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LATTICE OF c-STRUCTURES

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Abstract. In this paper, we study the properties of the lattice of c-structures. We define the concept of simple expansion and examine some of its properties. Additionally, we determine the automorphism group of the lattice of c-structures and explore the fixed points of the group of automorphisms.

1. Introduction

Connectedness is one of the fundamental features of objects. Widespread applications of connectedness include mathematical morphology, pattern recognition, image filtering, ultrasound image segmentation, tumor detection, biomedical image analysis, visual saliency estimation, digital topology, link theory, etc. In graph theory, connectedness is defined in terms of path, whereas connectedness in topology is described in terms of separation. Utilizing these two methodologies independently creates difficulties, and an integrated approach is required to resolve this limitation. The concept of c-spaces generalizes the notion of connectedness in both topology and graph theory. R. Börger[2] proposed the theory of connectivity classes or c-structures in 1983 and then studied by many authors. For more details, see [3, 4, 6, 7, 8, 9, 10, 12, 14]. Let X be a given set. Then the set of all c-structures on X is a lattice under the usual set inclusion. We denote this lattice by $LCS(X)$. Here, the join of two c-structures C_1 and C_2, is the c-structure generated by $C_1 \cup C_2$ and the meet of C_1 and C_2 is $C_1 \cap C_2$. The discrete c-structure D_X is the least element and the indiscrete c-structure $P(X)$ is the greatest element in $LCS(X)$.

Lattice theory has been demonstrated to be useful in many instances when studying the totality of mathematical systems of a given type and it provides a framework for organising the study of a specific class in mathematics. In [7] K. P. Ratheesh and N. M. Madhavan Namboothiri proved that the collection $LCS(X)$ forms a complete lattice and they studied some properties of $LCS(X)$. It is also proved that $LCS(X)$ is not dually atomic and the dual atoms in $LCS(X)$ are of the form $P(X) \setminus \{\{a, b\}\}$, where $a, b \in X$ and $a \neq b$. Here, we will continue these investigations by looking further into the lattice of all c-structures on a given set X and also determine the automorphism group of the lattice of c-structures.

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2. Preliminaries

First, we introduce various definitions and notations related to c-spaces and lattices that will be used later. See [1, 6] for more details.

A c-space [6] is an ordered pair (X, C_X) where C_X is a collection of subsets of X such that the following properties hold.

(1) $\emptyset \in C_X$ and $\{x\} \in C_X$ for all $x \in X$.
(2) If $\{C_i : i \in I\}$ is a collection of subsets in C_X with $\bigcap_{i\in I} C_i \neq \emptyset$, then $\bigcup_{i\in I} C_i \in C_X$.

Here, C_X is called a c-structure on X and elements of C_X are called connected sets.

Let (X, C_X) be a c-space and $A \subseteq X$. Then a point $x \in X$ is said to touch the set A if there is a nonempty subset $C \subseteq A$ such that $\{x\} \cup C$ is connected [6]. The set of all points touching the set A is denoted by $t(A)$. The set A is said to be t-closed [6] if $t(A) = A$. The t-closure of a set A [6] is defined to be the smallest t-closed set containing A and is denoted by $\overline{A}$. The set A is said to be c-dense [11] if $t(A) = X$. A subfamily $\mathbf{A}$ of $P(X)$ is said to form an upper family [11] if $B \in \mathbf{A}$ and $B \subseteq C \subseteq X$ implies $C \in \mathbf{A}$. Obviously $C_X = D_X \cup \mathbf{A}$ is a c-structure on X and is called an upper c-structure [11]. A c-space (X, C_X) is said to be C_1 [6] if distinct points of X are disconnected. That is, $\overline{\{x\}} = \{x\}$ for all $x \in X$. Let X be any set and $\mathcal{B} \subseteq P(X)$. Then the intersection of all c-structures on X containing $\mathcal{B}$ is a c-structure and is called the c-structure generated by $\mathcal{B}$ and is denoted by $\langle \mathcal{B} \rangle$ [7]. Let X and Y be two c-spaces and $f : X \to Y$ be a function. Then f is said to be c-continuous if it maps connected sets in X to connected sets in Y [6]. Now f is said to be a c-isomorphism or a catenomorphism if it is bijective and both f and f^{-1} are c-continuous [6]. A c-isomorphism from a c-space onto itself is called a c-automorphism. A c-space (X, C_X) is said to be completely homogeneous [11] if every bijection on X is a c-automorphism.

Let $(L, \leq)$ be a lattice with the smallest element 0 and the largest element 1. Then for $a, b \in L$, we say a is an upper neighbor of b or a covers b if $b \leq a$ and $a \neq b$ then for every $c \in L$ with $b \leq c \leq a$ we have either $c = b$ or $c = a$ [1]. An atom of the lattice L is an element that covers the smallest element 0. A lattice is atomic if every element other than the smallest element can be written as the join of atoms [1]. A dual atom or an anti-atom is an element which is covered by the largest element 1 in the lattice. A lattice is dually atomic if every element other than the largest element can be written as the meet of dual atoms. Let a, $b \in L$. Then a is said to be a complement of b if $a \vee b = 1$ and $a \wedge b = 0$ [1]. A sublattice of a lattice L is a subset M of L such that $a \in M$, $b \in M$ imply $a \wedge b \in M$ and $a \vee b \in M$. The lattices $(L_0, \leq)$ and $(L_1, \leq')$ are isomorphic and the map $\phi : L_0 \to L_1$ is an isomorphism if and only if ϕ is one to one and onto and $a \leq b$ in L_0 if and only if $\phi(a) \leq' \phi(b)$ in L_1. An isomorphism from a lattice onto itself is called an automorphism. A lattice L is called lower semi-modular [1] if and only if for any two distinct elements a and b in L which are both covered

by a third element c, it follows that a and b both cover $a \wedge b$. The dual notion is called upper semi-modular.

3. Lattice of c-structures

In [7], it is proved that $LCS(X)$ is an atomic lattice whose atoms are of the form $D_X \cup \{A\}$ where $A \subset X$, $|A| \geq 2$ and the dual atoms in $LCS(X)$ are of the form $P(X) \setminus \{\{a, b\}\}$, a, $b \in X$ and $a \neq b$. In general, $LCS(X)$ is not dually atomic. Here, we continue the study of the lattice of c-structures.

Definition 3.1. A c-structure $C_X \in LCS(X)$, where $C_X \neq P(X)$ is said to be dually atomic if it can be written as the meet of dual atoms in $LCS(X)$.

We can easily characterize dually atomic c-structures in $LCS(X)$.

Theorem 3.2. *Let $C_X \neq P(X)$ be a c-structure on X, where $|X| \geq 3$ and $\mathcal{B} = \{B \subseteq X : |B| \geq 3\}$. Then C_X is dually atomic if and only if $\mathcal{B} \subseteq C_X$.*

Proof. Suppose that C_X is dually atomic. Then C_X can be written as the meet of dual atoms. Since the dual atoms in X are of the form $P(X) \setminus \{\{a, b\}\}$ where $a, b \in X$ and $a \neq b$, the dual atom contains all subsets of X having cardinality greater than 2 and so is their meet. Hence $\mathcal{B} \subseteq C_X$. Conversely, assume that C_X contains all subsets of X having cardinality greater than 2. Let $\mathcal{A} = \{\{a, b\} : \{a, b\} \notin C_X\}$. Then we have $C_X = \bigwedge_{\{a,b\} \in \mathcal{A}} (P(X) \setminus \{\{a, b\}\})$. Hence C_X is dually atomic. □

Let X be a nonempty set and $C_X \in LCS(X)$. Then $C'_X \in LCS(X)$ is said to be a complement of C_X, if $C_X \vee C'_X = P(X)$ and $C_X \wedge C'_X = D_X$.

Let $C_X \neq D_X$ be a c-structure on X and there exists no $C \in C_X$ such that $|C| = 2$. If C_X has a complement, say C'_X, then C'_X contains all two element subsets of X and hence $C'_X = P(X)$, implies $C_X \wedge C'_X = C_X \neq D_X$. So C_X has no complement. That is, if there exists no two element subset of X in C_X, then C_X has no complement.

In the next theorem, we characterize the c-spaces for which complement exists.

Theorem 3.3. *Let $C_X \in LCS(X)$ and $\mathcal{C}$ be the collection of all two element subsets of X. Then C'_X exists if and only if there exists some $B \subseteq \mathcal{C}$ which generates C_X and $\mathcal{C} \setminus B$ generates $P(X) \setminus C_X$. Furthermore $C'_X = D_X \cup (P(X) \setminus C_X)$.*

Proof. Let $C_X \in LCS(X)$ such that C'_X exists. Since $C_X \vee C'_X = P(X)$, $C_X \cup C'_X$ contains all two element subsets of X. Also $C_X \wedge C'_X = D_X$, there exists no two element subset in common. Now let B be the collection of all two element subsets of X in C_X and B' be the collection of all two element subsets of X in C'_X. Then $B \cap B' = \emptyset$ and $B \cup B' = \mathcal{C}$. So $B' = \mathcal{C} \setminus B$. Suppose there exists some $C \in C_X$ with $|C| \geq 3$ such that $C \notin \langle B \rangle$. Then B generates some proper subsets of C. Otherwise, $C \in C'_X$, which is not possible.

Let A be a proper subset of C generated by B. Then there exists some $y \in C \setminus A$ such that $\{x, y\} \notin C_X$ for all $x \in A$. Otherwise, $A \cup \{y\}$ is connected for all

$y \in C \setminus A$ and hence $C \in C_X$, which is not possible. Let $\mathcal{A} = \{A \subsetneq C : A \in C_X$ and $A \in \langle B \rangle\}$ and for each $A \in \mathcal{A}$, define $C(A) = \bigcup\{\{x, y\} : \{x, y\} \notin C_X$ for all $x \in A$ and $y \in C \setminus A\}$. Clearly $C(A) \in C'_X$ for each $A \in \mathcal{A}$. We prove $C = C(A)$ for some $A \in \mathcal{A}$. Let $z \in C$. Then either $z \in A$ or $z \in C \setminus A$. If $z \in A$, then there exists some $y \in C \setminus A$ such that $\{z, y\} \notin C_X$. So $z \in C(A)$. If $z \in C \setminus A$, then either $\{x, z\} \in C_X$ for all $x \in A$ or there exists a $x \in A$ such that $\{x, z\} \notin C_X$. Now $\{x, z\} \notin C_X$ implies $z \in C(A)$. If $\{x, z\} \in C_X$ for all $x \in A$ then there exists some $A_1 \in \mathcal{A}$ where $A \subsetneq A_1$ such that $z \in A_1$ and $\{z, y\} \notin C_X$ for some $y \in C \setminus A_1$. So $z \in C(A_1)$. Hence $C = C(A_1)$. Thus $C \in C'_X$, which is a contradiction. Thus any $C \in C_X$ with $|C| \geq 3$ is generated by elements of B. Similarly, we can prove that any $C' \in P(X) \setminus C_X$ is generated by elements of $\mathcal{C} \setminus B$. Since $C_X \wedge C'_X = D_X$, C'_X contains none of the elements of C_X except D_X implies that $C'_X \subseteq D_X \cup (P(X) \setminus C_X)$. Since $C_X \vee C'_X = P(X)$, $C_X \cup C'_X$ contains all two element subsets of X. We have $C_X = \langle B \rangle$, it follows that C'_X contains $\mathcal{C} \setminus B$. So $\langle \mathcal{C} \setminus B \rangle \in C'_X$. But $\langle \mathcal{C} \setminus B \rangle = P(X) \setminus C_X$ implies $C'_X = D_X \cup (P(X) \setminus C_X)$.
The converse is trivial. Hence the theorem. □

From Theorem 3.3, it follows that, for any c-structure C_X, if the complement exists, then it is unique.

Corollary 3.4. *Let $C_X \neq D_X$ be a C_1 c-structure. Then C'_X does not exist.*

Proof. Since the c-space (X, C_X) is C_1, distinct points of X are disconnected. So there do not exist two element connected sets in C_X. It follows that C'_X does not exist. □

The converse of Corollary 3.4 does not hold in general. Observe that, the standard c-structure on $\mathbb{Z}$, that is, the c-structure generated by the intervals in $\mathbb{Z}$, does not have a complement and is not C_1.

Proposition 3.5. *Let C_X be a c-structure on X such that C'_X exists. Then the union of touching points of any subset of X with respect to C_X and C'_X is X.*

Proof. Let $A \subseteq X$. Suppose that x is not a touching point of A in C_X, since $C_X \vee C'_X = P(X)$, it follows that x is a touching point of A in C'_X. Since x is arbitrary, the proof follows. □

The collection of all C_1 c-structures on a given set X, denoted by $\mathcal{F}(X)$ is a sublattice of $LCS(X)$. Clearly, D_X is C_1 and is the least element of $\mathcal{F}(X)$. Let $\mathcal{B} = \{B \subseteq X : |B| = 2\}$. Then $(P(X) \setminus \mathcal{B}) \in \mathcal{F}(X)$ and is the largest element in $\mathcal{F}(X)$. Obviously, $\mathcal{F}(X)$ is an atomic lattice. The atoms are of the form $D_X \cup \{A\}$, $|A| \geq 3$.

Remark 3.6. The dual atoms in $\mathcal{F}(X)$ are of the form $P(X) \setminus (\mathcal{B} \cup \{\{a, b, c\}\})$, where a, b and c are distinct points of X. But $\mathcal{F}(X)$ is not dually atomic.

Example 3.7. Let $X = \{1, 2, 3, 4\}$. Then the dual atoms in $\mathcal{F}(X)$ are $D_X \cup \{\{2, 3, 4\}, \{1, 3, 4\}, \{1, 2, 4\}, X\}$, $D_X \cup \{\{1, 2, 3\}, \{1, 3, 4\}, \{1, 2, 4\}, X\}$, $D_X \cup \{\{1, 2, 3\}, \{2, 3, 4\}, \{1, 3, 4\}, X\}$, $D_X \cup \{\{1, 2, 3\}, \{2, 3, 4\}, \{1, 2, 4\}, X\}$. Note that $D_X \cup \{\{1, 2, 3\}\}$ can not be written as a meet of dual atoms.

Obviously, $\mathcal{F}(X)$ is not complemented.

4. Simple Expansion of c-structures

The simple expansion is the method used for generating finer c-structures from a given c-structure. More precisely, for a given c-structure C_X and a subset G of X, where $G \notin C_X$, by using the idea of simple expansion, we determine the smallest c-structure on X containing both C_X and G.

Theorem 4.1. *Let C_X be a c-structure on a nonempty set X and $G \notin C_X$. Let $\{C_i : i \in I\}$ be the collection of sets in C_X with $C_i \cap G \neq \emptyset$ for all $i \in I$. Then $C_X(G) = C_X \cup \{(\bigcup_{j\in J} C_j) \cup G$ for all $J \subseteq I\}$ is the smallest c-structure on X containing both C_X and G.*

Proof. Clearly, $C_X \subseteq C_X(G)$. Since every singleton in G intersects with G, $G \in C_X(G)$. Let $\{K_l : l \in L\}$ be a collection of sets in $C_X(G)$ with $\bigcap_{l\in L} K_l \neq \emptyset$. If $K_l \in C_X$ for all l, then $\bigcup_{l\in L} K_l \in C_X$. Otherwise, we have

$$\bigcup_{l\in L} K_l = K \cup ((\bigcup_{j\in J} C_j) \cup G)$$

where $K \in C_X$. Since $\bigcap_{l\in L} K_l \neq \emptyset$, $K \cap ((\bigcup_{j\in J} C_j) \cup G) \neq \emptyset$ and hence either $K \cap (\bigcup_{j\in J} C_j) \neq \emptyset$ or $K \cap G \neq \emptyset$.

If $K \cap G \neq \emptyset$, then $K \cup ((\bigcup_{j\in J} C_j) \cup G) \in C_X(G)$. If $K \cap (\bigcup_{j\in J} C_j) \neq \emptyset$, then there exists some $C_{j'}$, $j' \in J$ such that $K \cap C_{j'} \neq \emptyset$. It follows that $K \cup C_{j'} \in C_X$. Since $C_{j'} \cap G \neq \emptyset$, $(K \cup C_{j'}) \cap G \neq \emptyset$. So $\bigcup_{l\in L} K_l = (K \cup C_{j'}) \cup ((\bigcup_{\substack{j\in J \\ j\neq j'}} C_j) \cup G) \in C_X(G)$.

If $K_l \in C_X(G) \backslash C_X$ for all $l \in L$, then it is clear that $\bigcup_{l\in L} K_l \in C_X(G)$. Thus, $C_X(G)$ is a c-structure on X. Obviously, $C_X(G)$ is the smallest c-structure containing both C_X and G. This completes the proof. □

Definition 4.2. Let C_X be a c-structure on a nonempty set X and $G \subseteq X$ with $G \notin C_X$. Then $C_X(G)$ is called the simple expansion of C_X by G.

Corollary 4.3. $C_X(G) = C_X \vee (D_X \cup \{G\})$.

Example 4.4. Let $X = \{1, 2, 3, 4\}$ and $C_X = D_X \cup \{\{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$. Let $G = \{3, 4\} \notin C_X$. Then

$$C_X(G) = D_X \cup \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3\}, \{2, 3, 4\}, X\}.$$

Let $K_X = D_X \cup \{\{1, 2\}, \{2, 3\}, \{1, 2, 3\}, \{2, 3, 4\}, X\} \in LCS(X)$. Then $C_X \subsetneq K_X \subsetneq C_X(G)$. So $C_X(G)$ is not an upper neighbor of C_X.

Remark 4.5. Let (X, C_X) be a c-space and $G \subseteq X$. If $G \cap C = \emptyset$ for all non degenerate sets $C \in C_X$, then $C_X(G)$ is an upper neighbor. Since $G \cap C = \emptyset$ for all non degenerate sets C in C_X. It follows that $C_X(G) = C_X \cup \{G\}$, is an upper neighbor of C_X.

Theorem 4.6. *Let (X, C_X) be a c-space. Then every upper neighbor of C_X is a simple expansion of C_X.*

Proof. Let $\tilde{C_X}$ be an upper neighbor of C_X and $G \subseteq X$ such that $G \in \tilde{C_X} \setminus C_X$. Then $C_X(G)$ is the smallest c-structure containing both C_X and G. Thus $C_X \subset C_X(G) \subseteq \tilde{C_X}$. Since $\tilde{C_X}$ is an upper neighbor, $C_X(G) = \tilde{C_X}$. Hence every upper neighbor of C_X is a simple expansion of C_X. □

Let C_X be a c-structure such that C'_X exists. Then for any $G \notin C_X$, the complement of $C_X(G)$ denoted by $C'_X(G)$ need not exists.

Example 4.7. Let $X = \{1, 2, 3, 4\}$ and $C_X = D_X \cup \{\{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$. Let $G = \{1, 3, 4\} \notin C_X$. Then
$C_X(G) = D_X \cup \{\{1, 2\}, \{1, 3\}, \{1, 2, 3\}, \{1, 3, 4\}, X\}$ has no complement, since $\{1, 3, 4\}$ can not be generated by two element subsets of X in C_X.

Remark 4.8. Let C_X be a c-structure such that C'_X exists. If $C'_X(G)$ exists, then $|G| = 2$. But $|G| = 2$ need not implies $C'_X(G)$ exists.

Example 4.9. Consider the c-space $(\mathbb{Z}, C_\mathbb{Z})$, where $C_\mathbb{Z} = D_\mathbb{Z} \cup \{(n, \infty) : n \in \mathbb{Z}\}$. Let $G = \{1, 2, 3\}$. Then $C_\mathbb{Z}(G) = C_\mathbb{Z} \cup \{G\}$. Let $A = \{2, 3, 4\}$. Then $t(A) = A$ in $C_\mathbb{Z}$ and $t(A) = \{1, 2, 3, 4\}$ in $C_\mathbb{Z}(G)$. If $A = \{-2, -1, 0\}$, then $t(A) = A$ in both $C_\mathbb{Z}$ and $C_\mathbb{Z}(G)$.

Theorem 4.10. *Let (X, C_X) be a c-space, $G \notin C_X$, and $A \subseteq X$ such that $G \cap A = \emptyset$. Then $t(A)$ is the same in C_X and $C_X(G)$.*

Proof. Let $t(A) = Y$ in C_X, where $Y \subseteq X$. Suppose that $x \in X \setminus Y$ is a touching point of A in $C_X(G)$. Then there exists a nonempty subset $C \subseteq A$ such that $C \cup \{x\} \in C_X(G)$. Thus $C \cup \{x\}$ can be written as $C \cup \{x\} = (\bigcup_{j \in J} C_j) \cup G$, where $\{C_j : j \in J\} \subseteq C_X$ with $C_j \cap G \neq \emptyset$ for each $j \in J$, which implies $G \subseteq C \cup \{x\}$. It follows that $G \cap C \neq \emptyset$, so $G \cap A \neq \emptyset$, a contradiction. □

A simple expansion of a C_1 c-structure need not be C_1. But if we restrict the cardinality of G, we have the following theorem.

Theorem 4.11. *Let (X, C_X) be a C_1 c-space. Then $C_X(G)$ is C_1 if and only if $|G| \geq 3$.*

Proof. Suppose that $C_X(G)$ is C_1. Let $G = \{a, b\} \subseteq X$, where $a, b \in X$ such that $a \neq b$. Then $C_X(G)$ is not C_1. Conversely, suppose that $|G| \geq 3$. It is clear that $C_X(G)$ does not contains any two element subsets of X. Hence $C_X(G)$ is C_1. □

If C_X is an upper c-structure, then $C_X(G)$ need not be an upper c-structure.
Example: Let $X = \{1, 2, 3, 4\}$ and $C_X = D_X \cup \{\{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, X\}$. Let $G = \{3, 4\} \notin C_X$. Then
$C_X(G) = D_X \cup \{\{1, 2\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, X\}$ is not an upper c-structure, since $\{1, 3, 4\}$, $\{2, 3, 4\} \notin C_X$.

Theorem 4.12. *Let (X, C_X) be an upper c-space and $G \subseteq X$. Then $(X, C_X(G))$ is an upper c-space if and only if $G \cup \{x\} \in C_X$ for all $x \in X \setminus G$.*

Proof. Suppose that $G \cup \{x\} \in C_X$ for all $x \in X \setminus G$. This implies that $G \cup \{x\} \in C_X(G)$. Since C_X is an upper c-structure, $G' \in C_X$ for any G' such that $G \cup \{x\} \subseteq G' \subseteq X$. Thus $G' \in C_X(G)$. So any superset of $G \in C_X(G)$. It follows that $C_X(G)$ is an upper c-structure.

Conversely, let $C_X(G)$ be an upper c-structure. Since $G \in C_X(G)$, $G \cup \{x\} \in C_X(G)$ for all $x \in X \setminus G$. If $G \cup \{x\} \in C_X$ for all $x \in X \setminus G$, then the proof follows. If not, $G \cup \{x\}$ can be written as $G \cup \{x\} = (\bigcup_{j \in J} C_j) \cup G$, where $C_j \in C_X$ and $C_j \cap G \neq \emptyset$ for each $j \in J$. Thus $x \in \bigcup_{j \in J} C_j$, which implies $x \in C_{j'}$ for some $j' \in J$. Also, $\bigcup_{j \in J} C_j$ contains exactly one point which does not belong to G. So there exists some $G_1 \subset G$ such that $C_{j'}$ can be written as $C_{j'} = G_1 \cup \{x\}$. So $G_1 \cup \{x\} \in C_X$, since C_X is an upper c-structure, it follows that $G \cup \{x\} \in C_X$. This completes the proof. □

Theorem 4.13. *Let X be a finite set, $C_X \neq P(X)$ be a c-structure on X. Then there exists some $G \subseteq X$, $G \notin C_X$ such that $C_X(G) = C_X \cup \{G\}$.*

Proof. Let $|X| = n$. Here we prove $C_X(G) = C_X \cup \{G\}$ is a c-structure on X for some $G \subseteq X$. If $X \notin C_X$, then $C_X \cup \{X\}$ is a c-structure on X. If $X \in C_X$, then consider the collection, $\mathcal{G}_{n-1} = \{G \subset X : |G| = n-1\}$. If $\mathcal{G}_{n-1} \not\subseteq C_X$, then take some $G \in \mathcal{G}_{n-1}$ such that $G \notin C_X$. Consider $C_X \cup \{G\}$. Let C_1, $C_2 \in C_X \cup \{G\}$ with $C_1 \cap C_2 \neq \emptyset$. If C_1, $C_2 \in C_X$, then $C_1 \cup C_2 \in C_X$. If $C_1 \in C_X$ and $C_2 = G$ then $C_1 \cup C_2 = X$ or $C_1 \cup C_2 = G$, which implies $C_1 \cup C_2 \in C_X(G)$. Similarly, if $\mathcal{G}_{n-1} \subset C_X$ and $X \in C_X$, then consider $\mathcal{G}_{n-2} = \{G \subset X : |G| = n-2\}$. If $\mathcal{G}_{n-2} \not\subseteq C_X$, then take some $G \in \mathcal{G}_{n-2}$ such that $G \notin C_X$. Then $C_X \cup \{G\}$ is a c-structure on X. Proceeding like this, if C_X contains all sets with cardinality greater than 2 and its supersets and $C_X \neq P(X)$, there exists some $\{x, y\} \subset X$ with $\{x, y\} \notin C_X$, then we can take G as $\{x, y\}$ and $C_X \cup \{G\}$ is a c-structure on X. Hence the proof. □

Theorem 4.14. *Let C_X and $\tilde{C_X}$ be two c-structures on X. Then $\tilde{C_X}$ is an upper neighbor of C_X if and only if $\tilde{C_X} = C_X(G)$ for all $G \in \tilde{C_X} \setminus C_X$.*

Proof. Suppose that $\tilde{C_X}$ is an upper neighbor of C_X and $G \in \tilde{C_X} \setminus C_X$. Since $C_X(G)$ is the smallest c-structure containing C_X and G, we can write $C_X \subseteq C_X(G) \subseteq \tilde{C_X}$. But $\tilde{C_X}$ is an upper neighbor of C_X we get $\tilde{C_X} = C_X(G)$. Thus for all $G \in \tilde{C_X} \setminus C_X$ we get $\tilde{C_X} = C_X(G)$.

Conversely, let $\tilde{C_X} = C_X(G)$ for all $G \in \tilde{C_X} \setminus C_X$ and $\tilde{C_X}$ be not an upper neighbor of C_X. Then there exists some $K_X \in LCS(X)$ such that $C_X \subsetneq K_X \subsetneq \tilde{C_X}$. Since $K_X \neq C_X$, there exists some $H \in K_X$ such that $H \notin C_X$. Now $H \in K_X$ implies $H \in \tilde{C_X}$. So $H \in \tilde{C_X} \setminus C_X$ implies $\tilde{C_X} = C_X(H)$. That is, $\tilde{C_X}$ is the smallest c-structure containing both C_X and H. It follows that $K_X = \tilde{C_X}$, a contradiction. Thus $\tilde{C_X}$ is an upper neighbor of C_X. □

Theorem 4.15. *Let X be a set and C_X be a c-structure on X. Then every upper neighbor of C_X is of the form $C_X \cup \{G\}$, where $G \subseteq X$ and $G \notin C_X$.*

Proof. Suppose $\tilde{C_X} = C_X \cup \mathcal{B}$ is an upper neighbor of C_X, where $\mathcal{B}$ contains at least two subsets of X which do not belong to C_X. Now for a fixed set $G_k \in \mathcal{B}$, let $\mathcal{B}_k = \{G_h \in \mathcal{B} : G_h \subsetneq G_k\}$. First we consider the case $\mathcal{B}_k \neq \emptyset$. Let $\mathcal{B}'_k = \mathcal{B} \setminus \mathcal{B}_k$. We prove $C_X \cup \mathcal{B}'_k$ is a c-structure on X. Let $\{C_i : i \in I\}$ be a collection of sets in $C_X \cup \mathcal{B}'_k$ with $\bigcap_{i\in I} C_i \neq \emptyset$. If $C_i \in C_X$ for all i, then $\bigcup_{i\in I} C_i \in C_X$. Otherwise, let $\bigcup_{i\in I} C_i = C \cup (\bigcup_{j\in J} C_j)$ where $C \in C_X$, $J \subseteq I$ and $C_j \in \mathcal{B}'_k$ for each $j \in J$. If $\bigcup_{i\in I} C_i \notin C_X \cup \mathcal{B}'_k$, then $\bigcup_{i\in I} C_i \in \mathcal{B}_k$ which implies $C_j \in \mathcal{B}_k$ for each $j \in J$, which is a contradiction. So $\bigcup_{i\in I} C_i \in C_X \cup \mathcal{B}'_k$. If $C_i \in \mathcal{B}'_k$ for all i, then also we can prove that $\bigcup_{i\in I} C_i \in C_X \cup \mathcal{B}'_k$. Thus $C_X \cup \mathcal{B}'_k$ is a c-structure on X. If $\mathcal{B}_k = \emptyset$, then consider $\tilde{C_X} \setminus \{G_k\}$. Similarly, we can prove that $\tilde{C_X} \setminus \{G_k\}$ is a c-structure on X. In each of the cases, $\tilde{C_X}$ is not an upper neighbor of C_X. Thus an upper neighbor of C_X is of the form $C_X \cup \{G\}$, $G \notin C_X$. □

If X is finite, then by Theorem 4.13, every c-structure $C_X \neq P(X)$ on X has an upper neighbor. But in general, this is not true. The following example illustrates this.

Example 4.16. Let $C_{\mathbb{Z}}$ be a c-structure on $\mathbb{Z}$, the set of integers given by

$$C_{\mathbb{Z}} = D_{\mathbb{Z}} \cup \{P(2\mathbb{Z}) \cup P(2\mathbb{Z}+1)\} \cup \{C \subseteq \mathbb{Z} : C \cap \mathbb{Z} \text{ is a cofinite subset of } \mathbb{Z}\},$$

where $P(2\mathbb{Z})$ is the power set of $2\mathbb{Z}$ and $P(2\mathbb{Z}+1)$ is the power set of $2\mathbb{Z}+1$.

Let $G \notin C_{\mathbb{Z}}$ implies there exist some $x \in 2\mathbb{Z}$ and $y \in 2\mathbb{Z}+1$ such that $\{x, y\} \subseteq G$. Let $z \in G^c$. Then either $\{x, z\}$ or $\{y, z\} \in C_{\mathbb{Z}}$. So $G \cup \{z\} \in C_{\mathbb{Z}}(G)$. But $G \cup \{z\} \notin C_{\mathbb{Z}} \cup \{G\}$. Thus $C_{\mathbb{Z}} \cup \{G\}$ is not an upper neighbor. So $C_{\mathbb{Z}}$ has no upper neighbor.

Note that $LCS(X)$ is non-modular and hence non distributive when $|X| > 2$ [7]. But we can show that it is lower semi-modular.

Theorem 4.17. *For any set X, $LCS(X)$ is lower semi-modular.*

Proof. Suppose C_X, $K_X \in LCS(X)$ and $C_X \neq K_X$. Let $\mathcal{K}_X \in LCS(X)$ be an upper neighbor of both C_X and K_X. Then $\mathcal{K}_X$ is of the form $C_X \cup \{G\}$, $G \notin C_X$ or $K_X \cup \{H\}$, $H \notin K_X$ which implies $G \in K_X$ and $H \in C_X$. Thus $C_X \wedge K_X = C_X \setminus \{H\} = K_X \setminus \{G\}$, implies both C_X and K_X are upper neighbor of $C_X \wedge K_X$. □

Remark 4.18. $LCS(X)$ is not upper semi-modular.

Example 4.19. Let $X = \{1, 2, 3, 4\}$. Consider the c-structures

$$C_X = D_X \cup \{\{1, 2, 3\}\} \text{ and } K_X = D_X \cup \{\{1, 3, 4\}\}.$$

Here, both C_X and K_X are upper neighbors of D_X and

$$C_X \vee K_X = D_X \cup \{\{1, 2, 3\}, \{1, 3, 4\}, X\}.$$

But $C_X \vee K_X$ is neither an upper neighbor of C_X nor K_X.

Definition 4.20. [1] An ideal is a nonempty subset J of a lattice L with the properties

(1) $a \in J$, $x \in L$, $x \leq a$ implies $x \in J$.
(2) $a \in J$, $b \in J$ implies $a \vee b \in J$.

For any set X, consider $LCS(X)$. Let $C_X \in LCS(X)$. Then $J(X) = \{K_X \in LCS(X) : K_X \subseteq C_X\}$ is an ideal in $LCS(X)$.

Theorem 4.21. *The collection of all C_1 c-structures is an ideal.*

Proof. Let $\mathcal{F}(X)$ denotes the collection of all C_1 c-structures on a set X. Then $C_X \vee K_X \in \mathcal{F}(X)$ for any C_X, $K_X \in \mathcal{F}(X)$. Let $C_X \in \mathcal{F}(X)$, $\mathcal{K}_X \in LCS(X)$ such that $\mathcal{K}_X \subseteq C_X$. So $\mathcal{K}_X$ is also C_1, which implies $\mathcal{K}_X \in \mathcal{F}(X)$. Hence the proof. □

5. Automorphisms of $LCS(X)$

Here, we determine the automorphism group of $LCS(X)$. Throughout this section, $S(X)$ denotes the symmetric group on X.

Lemma 5.1. *Let $f \in S(X)$ and $C_X \in LCS(X)$. Then $f(C_X) = \{f(C) : C \in C_X\}$ is a c-structure on X, where $f(C) = \{f(x) : x \in C\}$.*

Proof. Since f is a bijection, $f(D_X) = D_X$ and hence $D_X \subseteq f(C_X)$. Let $\{C_i : i \in I\}$ be an arbitrary family of sets in $f(C_X)$ with $\bigcap_{i\in I} C_i \neq \emptyset$. Then $C_i = f(K_i)$ where $K_i \in C_X$. Since $\bigcap_{i\in I} C_i \neq \emptyset$, there exists some $c \in X$ such that $c \in C_i$ for all i. Then $f^{-1}(c) \in K_i$ for all i. That is $\bigcap_{i\in I} K_i \neq \emptyset$ and hence $\bigcup_{i\in I} K_i \in C_X$. Now $\bigcup_{i\in I} C_i = \bigcup_{i\in I} f(K_i) = f(\bigcup_{i\in I} K_i) \in f(C_X)$. Hence the proof. □

Lemma 5.2. *Let $f \in S(X)$. Define f_c, a map on $LCS(X)$ by $f_c(C_X) = f(C_X)$, $C_X \in LCS(X)$. Then f_c is an automorphism on $LCS(X)$.*

Proof. Let C_X and $K_X \in LCS(X)$ such that $f_c(C_X) = f_c(K_X)$. That is $f(C_X) = f(K_X)$. Now $C \in C_X$ if and only if $f(C) \in f(C_X)$ if and only if $f(C) \in f(K_X)$ if and only if $C \in K_X$. So $C_X = K_X$, which implies that f_c is injective.
Now for any $K_X \in LCS(X)$, let $C_X = \{f^{-1}(C) : C \in K_X\}$ where $f^{-1}(C) = \{f^{-1}(x) : x \in C\}$. Then $C_X = f^{-1}(K_X)$. It follows that $C_X = f_c^{-1}(K_X)$. So f_c is surjective and hence it is bijective.
Suppose that $C_X \subseteq K_X$. Let $K \in f_c(C_X)$. Then there exists some $C \in C_X$ such that $K = f_c(C)$ implies $K = f(C)$. Since $C \in C_X$, we have $C \in K_X$ which implies $K = f(C) \in f(K_X)$, it follows that $K \in f_c(K_X)$. Thus $f_c(C_X) \subseteq f_c(K_X)$. So f_c is an automorphism. □

It is easy to prove that an automorphism on $LCS(X)$ maps atoms to atoms and dual atoms to dual atoms.

Lemma 5.3. *Let F be an automorphism on $LCS(X)$. Then C_X, $K_X \in LCS(X)$ are complements to each other if and only if $F(C_X)$ and $F(K_X)$ are complements to each other.*

Proof. Since C_X and K_X are complements to each other, we have $C_X \vee K_X = P(X)$ and $C_X \wedge K_X = D_X$. Now

$$\begin{aligned} F(C_X) \vee F(K_X) &= F(C_X \vee K_X) \\ &= F(P(X)) \\ &= P(X) \end{aligned}$$

Similarly, $F(C_X) \wedge F(K_X) = F(C_X \wedge K_X) = F(D_X) = D_X$. □

Theorem 5.4. *The set of all automorphisms of $LCS(X)$ is precisely $\{f_c : f \in S(X)\}$.*

Proof. By Lemma 5.2, for all $f \in S(X)$, if we define $f_c(C_X) = f(C_X)$, then f_c is an automorphism on $LCS(X)$. Conversely, let F be an automorphism on $LCS(X)$ where $|X| \geq 3$. Let $\mathcal{A}$ denote the set of all atoms of the form $\alpha_A = D_X \cup \{A\}$, $A \subseteq X$, and $|A| = 2$. Consider the dual atom of α_A, which is denoted by α'_A and $\alpha'_A = P(X) \setminus \{A\}$. Since α_A and α'_A are complements to each other by Lemma 5.3, $F(\alpha_A)$ and $F(\alpha'_A)$ are complements to each other. Since $F(\alpha'_A)$ is a dual atom, then $F(\alpha'_A)$ is of the form $F(\alpha'_A) = P(X) \setminus \{B\}$, where $|B| = 2$ implies $B \in F(\alpha_A)$. Since $F(\alpha'_A) \vee F(\alpha_A) = P(X)$ implies $F(\alpha_A) = D_X \cup \{B\}$. So F maps $\mathcal{A}$ into $\mathcal{A}$.

Now for any $\alpha_G = D_X \cup \{G\} \in \mathcal{A}$, $|G| = 2$, consider the corresponding dual atom $\alpha'_G = P(X) \setminus \{G\}$. Since F is onto, there exists a dual atom $\alpha'_H \in LCS(X)$ where $H \subseteq X$, $|H| = 2$ such that $F(\alpha'_H) = \alpha'_G$, where $\alpha'_H = P(X) \setminus \{H\}$. Let $\alpha_H = D_X \cup \{H\}$. Now $F(\alpha_H) \vee \alpha'_G = F(\alpha_H) \vee F(\alpha'_H) = F(\alpha_H \vee \alpha'_H) = F(P(X)) = P(X)$, which implies $G \in F(\alpha_H)$. It follows that $F(\alpha_H) = \alpha_G$. Thus F maps $\mathcal{A}$ onto $\mathcal{A}$.

Claim: $F = f_c$.
Let $C_1 = D_X \cup \{A\}$, $C_2 = D_X \cup \{B\} \in \mathcal{A}$, $A \cap B \neq \emptyset$, $A \neq B$. Let $F(C_1) = D_X \cup \{C\}$, $F(C_2) = D_X \cup \{D\}$. Suppose that $C \cap D = \emptyset$. Then $F(C_1 \vee C_2) = F(C_1) \vee F(C_2) = D_X \cup \{C, D\}$. Now

$$\begin{aligned} F(C_1 \vee C_2) &= F(D_X \cup \{A, B, A \cup B)\} \\ &= F((D_X \cup \{A\}) \vee (D_X \cup \{B\}) \vee (D_X \cup \{A \cup B\})) \\ &= F(D_X \cup \{A\}) \vee F(D_X \cup \{B\}) \vee F(D_X \cup \{A \cup B\}) \\ &= (D_X \cup \{C\}) \vee (D_X \cup \{D\}) \vee F(D_X \cup \{A \cup B\}) \\ &= D_X \cup \{C, D\} \vee F(D_X \cup \{A \cup B\}) \end{aligned}$$

But $F(C_1 \vee C_2) = D_X \cup \{C, D\}$ implies that $F(D_X \cup \{A \cup B\})$ can be either of D_X, $D_X \cup \{C\}$, $D_X \cup \{D\}$ or $D_X \cup \{C, D\}$. In each case, we will get a contradiction. So $C \cap D \neq \emptyset$ and it follows that $F(D_X \cup \{A \cup B\}) = D_X \cup \{C \cup D\}$.

Now for any $x \in X$, there exist at least two distinct elements α_{A_i} and $\alpha_{A_j} \in \mathcal{A}$ such that $x \in A_i \cap A_j$. Let $F(\alpha_{A_i}) = \alpha_{B_i}$ and $F(\alpha_{A_j}) = \alpha_{B_j}$. Then obviously α_{B_i} and $\alpha_{B_j} \in \mathcal{A}$. Corresponding to each $x \in X$, we can find a $y \in X$ such that $y \in B_i \cap B_j$. Now for each $x \in X$, define $f : X \to X$ as $f(x) = y$. Let $A_i = \{x, a\}$, $B_i = \{y, c\}$, $A_j = \{x, b\}$, $B_j = \{y, d\}$. Suppose that $A_k = \{x, e\}$. Let $F(\alpha_{A_k}) = \alpha_{B_k}$, $|B_k| = 2$. If $y \notin B_k$, then $B_k = \{c, d\}$. Since $|B_j \cap B_k| = 1$

and $|B_i \cap B_k| = 1$, we have

$F(D_X \cup \{\{x, a, b\}\}) = D_X \cup \{\{y, c, d\}\}$ and $F(D_X \cup \{\{x, a, e\}\}) = D_X \cup \{\{y, c, d\}\}$,

which is a contradiction. Thus $y \in B_k$. So f is well defined. It is easy to see that f is a bijection. Thus the proof follows. □

Definition 5.5. [5] Let $f : X \to X$ be a function. A point $x \in X$ is said to be a fixed point of f, if $f(x) = x$.

Next, we determine the fixed points of the group of automorphisms of $LCS(X)$. That is, the set of all c-structures that are fixed by all automorphisms on $LCS(X)$. It is clear that for any set X, every automorphism on $LCS(X)$ maps D_X onto itself and $P(X)$ onto itself. So D_X and $P(X)$ are fixed points of any automorphism on $LCS(X)$.

Theorem 5.6. *Let $\mathcal{F}$ denote the group of all automorphisms on $LCS(X)$. Then the fixed points of $\mathcal{F}$ are precisely the completely homogeneous c-structures.*

Proof. Let $f \in S(X)$. By Theorem 5.4, $F = f_c$ for all $F \in \mathcal{F}$. Suppose that C_X is a fixed point of $\mathcal{F}$. Then $F(C_X) = C_X$ for all $F \in \mathcal{F}$. It follows that $f_c(C_X) = C_X$, which implies $f(C_X) = C_X$. That is, f maps C_X onto itself, which implies that f is a c-automorphism on X. Since f is arbitrary, each bijection on X is a c-automorphism, which implies that C_X is completely homogeneous. Thus the fixed points of $\mathcal{F}$ are the completely homogeneous c-structures. The converse part is obvious. □

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References

1. Birkhoff, G. (1984). Lattice Theory. American Mathematical Society.
2. Börger, R. (1983). Connectivity Spaces and Component Categories. Categorical Topology, Proceedings of the International Conference held at the University of Toledo, Ohio, U. S. A, 71-89.
3. Dugowson, S. (2010). On Connectivity Spaces. Cahiers de Topologie et Géométrie Différentielle Catégoriques, 51(4), 282-315.
4. Darsana, C., & Sini, P. (2023). Hereditarily Homogeneous c-spaces. Research in Mathematics, 10(1), 1-6. https://doi.org/10.1080/27684830.2023.2231607
5. Joshi, K. D. (2006). Introduction to General Topology. New Age International (P) Limited.
6. Muscat, J., & Buhagiar, D. (2006). Connective Spaces. Memoirs of the Graduate School of Science and Engineering, Shimane University Series B, Mathematical Sciences, 39, 1-13.
7. Ratheesh, K. P., & Madhavan Namboothiri, N. M. (2013). On c-spaces and Connective Spaces. South Asian Journal of Mathematics, 4(1), 48-56.
8. Santhosh, P. K. (2017). On the Quotients of c-spaces. Boletim da Sociedade Paranaense de Matemática, 35(1), 97–109. https://doi.org/10.5269/bspm.v35i1.24592
9. Santhosh, P. K. (2022). On a Stronger Notion of Connectedness in c-spaces. Algebra and Discrete Mathematics, 34(2), 317-325. http://doi.org/10.12958/adm1624
10. Serra, J. (1998). Connectivity on Complete Lattices. Journal of Mathematical Imaging and Vision, 9, 231-251. https://doi.org/10.1023/A:1008324520475

11. Sini, P. (2020). On Completely Homogeneous c-spaces. Heliyon, 6(12). https://doi.\org/10.1016/j.heliyon.2020.e05747
12. Sruthi, A. K., & Ramachandran, P. T. (2019). On c-spaces and Hypergraphs. Far East Journal of Mathematical Sciences, 110(1), 83-92. https://doi.org/10.17654/MS110010083
13. Sruthi, A. K., & Sini, P. (2023). Some Order Properties of c-spaces. Palestine Journal of Mathematics, 12(3), 180-185.
14. Stadler, B. M. R., & Stadler, P. F. (2015). Connectivity Spaces. Mathematics in Computer Science, 9, 409-436. https://doi.org/10.1007/s11786-015-0241-1

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