(2) $(H,\circ)$ *is a Basarab loop if and only if*

(a) $(Q,\cdot)$ *Basarab loop;*
(b) $(I,R_{x\delta^{-1}}R_x^{-1},R_{x\delta^{-1}}R_x^{-1}),(R_xR_{x\delta^{-1}}^{-1},L_x^{-1}L_{x\delta^{-1}},I)\in AUT(Q,\cdot)$*; and*
(c) $x(x^\lambda y\cdot x\delta^{-1})=y\cdot x\delta^{-1},\ \ x(y\cdot x^\rho\delta^{-1})\cdot x\delta^{-1}=xy.$
for all $x\in Q$ *and* $\delta\in A(Q)$.

Proof. (1) By Lemma 2.6(1), $(H,\circ)$ is a Basarab loop implies $(L_{x\alpha}R_x^{-1},L_x,L_{x\alpha})$ and $(R_{x\alpha},R_xL_{x\alpha}^{-1},R_x)$ are autotopisms of Q for all $x,y,z\in Q$ and $\alpha\in A(Q)$. Consequently, $(Q,\cdot)$ is a Basarab loop, hence $(L_xR_x^{-1},L_x,L_x)$ and $(R_x,R_xL_x^{-1},R_x)$ are autotopisms of Q for all $x\in Q$. Let

$$C=(L_{x\alpha}R_x^{-1},L_x,L_{x\alpha}),\ D=(R_{x\alpha},R_xL_{x\alpha}^{-1},R_x),\ E=(L_xR_x^{-1},L_x,L_x),$$
$$F=(R_x,R_xL_x^{-1},R_x).$$

$$C\in AUT(Q)\Rightarrow C^{-1}=(R_xL_{x\alpha}^{-1},L_x^{-1},L_{x\alpha}^{-1})\in AUT(Q)$$
$$D\in AUT(Q)\Rightarrow D^{-1}=(R_{x\alpha}^{-1},L_{x\alpha}R_x^{-1},R_x^{-1})\in AUT(Q)$$
$$E\in AUT(Q)\Rightarrow E^{-1}=(R_xL_x^{-1},L_x^{-1},L_x^{-1})\in AUT(Q)$$
$$F\in AUT(Q)\Rightarrow F^{-1}=(R_x^{-1},L_xR_x^{-1},R_x^{-1})\in AUT(Q)$$

Thus, $(H,\circ)$ is a Basarab loop implies

$$F^{-1}D=(R_x^{-1},L_xR_x^{-1},R_x^{-1})(R_{x\alpha},R_xL_{x\alpha}^{-1},R_x)$$
$$=(R_x^{-1}R_{x\alpha},L_xR_x^{-1}R_xL_{x\alpha}^{-1},R_x^{-1}R_x)=(R_x^{-1}R_{x\alpha},L_xL_{x\alpha}^{-1},I)$$
$$\text{and } CE^{-1}=(L_{x\alpha}R_x^{-1},L_x,L_{x\alpha})(R_xL_x^{-1},L_x^{-1},L_x^{-1})$$
$$=(L_{x\alpha}R_x^{-1}R_xL_x^{-1},L_xL_x^{-1},L_{x\alpha}L_x^{-1})=(L_{x\alpha}L_x^{-1},I,L_{x\alpha}L_x^{-1})$$

are autotopisms of Q for all $x\in Q$ and $\alpha\in A$. For the converse, do the reverse and then use Lemma 2.6(1).

(2) This is similar to 1. We shall show some of the basic steps involved.
Going by Lemma 2.6(2),

$$C_*=(L_xR_{x\delta^{-1}}^{-1},L_{x\delta^{-1}},L_x)\text{ and }D_*=(R_x,R_{x\delta^{-1}}L_x^{-1},R_{x\delta^{-1}}).\text{ So,}$$
$$C_*\in AUT(Q)\Rightarrow C_*^{-1}=(R_{x\delta^{-1}}L_x^{-1},L_{x\delta^{-1}}^{-1},L_x^{-1})\in AUT(Q)\text{ and}$$
$$D_*\in AUT(Q)\Rightarrow D_*^{-1}=(R_x^{-1},L_xR_{x\delta^{-1}}^{-1},R_{x\delta^{-1}}^{-1})\in AUT(Q).$$

Thus, $(H,\circ)$ is a Basarab loop implies that

$$D_*F^{-1}=(R_x,R_{x\delta^{-1}}L_x^{-1},R_{x\delta^{-1}})(R_x^{-1},L_xR_x^{-1},R_x^{-1})$$
$$=(R_xR_x^{-1},R_{x\delta^{-1}}L_x^{-1}L_xR_x^{-1},R_{x\delta^{-1}}R_x^{-1})$$
$$=(I,R_{x\delta^{-1}}R_x^{-1},R_{x\delta^{-1}}R_x^{-1})$$

Also,

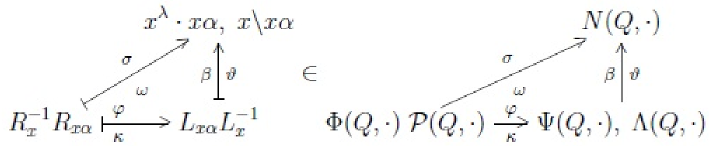
$$E^{-1}C_*=(R_xL_x^{-1},L_x^{-1},L_x^{-1})(L_xR_{x\delta^{-1}}^{-1},L_{x\delta^{-1}},L_x)$$
$$=(R_xL_x^{-1}L_xR_{x\delta^{-1}}^{-1},L_x^{-1}L_{x\delta^{-1}},I)$$
$$=(R_xR_{x\delta^{-1}}^{-1},L_x^{-1}L_{x\delta^{-1}},I)$$

are autotopisms of Q for all $x\in Q$ and $\delta\in A(Q)$.
For the converse, do the reverse and then use Lemma 2.6(2).

□

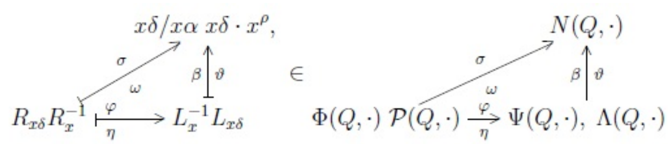
Corollary 2.11. *Let $(Q,\cdot)$ be a loop with $A(Q)$-holomorph $(H,\circ)$. If $(H,\circ)$ is a Basarab loop, then*

(1) *for all $x\in Q$, $\alpha\in A(Q)$, $\sigma=\varphi\beta$ and $\kappa=\sigma\vartheta^{-1}$, $R_x^{-1}R_{x\alpha}\overset{\kappa}{\approxeq}L_{x\alpha}L_x^{-1}$ so that*



(8)

(2) *for all $x\in Q$, $\delta\in A(Q)$, $\sigma=\varphi\beta$ and $\eta=\sigma\vartheta^{-1}$, $R_{x\delta}R_x^{-1}\overset{\eta}{\approxeq}L_x^{-1}L_{x\delta}$ so that*



(9)

Proof. (1) This follows from Theorem 2.3(1) and Corollary 2.6(1,2).
(2) This follows from Theorem 2.3(2) and Corollary 2.2(1,2). ☐

Theorem 2.4. (1) *Let $(Q,\cdot)$ be a loop with $A(Q)$-holomorph $(H,\circ)$ such that $x^\lambda\cdot x\alpha=(x\alpha\backslash x)^\lambda=x\backslash x\alpha$ for all $x\in Q$ and $\alpha\in A(Q)$. Then, $(H,\circ)$ is a Basarab loop if and only if :*
(a) *$(Q,\cdot)$ is a Basarab loop;*
(b) *every automorphism of Q is left nuclear; and*
(c) *$T_{(x,\alpha)}^{-1}=R_{x^\rho}L_{x\alpha}$ and $T_{(x,\alpha)}=L_{x^\lambda\alpha}R_x$ for all $x\in Q$ and $\alpha\in A(Q)$.*

(2) *Let $(Q,\cdot)$ be a loop with $A(Q)$-holomorph $(H,\circ)$ such that $x\delta^{-1}\cdot x^\rho=(x/x\delta^{-1})^\rho=x\delta^{-1}/x$ for all $x\in Q$ and $\delta\in A(Q)$. Then, $(H,\circ)$ is a Basarab loop if and only if :*
(a) *$(Q,\cdot)$ is a Basarab loop;*
(b) *every automorphism of Q is right nuclear; and*
(c) *$T_{(x\delta^{-1},\delta)}^{-1}=R_{x^\rho\delta^{-1}}L_x$ and $T_{(x\delta^{-1},\delta)}=L_{x^\lambda}R_{x\delta^{-1}}$ for all $x\in Q$ and $\delta\in A(Q)$.*

Proof. (1) Let $(H,\circ)$ be a Basarab loop. By Lemma 2.6(1) and Theorem 2.3(1), it follows that

$$C=(L_{x\alpha}R_x^{-1},L_x,L_{x\alpha}),\ D=(R_{x\alpha},R_xL_{x\alpha}^{-1},R_x)\in AUT(Q),$$
$$K=(R_x^{-1}R_{x\alpha},L_xL_{x\alpha}^{-1},I),\ P=(L_{x\alpha}L_x^{-1},I,L_{x\alpha}L_x^{-1})\in AUT(Q),$$
$$(x\alpha\cdot yx^\rho)x=x\alpha\cdot y\ \text{ and }\ x\alpha\cdot(x^\lambda\alpha\cdot y)x=yx \tag{10}$$

From (10), (c) is true. On the other hand,

$$P^{-1}C=(L_xL_{x\alpha}^{-1},I,L_xL_{x\alpha}^{-1})(L_{x\alpha}R_x^{-1},L_x,L_{x\alpha})=(L_xR_x^{-1},L_x,L_x)$$
and $DK^{-1}=(R_{x\alpha},R_xL_{x\alpha}^{-1},R_x)(R_{x\alpha}^{-1}R_x,L_{x\alpha}L_x^{-1},I)=(R_x,R_xL_x^{-1},R_x)$.

Then $P^{-1}C$ and DK^{-1} are autotopisms of (Q), hence $(Q,\cdot)$ is a Basarab loop and so (a) is true. Since $(H,\circ)$ is a Basarab loop, the autotopism $K=(R_x^{-1}R_{x\alpha},L_xL_{x\alpha}^{-1},I)$ implies for all $x,u,v\in Q$, $uR_x^{-1}R_{x\alpha}\cdot vL_xL_{x\alpha}^{-1}=(uv)I$.

If $v=e$, then

$$\begin{aligned} uR_x^{-1}R_{x\alpha}\cdot eL_xL_{x\alpha}^{-1}=u &\Rightarrow uR_x^{-1}R_{x\alpha}R_{x\alpha\backslash x}=u &(11)\\ &\Rightarrow R_x^{-1}R_{x\alpha}=R_{x\alpha\backslash x}^{-1} \quad (\text{by } x^\lambda\cdot x\alpha=(x\alpha\backslash x)^\lambda)\\ &\Rightarrow R_x^{-1}R_{x\alpha}=R_{(x^\lambda\cdot x\alpha)} &(12)\end{aligned}$$

Also, if $u=e$ in $uR_x^{-1}R_{x\alpha}\cdot vL_xL_{x\alpha}^{-1}=(uv)I$, then

$$\begin{aligned} eR_x^{-1}R_{x\alpha}\cdot vL_xL_{x\alpha}^{-1}=vI &\Rightarrow vL_xL_{x\alpha}^{-1}L_{(x^\lambda\cdot x\alpha)}=v &(13)\\ &\Rightarrow L_xL_{x\alpha}^{-1}=L_{(x^\lambda\cdot x\alpha)}^{-1}\\ &\Rightarrow L_{(x\alpha)}L_x^{-1}=L_{(x^\lambda\cdot x\alpha)} &(14)\end{aligned}$$

Then by (14), the autotopism P becomes $\psi=(L_{(x^\lambda\cdot x\alpha)},I,L_{(x^\lambda\cdot x\alpha)})$ for all $x\in Q$ and $\alpha\in A$. So,

$$(L_{(x^\lambda\cdot x\alpha)},I,L_{(x^\lambda\cdot x\alpha)})\Rightarrow uL_{(x^\lambda\cdot x\alpha)}\cdot vI=(uv)L_{(x^\lambda\cdot x\alpha)}\Rightarrow(x^\lambda\cdot x\alpha)u\cdot v=(x^\lambda\cdot x\alpha)\cdot(uv)$$

for all $u,v\in Q$. Thus, $x^\lambda\cdot x\alpha\in N$. Therefore, α is left nuclear, hence, (b) is true.

For the converse, assume that (a), (b) and (c) are true. Then, $(Q,\cdot)$ is a Basarab loop, $x^\lambda\cdot x\alpha\in N(Q)$ for all $x\in Q$ and $\alpha\in A(Q)$ and (10) is satisfied.

Now, $((x^{-1}\cdot x\alpha)\cdot u)\cdot v=(x^\lambda\cdot x\alpha)\cdot(u\cdot v)$ for all $u,v\in Q$ and so $\psi=(L_{(x^\lambda\cdot x\alpha)},I,L_{(x^\lambda\cdot x\alpha)})\in AUT(Q)$ for all $x\in Q$. Now,

$$\begin{aligned} x^\lambda\cdot x\alpha&=x^\lambda\cdot x\alpha \quad (\text{by } x^\lambda\cdot x\alpha=x\backslash x\alpha)\\ \Rightarrow x\alpha=x(x^\lambda\cdot x\alpha)&\Rightarrow x\alpha\cdot y=x(x^\lambda\cdot x\alpha)\cdot y \text{ and } y\cdot x\alpha=y\cdot x(x^\lambda\cdot x\alpha) &(15)\\ &\Rightarrow x\alpha\cdot y=x\cdot(x^\lambda\cdot x\alpha)y \text{ and } y\cdot x\alpha=yx\cdot(x^\lambda\cdot x\alpha)\\ &\Rightarrow yL_{x\alpha}=yL_{(x^\lambda\cdot x\alpha)}L_x \text{ and } yR_{x\alpha}=yR_xR_{(x^\lambda\cdot x\alpha)}\\ &\Rightarrow L_{x\alpha}L_x^{-1}=L_{(x^\lambda\cdot x\alpha)} \text{ and } R_x^{-1}R_{x\alpha}=R_{(x^\lambda\cdot x\alpha)} &(16)\end{aligned}$$

Thus, by (16), it follows that $\psi=P=(L_{x\alpha}L_x^{-1},I,L_{x\alpha}L_x^{-1})\in AUT(Q)$. Also,

$$\begin{aligned}(u\cdot v)\cdot(x^\lambda\cdot x\alpha)=&u\cdot[v\cdot(x^\lambda\cdot x\alpha)]\Rightarrow(u\cdot v)=[u\cdot[v\cdot(x^\lambda\cdot x\alpha)]][(x^\lambda\cdot x\alpha)^{-1}]\\ \Rightarrow(u\cdot v)=&[u(x^\lambda\cdot x\alpha)^{-1}(x^\lambda\cdot x\alpha)\cdot[v\cdot(x^\lambda\cdot x\alpha)]][(x^\lambda\cdot x\alpha)^{-1}]\\ =&[u\cdot(x^\lambda\cdot x\alpha)^{-1}]\cdot[(x^\lambda\cdot x\alpha)\cdot[[v\cdot(x^\lambda\cdot x\alpha)](x^\lambda\cdot x\alpha)^{-1}]]\\ =&(u\cdot(x^\lambda\cdot x\alpha)^{-1})\cdot((x^\lambda\cdot x\alpha)\cdot v)\\ =&uR_{(x^\lambda\cdot x\alpha)}^{-1}\cdot vL_{(x^\lambda\cdot x\alpha)}\end{aligned}$$

This implies, $w=(R_{(x^\lambda\cdot x\alpha)}^{-1},L_{(x^\lambda\cdot x\alpha)},I)\in AUT(Q)$

$$\Rightarrow w^{-1}=(R_{(x^\lambda\cdot x\alpha)},L_{(x^\lambda\cdot x\alpha)}^{-1},I)\in AUT(Q).$$

By (16), it follows that $w^{-1} = (R_x^{-1}R_{x\alpha}, L_xL_{x\alpha}^{-1}, I) \in AUT(Q)$. Therefore, $(H, \circ)$ is a Basarab loop going by Theorem 2.3.

(2) This is similar to the proof of 1 by using Lemma 2.6(2) and Theorem 2.3(2).

□

Theorem 2.5. (1) *Let $(Q,\cdot)$ be a loop.*

(a) *If $V_1(Q,\cdot) = \{\alpha \in A(Q,\cdot) \mid P(\alpha, x) = (L_{x\alpha}L_x^{-1}, I, L_{x\alpha}L_x^{-1}) \in AUT(Q,\cdot)\}$, then $V_1(Q,\cdot) \lhd A(Q,\cdot)$.*

(b) *If $(Q,\cdot)$ is a Basarab loop such that $\big(x, x^\lambda, x\alpha\big) = e$ for all $x \in Q$ and $\alpha \in A(Q)$ and $V_2(Q,\cdot) = \{\alpha \in A(Q,\cdot) \mid (L_{x^\lambda\cdot x\alpha}, I, L_{x^\lambda\cdot x\alpha}) \in AUT(Q,\cdot)\}$ $= \big\{\alpha \in A(Q,\cdot) \mid x^\lambda \cdot x\alpha \in N(Q,\cdot)\big\}$. Then $V_1(Q,\cdot) = V_2(Q,\cdot)$.*

(c) *If $(H,\circ)$ is a Basarab loop such that $\big(x, x^\lambda, x\alpha\big) = e$ for all $x \in Q$ and $\alpha \in A(Q)$, then $A(Q,\cdot) = V_1(Q,\cdot) = V_2(Q,\cdot)$.*

(2) *Let $(Q,\cdot)$ be a loop.*

(a) *If $V_3(Q,\cdot) = \{\delta \in A(Q,\cdot) \mid B(\delta, x) =$ $(I, R_{x\delta^{-1}}R_x^{-1}, R_{x\delta^{-1}}R_x^{-1}) \in AUT(Q,\cdot)\}$, then $V_3(Q,\cdot) \lhd A(Q,\cdot)$.*

(b) *If $(Q,\cdot)$ is a Basarab loop such that $(x\delta^{-1}, x^\rho, x) = e$ for all $x \in Q$ and $\delta \in A(Q)$ and $V_4(Q,\cdot) =$ $\big\{\delta \in A(Q,\cdot) \mid \big(I, R_{(x\delta^{-1}\cdot x^\rho)}, R_{(x\delta^{-1}\cdot x^\rho)}\big) \in AUT(Q,\cdot)\big\} =$ $\{\delta \in A(Q,\cdot) \mid x\delta^{-1}\cdot x^\rho \in N(Q,\cdot)\}$. Then $V_3(Q,\cdot) = V_4(Q,\cdot)$.*

(c) *If $(H,\circ)$ is a Basarab loop such that $(x\delta^{-1}, x^\rho, x) = e$ for all $x \in Q$ and $\delta \in A(Q)$, then $A(Q,\cdot) = V_3(Q,\cdot) = V_4(Q,\cdot)$.*

Proof. (1) (a) Let $V_1(Q,\cdot) = \{\alpha \in A(Q,\cdot) \mid (L_{x\alpha}L_x^{-1}, I, L_{x\alpha}L_x^{-1}) \in AUT(Q,\cdot)\}$. With $\alpha = I$, $(L_{xI}L_x^{-1}, I, L_{xI}L_x^{-1}) = (I, I, I) \in AUT(Q,\cdot)$. So, $V_1(Q,\cdot) \neq \emptyset$.
Let $\alpha, \beta \in V_1(Q)$, then

$$P(\beta, x\alpha)P(\alpha, x) = (L_{x\alpha\beta}L_{x\alpha}^{-1}, I, L_{x\alpha\beta}L_{x\alpha}^{-1})(L_{x\alpha}L_x^{-1}, I, L_{x\alpha}L_x^{-1}) =$$
$$(L_{x\alpha\beta}L_{x\alpha}^{-1}L_{x\alpha}L_x^{-1}, I, L_{x\alpha\beta}L_{x\alpha}^{-1}L_{x\alpha}L_x^{-1}) = (L_{x\alpha\beta}L_x^{-1}, I, L_{x\alpha\beta}L_x^{-1}) \in AUT(Q,\cdot)$$
$$\Rightarrow \alpha\beta \in V_1(Q).$$

Next, we show that $\alpha^{-1} \in V_1(Q)$ for all $\alpha \in V_1(Q)$. Let $\alpha \in V_1(Q)$, then

$$\begin{aligned}
P(\alpha, x) =& (L_{x\alpha}L_x^{-1}, I, L_{x\alpha}L_x^{-1}) \in AUT(Q,\cdot) \Rightarrow\\
P(\alpha, x)^{-1} =& (L_xL_{x\alpha}^{-1}, I, L_xL_{x\alpha}^{-1}) \in AUT(Q,\cdot) \Rightarrow\\
P(\alpha, x\alpha^{-1})^{-1} =& \big(L_{x\alpha^{-1}}L_{x\alpha^{-1}\alpha}^{-1}, I, L_{x\alpha^{-1}}L_{x\alpha^{-1}\alpha}^{-1}\big) \in AUT(Q,\cdot) \Rightarrow\\
P(\alpha, x\alpha^{-1})^{-1} =& (L_{x\alpha^{-1}}L_x^{-1}, I, L_{x\alpha^{-1}}L_x^{-1}) \in AUT(Q,\cdot) \Rightarrow \alpha^{-1} \in V_1(Q).
\end{aligned}$$

If $\alpha \in V_1(Q)$ and $\theta \in A(Q)$, then for all $x, a, b \in Q$,

$$P(\alpha, x) = (L_{x\alpha}L_x^{-1}, I, L_{x\alpha}L_x^{-1}) \in AUT(Q, \cdot) \Leftrightarrow x\backslash(x\alpha \cdot a) \cdot b = x\backslash\,(x\alpha \cdot (ab)) \Leftrightarrow$$
$$(x\backslash(x\alpha \cdot a) \cdot b)\,\theta = (x\backslash\,(x\alpha \cdot (ab)))\,\theta \Leftrightarrow$$
$$x\theta\backslash(x\alpha\theta \cdot a\theta) \cdot b\theta = x\theta\backslash\,(x\alpha\theta \cdot (a\theta \cdot b\theta)) \quad (\text{set } \ x \mapsto x\theta^{-1})$$
$$\Leftrightarrow x\theta^{-1}\theta\backslash(x\theta^{-1}\alpha\theta \cdot a\theta) \cdot b\theta = x\theta^{-1}\theta\backslash\left(x\theta^{-1}\alpha\theta \cdot (a\theta \cdot b\theta)\right) \Leftrightarrow$$
$$x\backslash(x\theta^{-1}\alpha\theta \cdot a\theta) \cdot b\theta = x\backslash\left(x\theta^{-1}\alpha\theta \cdot (a\theta \cdot b\theta)\right) \Leftrightarrow (L_{x\theta^{-1}\alpha\theta}L_x^{-1}, I, L_{x\theta^{-1}\alpha\theta}L_x^{-1}) \in$$
$$AUT(Q, \cdot) \Leftrightarrow P(\theta^{-1}\alpha\theta, x) \in AUT(Q, \cdot) \Leftrightarrow \theta^{-1}\alpha\theta \in V_1(Q).$$

Therefore, $V_1(Q, \cdot) \lhd A(Q, \cdot)$.

(b) Let $\alpha \in V_1(Q)$, then for all $x \in Q$, $P(\alpha, x) = (L_{x\alpha}L_x^{-1}, I, L_{x\alpha}L_x^{-1}) \in AUT(Q, \cdot)$. Consequently,

$$eL_{x\alpha}L_x^{-1} \cdot yI = (e \cdot y)L_{x\alpha}L_x^{-1} \Rightarrow (x\backslash x\alpha)y = yL_{x\alpha}L_x^{-1} \Rightarrow yL_{(x\backslash x\alpha)} = yL_{x\alpha}L_x^{-1} \Rightarrow$$
$$L_{(x\backslash x\alpha)} = L_{x\alpha}L_x^{-1} \Rightarrow L_{x\alpha}L_x^{-1} = L_{(x^\lambda \cdot x\alpha)} \Rightarrow$$
$$P(\alpha, x) = \left(L_{(x^\lambda \cdot x\alpha)}, I, L_{(x^\lambda \cdot x\alpha)}\right) \in AUT(Q, \cdot) \Rightarrow V_1(Q, \cdot) \subseteq V_2(Q, \cdot).$$

Let $\alpha \in V_2(Q)$, then for all $x \in Q$, $\left(L_{(x^\lambda \cdot x\alpha)}, I, L_{(x^\lambda \cdot x\alpha)}\right) \in AUT(Q, \cdot)$. Consequently,

$$x^\lambda \cdot x\alpha = x^\lambda \cdot x\alpha \quad (\text{by } \ x^\lambda \cdot x\alpha = x\backslash x\alpha)$$
$$\Rightarrow x\alpha = x(x^\lambda \cdot x\alpha) \Rightarrow x\alpha \cdot y = x(x^\lambda \cdot x\alpha) \cdot y \Rightarrow x\alpha \cdot y = x \cdot (x^\lambda \cdot x\alpha)y$$
$$\Rightarrow yL_{x\alpha} = yL_{(x^\lambda \cdot x\alpha)}L_x \Rightarrow L_{x\alpha}L_x^{-1} = L_{(x^\lambda \cdot x\alpha)} \Rightarrow$$
$$P(\alpha, x) = (L_{x\alpha}L_x^{-1}, I, L_{x\alpha}L_x^{-1}) \in AUT(Q, \cdot) \Rightarrow V_2(Q, \cdot) \subseteq V_1(Q, \cdot).$$

Thus, $V_1(Q, \cdot) = V_2(Q, \cdot)$.

(c) Going by (14) of Theorem 2.4(1), $x^\lambda \cdot x\alpha \in N(Q)$ for all $x \in Q$ and $\alpha \in A(Q)$. Thus by (a) and (b), $A(Q, \cdot) = V_1(Q, \cdot) = V_2(Q, \cdot)$.

(2) (a) Let $V_3(Q, \cdot) = \{\delta \in A(Q, \cdot) \mid (I, R_{x\delta^{-1}}R_x^{-1}, R_{x\delta^{-1}}R_x^{-1}) \in AUT(Q, \cdot)\}$. With $\delta = I$, $(I, R_{xI}R_x^{-1}, R_{xI}R_x^{-1}) = (I, I, I) \in AUT(Q, \cdot)$. So, $V_3(Q, \cdot) \neq \emptyset$.

Let $\delta, \beta \in V_3(Q)$, then

$$B(\beta, x\delta^{-1})B(\delta, x) = (I, R_{x\delta^{-1}\beta^{-1}}R_{x\delta^{-1}}^{-1}, R_{x\delta^{-1}\beta^{-1}}R_{x\delta^{-1}}^{-1})(I, R_{x\delta^{-1}}R_x^{-1}, R_{x\delta^{-1}}R_x^{-1}) =$$
$$(I, R_{x(\beta\delta)^{-1}}R_{x\delta^{-1}}^{-1}R_{x\delta^{-1}}R_x^{-1}, R_{x(\beta\delta)^{-1}}R_{x\delta^{-1}}^{-1}R_{x\delta^{-1}}R_x^{-1}) =$$
$$(I, R_{x(\beta\delta)^{-1}}R_x^{-1}, R_{x(\beta\delta)^{-1}}R_x^{-1}) \in AUT(Q, \cdot) \Rightarrow \beta\delta \in V_3(Q).$$

Next, we show that $\delta^{-1} \in V_3(Q)$ for all $\delta \in V_3(Q)$. Let $\delta \in V_3(Q)$, then

$$\begin{aligned}
B(\delta, x) =& (I, R_{x\delta^{-1}}R_x^{-1}, R_{x\delta^{-1}}R_x^{-1}) \in AUT(Q, \cdot) \Rightarrow\\
B(\delta, x)^{-1} =& (I, R_xR_{x\delta^{-1}}^{-1}, R_xR_{x\delta^{-1}}^{-1}) \in AUT(Q, \cdot) \Rightarrow\\
B(\delta, x\delta)^{-1} =& (I, R_{x\delta}R_{x\delta\delta^{-1}}^{-1}, R_{x\delta}R_{x\delta\delta^{-1}}^{-1}) \in AUT(Q, \cdot) \Rightarrow\\
B(\delta, x\delta)^{-1} =& \left(I, R_{x(\delta^{-1})^{-1}}R_x^{-1}, R_{x(\delta^{-1})^{-1}}R_x^{-1}\right) \in AUT(Q, \cdot) \Rightarrow \delta^{-1} \in V_3(Q).
\end{aligned}$$

If $\delta \in V_3(Q)$ and $\theta \in A(Q)$, then for all $x, a, b \in Q$,

$$B(\delta, x) = (I, R_{x\delta^{-1}}R_x^{-1}, R_{x\delta^{-1}}R_x^{-1}) \in AUT(Q, \cdot) \Leftrightarrow$$
$$a \cdot \left(b \cdot x\delta^{-1}\right)/x = \left(ab \cdot x\delta^{-1}\right)/x \Leftrightarrow$$
$$\left(a \cdot \left(b \cdot x\delta^{-1}\right)/x\right)\theta = \left(\left(ab \cdot x\delta^{-1}\right)/x\right)\theta \Leftrightarrow$$
$$a\theta \cdot \left(b\theta \cdot x\delta^{-1}\theta\right)/x\theta = \left((a\theta \cdot b\theta) \cdot x\delta^{-1}\theta\right)/x\theta \quad (\text{set } \ x \mapsto x\theta^{-1})$$
$$\Leftrightarrow a\theta \cdot \left(b\theta \cdot x\theta^{-1}\delta^{-1}\theta\right)/x\theta^{-1}\theta = \left((a\theta \cdot b\theta) \cdot x\theta^{-1}\delta^{-1}\theta\right)/x\theta^{-1}\theta \Leftrightarrow$$
$$a\theta \cdot \left(b\theta \cdot x\left(\theta^{-1}\delta\theta\right)^{-1}\right)/x = \left((a\theta \cdot b\theta) \cdot x\left(\theta^{-1}\delta\theta\right)^{-1}\right)/x \Leftrightarrow$$
$$\left(I, R_{x(\theta^{-1}\delta\theta)^{-1}}R_x^{-1}, R_{x(\theta^{-1}\delta\theta)^{-1}}R_x^{-1}\right) \in AUT(Q, \cdot)$$
$$\Leftrightarrow P(\theta^{-1}\delta\theta, x) \in AUT(Q, \cdot) \Leftrightarrow \theta^{-1}\delta\theta \in V_3(Q).$$

Therefore, $V_3(Q, \cdot) \lhd A(Q, \cdot)$.

(b) Let $\delta \in V_3(Q)$, then for all $x \in Q$, $B(\delta, x) = (I, R_{x\delta^{-1}}R_x^{-1}, R_{x\delta^{-1}}R_x^{-1}) \in AUT(Q, \cdot)$. Consequently,

$$y \cdot eR_{x\delta^{-1}}R_x^{-1} = (y \cdot e)R_{x\delta^{-1}}R_x^{-1} \Rightarrow y(x\delta^{-1}/x) = yR_{x\delta^{-1}}R_x^{-1} \Rightarrow$$
$$yR_{(x\delta^{-1}/x)} = yR_{x\delta^{-1}}R_x^{-1} \Rightarrow$$
$$R_{(x\delta^{-1}/x)} = R_{x\delta^{-1}}R_x^{-1} \Rightarrow R_{x\delta^{-1}}R_x^{-1} = R_{(x\delta^{-1} \cdot x^\rho)} \Rightarrow$$
$$B(\delta, x) = \left(I, R_{(x\delta^{-1} \cdot x^\rho)}, R_{(x\delta^{-1} \cdot x^\rho)}\right) \in AUT(Q, \cdot) \Rightarrow V_3(Q, \cdot) \subseteq V_4(Q, \cdot).$$

Let $\delta \in V_4(Q)$, then for all $x \in Q$, $\left(I, R_{(x\delta^{-1} \cdot x^\rho)}, R_{(x\delta^{-1} \cdot x^\rho)}\right) \in AUT(Q, \cdot)$. Consequently,

$$x\delta^{-1} \cdot x^\rho = x\delta^{-1} \cdot x^\rho \quad (\text{by } \ x\delta^{-1} \cdot x^\rho = x\delta^{-1}/x)$$
$$\Rightarrow (x\delta^{-1} \cdot x^\rho)x = x\delta^{-1} \Rightarrow y \cdot (x\delta^{-1} \cdot x^\rho)x = y \cdot x\delta^{-1} \Rightarrow y(x\delta^{-1} \cdot x^\rho) \cdot x = y \cdot x\delta^{-1}$$
$$\Rightarrow yR_{(x\delta^{-1} \cdot x^\rho)}R_x = yR_{x\delta^{-1}} \Rightarrow R_{x\delta^{-1}}R_x^{-1} = R_{(x\delta^{-1} \cdot x^\rho)} \Rightarrow$$
$$B(\delta, x) = \left(I, R_{(x\delta^{-1} \cdot x^\rho)}, R_{(x\delta^{-1} \cdot x^\rho)}\right) \in AUT(Q, \cdot) \Rightarrow V_4(Q, \cdot) \subseteq V_3(Q, \cdot).$$

Thus, $V_3(Q, \cdot) = V_4(Q, \cdot)$.

(c) Going by Theorem 2.4(2), $x\delta^{-1} \cdot x^\rho \in N(Q)$ for all $x \in Q$ and $\delta \in A(Q)$. Thus by (a) and (b), $A(Q, \cdot) = V_3(Q, \cdot) = V_4(Q, \cdot)$.

□

Theorem 2.6. (1) *Let $(Q, \cdot)$ be a loop.*

(a) *If $W_1(Q, \cdot) = \{\alpha \in A(Q, \cdot) \mid K(\alpha, x) = (R_x^{-1}R_{x\alpha}, L_xL_{x\alpha}^{-1}, I) \in AUT(Q, \cdot)\}$, then $W_1(Q, \cdot) \lhd A(Q, \cdot)$.*

(b) *If $(Q, \cdot)$ is a Basarab loop such that $x^\lambda \cdot x\alpha = (x\alpha\backslash x)^\lambda = x\backslash x\alpha$ for all $x \in Q$ and $\alpha \in A(Q)$ and*

$$W_2(Q, \cdot) = \left\{\alpha \in A(Q, \cdot) \mid (R_x^{-1}R_{x\alpha}, L_xL_{x\alpha}^{-1}, I) \in AUT(Q, \cdot)\right\} =$$
$$\left\{\alpha \in A(Q, \cdot) \mid x^\lambda \cdot x\alpha \in N(Q, \cdot)\right\}. \ \textit{Then, } \ W_1(Q, \cdot) = W_2(Q, \cdot).$$

.

(c) *If $(H, \circ)$ is a Basarab loop such that $x^\lambda \cdot x\alpha = (x\alpha\backslash x)^\lambda = x\backslash x\alpha$ for all $x \in Q$ and $\alpha \in A(Q)$, then $A(Q, \cdot) = W_1(Q, \cdot) = W_2(Q, \cdot)$.*

(2) *Let* $(Q,\cdot)$ *be a loop.*

(a) *If*

$$W_3(Q,\cdot)=\left\{\delta\in A(Q,\cdot)\mid W(\delta,x)=(R_xR^{-1}_{x\delta^{-1}},L^{-1}_xL_{x\delta^{-1}},I)\in AUT(Q,\cdot)\right\},$$

then $V_3(Q,\cdot)\lhd A(Q,\cdot)$.

(b) *If* $(Q,\cdot)$ *is a Basarab loop such that* $x\delta^{-1}\cdot x^\rho=(x/x\delta^{-1})^\rho=x\delta^{-1}/x$ *for all* $x\in Q$ *and* $\delta\in A(Q)$ *and*

$$W_4(Q,\cdot)=\left\{\delta\in A(Q,\cdot)\mid (R_xR^{-1}_{x\delta^{-1}},L^{-1}_xL_{x\delta^{-1}},I)\in AUT(Q,\cdot)\right\}=$$
$$\left\{\delta\in A(Q,\cdot)\mid x\delta^{-1}\cdot x^\rho\in N(Q,\cdot)\right\}.\ \textit{Then, } W_3(Q,\cdot)=W_4(Q,\cdot).$$

(c) *If* $(H,\circ)$ *is a Basarab loop such that* $x\delta^{-1}\cdot x^\rho=(x/x\delta^{-1})^\rho=x\delta^{-1}/x$ *for all* $x\in Q$ *and* $\delta\in A(Q)$*, then* $A(Q,\cdot)=W_3(Q,\cdot)=W_4(Q,\cdot)$.

Proof. (1) (a) Let $W_1(Q,\cdot)=\{\alpha\in A(Q,\cdot)\mid (R_x^{-1}R_{x\alpha},L_xL^{-1}_{x\alpha},I)\in AUT(Q,\cdot)\}$. With $\alpha=I$, $(R_x^{-1}R_{xI},L_xL^{-1}_{xI},I)=(I,I,I)\in AUT(Q,\cdot)$. So, $W_1(Q,\cdot)\neq\emptyset$.

Let $\alpha,\beta\in W_1(Q)$, then

$$K(\alpha,x\alpha)K(\beta,x\alpha)=(R_x^{-1}R_{x\alpha},L_xL^{-1}_{x\alpha},I)(R^{-1}_{x\alpha}R_{x\alpha\beta},L_{x\alpha}L^{-1}_{x\alpha\beta},I)=$$
$$(R_x^{-1}R_{x\alpha}R^{-1}_{x\alpha}R_{x\alpha\beta},L_xL^{-1}_{x\alpha}L_{x\alpha}L^{-1}_{x\alpha\beta},I)=(R_x^{-1}R_{x\alpha\beta},L_xL^{-1}_{x\alpha\beta},I)\in AUT(Q,\cdot)$$
$$\Rightarrow\alpha\beta\in W_1(Q).$$

Next, we show that $\alpha^{-1}\in W_1(Q)$ for all $\alpha\in W_1(Q)$. Let $\alpha\in W_1(Q)$, then

$$K(\alpha,x)=(R_x^{-1}R_{x\alpha},L_xL^{-1}_{x\alpha},I)\in AUT(Q,\cdot)\Rightarrow$$
$$K(\alpha,x)^{-1}=(R^{-1}_{x\alpha}R_x,L_{x\alpha}L_x^{-1},I)\in AUT(Q,\cdot)\Rightarrow$$
$$K(\alpha,x\alpha^{-1})=(R^{-1}_{x\alpha^{-1}}R_{x\alpha^{-1}\alpha},L_{x\alpha^{-1}}L^{-1}_{x\alpha^{-1}\alpha},I)=$$
$$(R^{-1}_{x\alpha^{-1}}R_x,L_{x\alpha^{-1}}L_x^{-1},I)\in AUT(Q,\cdot)\Rightarrow K(\alpha,x\alpha^{-1})^{-1}=$$
$$(R_x^{-1}R_{x\alpha^{-1}},L_xL^{-1}_{x\alpha^{-1}},I)\in AUT(Q,\cdot)\Rightarrow\alpha^{-1}\in W_1(Q).$$

If $\alpha\in W_1(Q)$ and $\theta\in A(Q)$, then for all $x,a,b\in Q$,

$$K(\alpha,x)=(R_x^{-1}R_{x\alpha},L_xL^{-1}_{x\alpha},I)\in AUT(Q,\cdot)\Leftrightarrow(a/x\cdot x\alpha)\cdot x\alpha\backslash(xb)=ab\Leftrightarrow$$
$$((a/x\cdot x\alpha)\cdot x\alpha\backslash(xb))\,\theta=(ab)\theta\Leftrightarrow$$
$$(a\theta/x\theta\cdot x\alpha\theta)\cdot x\alpha\theta\backslash(x\theta\cdot b\theta)=a\theta\cdot b\theta\quad(\text{set }\ x\mapsto x\theta^{-1})$$
$$\Leftrightarrow(a\theta/x\theta^{-1}\theta\cdot x\theta^{-1}\alpha\theta)\cdot x\theta^{-1}\alpha\theta\backslash(x\theta^{-1}\theta\cdot b\theta)=a\theta\cdot b\theta\Leftrightarrow$$
$$(a\theta/x\cdot x\theta^{-1}\alpha\theta)\cdot x\theta^{-1}\alpha\theta\backslash(x\cdot b\theta)=a\theta\cdot b\theta\Leftrightarrow$$
$$K(\theta^{-1}\alpha\theta,x)=(R_x^{-1}R_{x\theta^{-1}\alpha\theta},L_xL^{-1}_{x\theta^{-1}\alpha\theta},I)\in AUT(Q,\cdot)$$
$$\Leftrightarrow K(\theta^{-1}\alpha\theta,x)\in AUT(Q,\cdot)\Leftrightarrow\theta^{-1}\alpha\theta\in W_1(Q).$$

Therefore, $W_1(Q,\cdot)\lhd A(Q,\cdot)$.

(b) Let $\alpha\in W_1(Q)$, then for all $x\in Q$, $K(\alpha,x)=(R_x^{-1}R_{x\alpha},L_xL^{-1}_{x\alpha},I)\in AUT(Q,\cdot)$. With the condition $x^\lambda\cdot x\alpha=(x\alpha\backslash x)^\lambda$ and the arguments from (11) to (12) of the proof of Theorem 2.4(1), $R_x^{-1}R_{x\alpha}=R_{(x^\lambda\cdot x\alpha)}$.

Also, using the arguments from (13) to (14) of the proof of Theorem 2.4(1), $L_xL_{x\alpha}^{-1} = L_{(x^\lambda \cdot x\alpha)}^{-1}$. So,

$$K(\alpha, x) = (R_x^{-1}R_{x\alpha}, L_xL_{x\alpha}^{-1}, I) = \left(R_{(x^\lambda \cdot x\alpha)}, L_{(x^\lambda \cdot x\alpha)}^{-1}, I\right) \in AUT(Q, \cdot) \Rightarrow$$
$$W_1(Q, \cdot) \subseteq W_2(Q, \cdot).$$

Let $\alpha \in W_2(Q)$, then for all $x \in Q$, $\left(R_{(x^\lambda \cdot x\alpha)}, L_{(x^\lambda \cdot x\alpha)}^{-1}, I\right) \in AUT(Q, \cdot)$. With the condition $x^\lambda \cdot x\alpha = x\backslash x\alpha$ and the arguments from (15) to (16) of the proof of Theorem 2.4(1),

$$L_{x\alpha}L_x^{-1} = L_{(x^\lambda \cdot x\alpha)} \text{ and } R_x^{-1}R_{x\alpha} = R_{(x^\lambda \cdot x\alpha)} \Rightarrow$$
$$\left(R_{(x^\lambda \cdot x\alpha)}, L_{(x^\lambda \cdot x\alpha)}^{-1}, I\right) = (R_x^{-1}R_{x\alpha}, L_xL_{x\alpha}^{-1}, I)W_2(Q, \cdot) \subseteq W_1(Q, \cdot).$$

Thus, $W_1(Q, \cdot) = W_2(Q, \cdot)$.

(c) Going by (14) of Theorem 2.4(1), $x^\lambda \cdot x\alpha \in N(Q)$ for all $x \in Q$ and $\alpha \in A(Q)$. Thus by (a) and (b), $A(Q, \cdot) = V_1(Q, \cdot) = V_2(Q, \cdot)$.

(2) This is similar to the proof of 1.

□

2.4. **Conclusion.** In this study, we investigated the holomorphy of Basarab loops. Some necessary and sufficient conditions for a loop with $A(Q)$-holomorph to be a left (right) Basarab loop, and Basarab loop were established.

From Corollary 2.2, it can be deduced that the $\Lambda(Q, \cdot)$-holomorph $H(Q, \cdot)$ of a left Basarab loop $(Q, \cdot)$ is precisely a left Basarab loop. Based on Corollary 2.2, we could not come up with a commutative diagram when the $A(Q)$-holomorph $H(Q, \cdot)$ is a left Basarab loop. Whereas, Corollary 2.3 resulted in the commutative diagram (5) when the $A(Q)$-holomorph $H(Q, \cdot)$ is a LIP left Basarab loop.

Similarly, from Corollary 2.6, it can be deduced that the $\mathcal{P}(Q, \cdot)$-holomorph $H(Q, \cdot)$ of a right Basarab loop $(Q, \cdot)$ is precisely a right Basarab loop. Based on Corollary 2.6, we could not come up with a commutative diagram when the $A(Q)$-holomorph $H(Q, \cdot)$ is a right Basarab loop. Whereas, Corollary 2.7 resulted in the commutative diagram (7) when the $A(Q)$-holomorph $H(Q, \cdot)$ is a RIP right Basarab loop.

From Corollary 2.10, it can be deduced that the $\Lambda(Q, \cdot) \cap \mathcal{P}(Q, \cdot)$-holomorph $H(Q, \cdot)$ of a Basarab loop $(Q, \cdot)$ is precisely a Basarab loop.

Theorem 2.3 alongside Corollary 2.2 and Corollary 2.6 resulted in the commutative diagrams (8) and (9) when the $A(Q)$-holomorph $H(Q, \cdot)$ is a Basarab loop and this resulted into the existence of mappings $\kappa, \eta : \mathcal{P}(Q, \cdot) \to \Lambda(Q, \cdot)$ such that $R_x^{-1}R_{x\alpha} \overset{\kappa}{\approxeq} L_{x\alpha}L_x^{-1}$ and $R_{x\alpha}R_x^{-1} \overset{\eta}{\approxeq} L_x^{-1}L_{x\alpha}$ for all $x \in Q,\ \alpha \in A(Q)$.

Theorem 2.5 and Theorem 2.6 assures us that in any left (right) Basarab loop or Basarab loop Q, the set of automorphisms α with autotopic characterization of any of the types $P(\alpha, x), B(\alpha, x), K(\alpha, x), W(\alpha, x)$ actually forms a normal subgroup in $A(Q)$.

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[1] Department of Computer Science and Mathematics, Hezekiah University, Umudi, PMB 3002 Nkwerre, Nigeria.
Email address: gideonoeffiong@gmail.com, g.effiong@hezekiah.edu.ng.

[2] Department of Mathematics, University of Lagos, Akoka 100213, Nigeria.
Email address: tgjaiyeola@unilag.edu.ng

[3] Department of Mathematics, Obafemi Awolowo University, Ile Ife 220005, Nigeria.
Email address: tjayeola@oauife.edu.ng

[4] Department of Mathematics, Federal University of Technology, Owerri, PMB 1526 Owerri, Nigeria.
Email address: martins.obi@futo.edu.ng

[5] Department of Mathematics, Federal University Oye-Ekiti, Oye-Ekiti, Nigeria.

Email address: lukman.akinola@fuoye.edu.ng