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THE POWER SERIESWISE ARMENDARIZ GRAPH OF A COMMUTATIVE RING

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Abstract. The rings considered in this article are commutative with non-zero identity which are not integral domains. Let R be a ring. Let $Z(R)$ denote the set of all zero-divisors of R and we denote $Z(R)\backslash\{0\}$ by $Z(R)^*$. In this article, we introduce and investigate the power serieswise Armendariz graph of R denoted by $\mathbb{PA}(R)$. It is the undirected graph whose vertex set is $Z(R[[X]])^*$ and distinct vertices $f(X)=\sum_{i=0}^{\infty} a_iX^i$ and $g(X)=\sum_{j=0}^{\infty} b_jX^j$ are adjacent in $\mathbb{PA}(R)$ if and only if $a_ib_j=0$ for all i and j. The aim of this article is to study the interplay between the ring-theoretic properties of R and the graph-theoretic properties of $\mathbb{PA}(R)$. We discuss some results on diameter, clique, and girth of $\mathbb{PA}(R)$.

1. Introduction

The rings considered in this article are commutative with non-zero identity which are not integral domains. Let R be a ring. We denote the set of all zero-divisors of R by $Z(R)$ and the set $Z(R)\backslash\{0\}$ by $Z(R)^*$. For a ring R, throughout this article, we assume that $Z(R)^* \neq \emptyset$. The study of interplay between ring theory and graph theory was initiated by Beck in [7]. Recall from [2] that the *zero-divisor graph* of R denoted by $\Gamma(R)$ is an undirected graph whose vertex set is $Z(R)^*$ and distinct vertices x, y are adjacent in $\Gamma(R)$ if and only if $xy=0$. For a ring R, we denote the polynomial ring and the power series ring in one variable X over R by $R[X]$ and $R[[X]]$ respectively. In [4], Axtell et al. investigated interesting results on zero-divisor graphs of polynomials and power series over commutative rings. It is easy to verify that $\Gamma(R)$ is a subgraph of $\Gamma(R[X])$ and $\Gamma(R[X])$ is a subgraph of $\Gamma(R[[X]])$. A lot of research articles on the zero-divisor graphs of commutative rings were published by eminent researchers. Moreover, the zero-divisor graph of a power series ring was also studied in [10, 14]. Furthermore, several researchers have studied different graphs associated with commutative rings. For an interesting study of such articles, the reader is referred to [15,16,19].

Recall from [18] that a ring R is said to be *Armendariz* if $f(X)=\sum_{i=0}^{n} a_iX^i$ and $g(X)=\sum_{j=0}^{m} b_jX^j$ are such that $f(X)g(X)=0$, then $a_ib_j=0$ for all $i \in \{0,1,\ldots,n\}$ and $j \in \{0,1,\ldots,m\}$. Moreover, Rege and Buhphang proved some

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useful properties of Armendariz ring in [17]. Furthermore, Bakkari investigated some useful results on power Armendariz rings in [5]. The concept of Armendariz graph of a ring was introduced by Abdioğlu et al. in [1]. Recall from [1] that the *Armendariz graph* of a ring R denoted by $A(R)$ is an undirected graph whose vertex set is $Z(R[X])^*$ and distinct vertices $f(X) = \sum_{i=0}^{n} a_i X^i$ and $g(X) = \sum_{j=0}^{m} b_j X^j$ are adjacent in $A(R)$ if and only if $a_i b_j = 0$ for all $i \in \{0, 1, \ldots, n\}$ and $j \in \{0, 1, \ldots, m\}$. Recently, we have studied some remarkable properties of the complement of the Armendariz graph of a commutative ring in [20].

In [12], Kim et al. introduced the concept of the power serieswise Armendariz ring. Recall from [12] that a ring R is said to be *power serieswise Armendariz* if $f(X) = \sum_{i=0}^{\infty} a_i X^i$ and $g(X) = \sum_{j=0}^{\infty} b_j X^j$ are such that $f(X)g(X) = 0$, then $a_i b_j = 0$ for all i and j. Observe that power serieswise Armendariz rings are Armendariz rings but the converse is not true by [12, Example 2.1]. A ring R is said to be *reduced* if R has no non-zero nilpotent elements. Note that any reduced ring is power serieswise Armendariz ring [12, Lemma 2.3(1)]. Moreover, Ouarrachi and Mahdou proved some interesting results on power serieswise Armendariz rings in [13].

This article is motivated by the interesting results proved on $A(R)$ and $(A(R))^c$ in [1] and [20] respectively. The purpose of this article is to generalize the concept of $A(R)$ by introducing the power serieswise Armendariz graph of R. The *power serieswise Armendariz graph* of R denoted by $\mathbb{PA}(R)$ is an undirected graph whose vertex set is $Z(R[[X]])^*$ and distinct vertices $f(X) = \sum_{i=0}^{\infty} a_i X^i$ and $g(X) = \sum_{j=0}^{\infty} b_j X^j$ are adjacent in $\mathbb{PA}(R)$ if and only if $a_i b_j = 0$ for all i and j. Note that $V(\mathbb{PA}(R)) = V(\Gamma(R[[X]])) = Z(R[[X]])^*$. Observe that $A(R)$ is a subgraph of $\mathbb{PA}(R)$. In this article, we study some useful properties of $\mathbb{PA}(R)$. Now, it is desirable to recall the following basic concepts from commutative ring theory.

A local ring R with unique maximal ideal $\mathfrak{m}$ is denoted by $(R, \mathfrak{m})$. The *nilradical* of R denoted by $nil(R)$ is the set of all nilpotent elements of R. We denote the set of all prime ideals and the set of all minimal prime ideals of R by $Spec(R)$ and $Min(R)$ respectively. Let $n \in \mathbb{N}$ with $n \geq 2$. We denote the ring of integer modulo n by $\mathbb{Z}_n$. The cardinality of a set A is denoted by $|A|$. Further results of commutative ring theory can be referred from the standard textbooks like [3, 11].

We next mention some basic concepts from graph theory. The graphs considered in this article are undirected and simple. Let $G = (V, E)$ be a connected graph. For any $u, v \in V$, we denote the distance between u and v in G by $d(u, v)$. We denote the diameter and the girth of G by $diam(G)$ and $gr(G)$ respectively. Recall that a *clique* of G is a complete subgraph of G. Moreover, the clique number of G is denoted by $\omega(G)$. We write $\omega(G) = \infty$ if G contains a clique on n vertices for all $n \geq 1$. Recall that the chromatic number of G is denoted by $\chi(G)$. A subgraph H of G is said to be a *spanning subgraph* of G if $V(G) = V(H)$. Any unexplained definitions and notations of graph theory can be referred from [6].

We now give brief outline of this article. This article consists of five sections. Section 1 is on introduction. In Section 2, we discuss some basic properties of $\mathbb{PA}(R)$. We prove some useful results on the diameter of $\mathbb{PA}(R)$ in Section 3. In

Section 4, we provide several useful results on the clique of $\mathbb{PA}(R)$. Finally, we prove some interesting properties on the girth of $\mathbb{PA}(R)$ in Section 5. We also present several examples to illustrate the results proved in this article.

2. Basic properties of $\mathbb{PA}(R)$

Let R be a ring such that $Z(R)^* \neq \emptyset$. The aim of this section is to study some basic properties of $\mathbb{PA}(R)$. We start this section with the following lemma.

Lemma 2.1. *Let R be a ring. Then $Z(R[[X]])^*$ is infinite.*

Proof. As $R[[X]]$ is infinite and is not an integral domain, it follows from [9, Theorem I] that $Z(R[[X]])^*$ is infinite. □

Lemma 2.2. *Let R be a Noetherian ring. Then no vertex of $\mathbb{PA}(R)$ is an isolated vertex.*

Proof. Let $f(X) \in Z(R[[X]])^*$. Since R is Noetherian, there exists $r \in R\backslash\{0\}$ such that $rf(X) = 0$ by [8, Theorem 5]. If $f(X) \neq r$, then $f(X) - r$ is an edge of $\mathbb{PA}(R)$. Suppose that $f(X) = r$. Then $r^2 = 0$. Let $g(X) = r + rX$. Then $r - g(X)$ is an edge of $\mathbb{PA}(R)$. This proves that no vertex of $\mathbb{PA}(R)$ is an isolated vertex. □

Lemma 2.3. *Let R be a ring. Then $\mathbb{PA}(R)$ is a spanning subgraph of $\Gamma(R[[X]])$.*

Proof. We know from Lemma 2.1 that $Z(R[[X]])^*$ is infinite. Let $f(X), g(X) \in Z(R[[X]])^*$ be distinct. Let $f(X) = \sum_{i=0}^{\infty} a_i X^i$ and let $g(X) = \sum_{j=0}^{\infty} b_j X^j$. Suppose that $f(X)$ and $g(X)$ are adjacent in $\mathbb{PA}(R)$. Therefore, $a_i b_j = 0$ for all i and j. This implies that $f(X)g(X) = 0$. Hence, $f(X)$ and $g(X)$ are adjacent in $\Gamma(R[[X]])$. As $V(\mathbb{PA}(R)) = V(\Gamma(R[[X]]))$, it follows that $\mathbb{PA}(R)$ is a spanning subgraph of $\Gamma(R[[X]])$. □

Remark 2.4. It is easy to verify from [8, Example 3] that Lemma 2.2 fails when R is not Noetherian. Moreover, the converse of Lemma 2.3 is not true in general. This can be verified from the following example.

Example 2.5. Consider the direct product of $\mathbb{Z}_8$ and $\mathbb{Z}_8$, that is, $\mathbb{Z}_8 \times \mathbb{Z}_8 = \{(r, m) \mid r, m \in \mathbb{Z}_8\}$. This can be made into a commutative ring by defining the multiplication as follows: $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$ for any $(r_1, m_1), (r_2, m_2) \in \mathbb{Z}_8 \times \mathbb{Z}_8$. The ring obtained in this way is called the ring obtained by using *Nagata's principle of idealization* and is denoted by $\mathbb{Z}_8(+)\mathbb{Z}_8$. For convenience, let us denote it by $R = \mathbb{Z}_8(+)\mathbb{Z}_8$. Show that $\mathbb{PA}(R)$ and $\Gamma(R[[X]])$ are different graphs.

Proof. Let $f(X), g(X) \in Z(R[X])^*$ be given by $f(X) = (4, 2) + (4, 1)X$ and $g(X) = (4, 0) + (4, 1)X$. Note that $f(X)g(X) = 0$. This implies that $f(X)$ and $g(X)$ are adjacent in $\Gamma(R[X])$ and so, $f(X)$ and $g(X)$ are adjacent in $\Gamma(R[[X]])$. As $(4, 1)(4, 0) = (0, 4) \neq (0, 0)$, it follows that $f(X)$ and $g(X)$ are not adjacent in $\mathbb{PA}(R)$. This shows that $\mathbb{PA}(R)$ and $\Gamma(R[[X]])$ are different graphs. □

Lemma 2.6. *Let R be a power serieswise Armendariz ring. Then $\mathbb{PA}(R)$ and $\Gamma(R[[X]])$ coincide. In particular, for a reduced ring R, $\mathbb{PA}(R)$ and $\Gamma(R[[X]])$ coincide.*

Proof. By hypothesis, R is a power serieswise Armendariz ring. We know from Lemma 2.3 that $\mathbb{PA}(R)$ is a spanning subgraph of $\Gamma(R[[X]])$. Therefore, it is enough to prove that every edge of $\Gamma(R[[X]])$ is an edge of $\mathbb{PA}(R)$. Let $f(X), g(X) \in Z(R[[X]])^*$ be distinct. Let $f(X) = \sum_{i=0}^{\infty} a_i X^i$ and let $g(X) = \sum_{j=0}^{\infty} b_j X^j$. Suppose that $f(X)$ and $g(X)$ are adjacent in $\Gamma(R[[X]])$. This implies that $f(X)g(X) = 0$. As R is a power serieswise Armendariz ring, it follows that $a_i b_j = 0$ for all i and j. Hence, $f(X)$ and $g(X)$ are adjacent in $\mathbb{PA}(R)$. Thus, $\mathbb{PA}(R)$ and $\Gamma(R[[X]])$ coincide. In particular, if R is reduced, then we know from [12, Lemma 2.3(1)] that R is the power serieswise Armendariz ring. This proves that $\mathbb{PA}(R)$ and $\Gamma(R[[X]])$ coincide. □

Theorem 2.7. *Let R be a Noetherian ring. Then $\mathbb{PA}(R)$ has a universal vertex if and only if $Z(R)$ is an annihilator ideal of R.*

Proof. If $|Z(R)^*| = 1$, then $R \cong \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2[X]/X^2\mathbb{Z}_2[X]$. If $R = \mathbb{Z}_4$, then $Z(R) = 2\mathbb{Z}_4 = ann(2)$ and $f(X) = 2$ is a universal vertex of $\mathbb{PA}(R)$. If $R = \mathbb{Z}_2[X]/X^2\mathbb{Z}_2[X]$, then $Z(R) = (X + X^2\mathbb{Z}_2[X])R = ann(X + X^2\mathbb{Z}_2[X])$ and $f(X) = X + X^2\mathbb{Z}_2[X]$ is a universal vertex of $\mathbb{PA}(R)$. Hence, we can assume that R has at least two non-zero zero-divisors.

Let $f(X) = \sum_{i=0}^{\infty} a_i X^i \in Z(R[[X]])^*$ be a universal vertex of $\mathbb{PA}(R)$. Let i be the first non-negative integer such that $a_i \neq 0$. Let $r \in Z(R)^* \backslash \{a_i\}$. Then $rX^i \in Z(R[[X]])^*$ and $rX^i \neq f(X)$. As $f(X)$ is a universal vertex of $\mathbb{PA}(R)$, it follows that $f(X)$ and rX^i are adjacent in $\mathbb{PA}(R)$. Therefore, $ra_i = 0$ and this shows that a_i is a universal vertex of $\Gamma(R)$. This implies that $R \cong \mathbb{Z}_2 \times A$, where A is an integral domain or $Z(R)$ is an annihilator ideal of R [2, Theorem 2.5]. If $R \cong \mathbb{Z}_2 \times A$, then we identify R with $\mathbb{Z}_2 \times A$. Let $i \geq 0$ be least such that $a_i \neq 0$. Note that either $a_i = (1, 0)$ or $a_i = (0, a)$ for some non-zero $a \in A$. Assume that $a_i = (1, 0)$. Then $g(X) = (1, 0)X^{i+1} \in Z(R[[X]])^*$ and $f(X) \neq g(X)$. Note that $f(X)$ and $g(X)$ are not adjacent in $\mathbb{PA}(R)$. This is a contradiction. If $a_i = (0, a)$ for some non-zero $a \in A$, then $g(X) = (0, a)X^{i+1}$ is such that $g(X) \in Z(R[[X]])^*$, $f(X) \neq g(X)$. Observe that $f(X)$ and $g(X)$ are not adjacent in $\mathbb{PA}(R)$. This is again a contradiction and so, $Z(R)$ is an annihilator ideal of R. Note that this part of the proof does need the hypothesis that R is Noetherian.

Conversely, suppose that $Z(R)$ is an annihilator ideal of R. Then there exists $b \in R \backslash \{0\}$ such that $Z(R) = ann(b)$. Let $f(X) = b$. Then $f(X)$ is adjacent to all other vertices of $\mathbb{PA}(R)$. Hence, $f(X)$ is a universal vertex of $\mathbb{PA}(R)$. Thus, $\mathbb{PA}(R)$ has a universal vertex. □

Example 2.8. Let $(R, \mathfrak{m})$ be a quasilocal ring which is not a field. If $\mathfrak{m}$ is nilpotent, then $\mathbb{PA}(R)$ has a universal vertex.

Proof. Note that there exists $n \geq 2$ least with the property that $\mathfrak{m}^n = (0)$. It is clear that $\mathfrak{m}$ is maximal with respect to the property of being contained in

$Z(R)$. Let $r \in \mathfrak{m}^{n-1}$ be such that $r \neq 0$. Observe that $Z(R) = \mathfrak{m} = ann(r)$. Let $f(X) = r$. As $\mathfrak{m}$ is nilpotent, it follows that $Z(R[[X]]) = \mathfrak{m}[[X]]$. Hence, $f(X)$ is a universal vertex. □

Theorem 2.9. *Let R_1 and R_2 be two integral domains and let $R = R_1 \times R_2$. Then $\mathbb{PA}(R)$ is complete bipartite.*

Proof. Note that R is reduced. Therefore, we obtain from Lemma 2.6 that $\mathbb{PA}(R) = \Gamma(R[[X]])$. Observe that $R[[X]]$ is the direct product of two integral domains. Hence, $\Gamma(R[[X]])$ is complete bipartite and so, $\mathbb{PA}(R)$ is complete bipartite. □

3. On the diameter of $\mathbb{PA}(R)$

The aim of this section is to prove some results on the diameter of $\mathbb{PA}(R)$. For a Noetherian ring R, we prove that $\mathbb{PA}(R)$ is connected and $diam(\mathbb{PA}(R)) \leq 3$. Moreover, we determine the relationship among diameters of $\mathbb{PA}(R)$, $\Gamma(R[[X]])$, and $\Gamma(R)$. We start this section with the following theorem.

Theorem 3.1. *Let R be a Noetherian ring. Then $\mathbb{PA}(R)$ is connected and $diam(\mathbb{PA}(R)) \leq 3$.*

Proof. Let $f(X), g(X) \in Z(R[[X]])^*$ be distinct. Let $f(X) = \sum_{i=0}^{\infty} a_i X^i$ and let $g(X) = \sum_{j=0}^{\infty} b_j X^j$. If $f(X)$ and $g(X)$ are adjacent in $\mathbb{PA}(R)$, then $d(f(X), g(X)) = 1$. Suppose that $f(X)$ and $g(X)$ are not adjacent in $\mathbb{PA}(R)$. We know from [8, Theorem 5] that there exist $r, s \in R \backslash \{0\}$ such that $rf(X) = 0 = sg(X)$. Now, we consider the following cases.

Case (1): Suppose that $rs \neq 0$. Let $h(X) = rs$. This implies that $f(X)h(X) = g(X)h(X) = 0$. This shows that $f(X) - h(X) - g(X)$ is a path of length two from $f(X)$ to $g(X)$ in $\mathbb{PA}(R)$.

Case (2): Suppose that $rs = 0$ and $r = s$. Let $q(X) = r = s$. This implies that $f(X) - q(X) - g(X)$ is a path of length two from $f(X)$ to $g(X)$ in $\mathbb{PA}(R)$.

Case (3): Suppose that $rs = 0$ and $r \neq s$. Let $k(X) = r$ and $t(X) = s$. This implies that $f(X)k(X) = 0 = k(X)t(X) = g(X)t(X)$. This shows that $f(X) - k(X) - t(X) - g(X)$ is a path of length three from $f(X)$ to $g(X)$ in $\mathbb{PA}(R)$.

From the above discussion, we obtain that $d(f(X), g(X)) \leq 3$. Hence, $\mathbb{PA}(R)$ is connected and $diam(\mathbb{PA}(R)) \leq 3$. □

Remark 3.2. Theorem 3.1 need not be true if R is not Noetherian. Suppose that R is not Noetherian. Then it is easy to verify from [8, Example 3] that $\mathbb{PA}(R)$ is not connected. We next give an example to compare the diameter of $\Gamma(R)$ and that of $\mathbb{PA}(R)$.

Example 3.3. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2$. Show that $diam(\Gamma(R)) \neq diam(\mathbb{PA}(R)) = diam(\Gamma(R[[X]]))$.

Proof. Note that $Z(R)^* = \{(1,0), (0,1)\}$. As $(1,0)(0,1) = (0,0)$, it follows that $\Gamma(R)$ is complete. We know that $\mathbb{Z}_2$ is Noetherian. This implies that R is Noetherian. As $Z(R)^* \neq \emptyset$, it follows from Lemma 2.1 that $Z(R[[X]])^*$ is infinite. We know from Theorem 3.1 that $\mathbb{PA}(R)$ is connected and $diam(\mathbb{PA}(R)) \leq 3$. Let $f(X), g(X) \in Z(R[[X]])^*$ be given by $f(X) = \sum_{i=0}^{\infty}(0,1)X^i$ and $g(X) = (0,1) + \sum_{i=0}^{\infty}(0,1)X^{i+2}$. Let $h(X) = (1,0)$. Hence, $f(X) - h(X) - g(X)$ is a path of length two from $f(X)$ to $g(X)$ in $\mathbb{PA}(R)$ but $f(X)$ and $g(X)$ are not adjacent in $\mathbb{PA}(R)$. Note that R is reduced. Therefore, we obtain from Lemma 2.6 that $\mathbb{PA}(R) = \Gamma(R[[X]])$. Observe that $R[[X]]$ is the direct product of two integral domains, that is, $R[[X]] \cong \mathbb{Z}_2[[X]] \times \mathbb{Z}_2[[X]]$. Hence, $\Gamma(R[[X]])$ is complete bipartite with infinite parts. As $\mathbb{PA}(R) = \Gamma(R[[X]])$, it follows that $diam(\mathbb{PA}(R)) = diam(\Gamma(R[[X]])) = 2$. Thus, $diam(\Gamma(R)) \neq diam(\mathbb{PA}(R)) = diam(\Gamma(R[[X]]))$. Now, it is clear that $\mathbb{PA}(R)$ and $\Gamma(R[[X]])$ are infinite complete bipartite graphs but $\Gamma(R)$ is a finite complete graph. However, there is a relationship between $\Gamma(R)$ and $\mathbb{PA}(R)$, where $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$. This relationship is established in the next theorem. □

Theorem 3.4. *Let R be a commutative ring and $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then the following statements are equivalent:*

(1) $\Gamma(R[[X]])$ *is complete.*
(2) $\mathbb{PA}(R)$ *is complete.*
(3) $A(R)$ *is complete.*
(4) $\Gamma(R)$ *is complete.*

Proof. Note that $\Gamma(R)$ is a subgraph of $A(R)$, $A(R)$ is a subgraph of $\mathbb{PA}(R)$, and $\mathbb{PA}(R)$ is a spanning subgraph of $\Gamma(R[[X]])$. Therefore, (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are clear. Moreover, (4) $\Rightarrow$ (1) follows from (3) $\Rightarrow$ (1) of [4, Theorem 3.2]. □

Theorem 3.5. *Let R be a Noetherian ring such that $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $diam(\mathbb{PA}(R)) = 2$ if and only if $diam(\Gamma(R[[X]])) = 2$.*

Proof. Suppose that $diam(\mathbb{PA}(R)) = 2$. As $\mathbb{PA}(R)$ is a spanning subgraph of $\Gamma(R[[X]])$, it follows that $diam(\Gamma(R[[X]])) \leq 2$. We claim that $diam(\Gamma(R[[X]])) = 2$. Suppose that $diam(\Gamma(R[[X]])) = 1$. Then it follows from (1) $\Rightarrow$ (2) of Theorem 3.4 that $diam(\mathbb{PA}(R)) = 1$. This is a contradiction. Therefore, we obtain that $diam(\Gamma(R[[X]])) = 2$.

Conversely, suppose that $diam(\Gamma(R[[X]])) = 2$. In this case, we know from (4) $\Rightarrow$ (5) of [4, Theorem 3.11] that either $Z(R)$ is the union of two primes with intersection (0) or $Z(R)$ is prime and $(Z(R))^2 \neq (0)$. As $diam(\Gamma(R[[X]])) = 2$, it follows from (2) $\Rightarrow$ (1) of Theorem 3.4 that $diam(\mathbb{PA}(R)) \geq 2$. Let $f(X) = \sum_{i=0}^{\infty} a_i X^i$, $g(X) = \sum_{j=0}^{\infty} b_j X^j \in Z(R[[X]])^*$ be two non-adjacent vertices of $\mathbb{PA}(R)$.

Suppose that $Z(R)$ is the union of two primes say $\mathfrak{p}_1$ and $\mathfrak{p}_2$ with their intersection (0). This implies that $Z(R) = \mathfrak{p}_1 \cup \mathfrak{p}_2$ and $\mathfrak{p}_1 \cap \mathfrak{p}_2 = (0)$. It is easy to verify that $Z(R[[X]]) = \mathfrak{p}_1[[X]] \bigcup \mathfrak{p}_2[[X]]$ and $\mathfrak{p}_1[[X]] \bigcap \mathfrak{p}_2[[X]] = (0)$. Hence, we obtain that $f(X), g(X) \in \mathfrak{p}_1[[X]] \bigcup \mathfrak{p}_2[[X]]$. By hypothesis, $f(X)$ and $g(X)$ are non-adjacent vertices of $\mathbb{PA}(R)$. This implies that either $f(X), g(X) \in \mathfrak{p}_1[[X]] \backslash \mathfrak{p}_2[[X]]$ or $f(X), g(X) \in \mathfrak{p}_2[[X]] \backslash \mathfrak{p}_1[[X]]$. Without loss of generality, suppose that $f(X), g(X) \in \mathfrak{p}_1[[X]] \backslash \mathfrak{p}_2[[X]]$. As R is Noetherian, it follows from [8, Theorem 5] that there exist $r, s \in R \backslash \{0\}$ such that

$rf(X) = 0 = sg(X)$. From $rf(X) = 0$ and $f(X) \notin \mathfrak{p}_2[[X]]$, we obtain that $r \in \mathfrak{p}_2 \backslash \mathfrak{p}_1$. Similarly, we can prove that $s \in \mathfrak{p}_2 \backslash \mathfrak{p}_1$. This shows that $rs \neq 0$. Let $h(X) = rs$. Hence, $f(X) - h(X) - g(X)$ is a path of length two from $f(X)$ to $g(X)$ in $\mathbb{PA}(R)$. This proves that $diam(\mathbb{PA}(R)) = 2$.

Suppose that $Z(R)$ is prime and $(Z(R))^2 \neq (0)$. As R is Noetherian, it follows that $Z(R)$ is finitely generated. Hence, we obtain from [11, Theorem 82] that there exists $r \in Z(R)^*$ such that $rZ(R) = (0)$. As $Z(R[[X]]) = Z(R)[[X]]$, we get that $rZ(R[[X]]) = (0)$. Let $t(X) = r$. This shows that $f(X) - t(X) - g(X)$ is a path of length two from $f(X)$ to $g(X)$ in $\mathbb{PA}(R)$. This proves that $diam(\mathbb{PA}(R)) = 2$. □

Theorem 3.6. *Let R be a Noetherian ring such that $R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then the following statements are equivalent:*

(1) *$diam(\Gamma(R)) = 2$.*
(2) *$diam(\Gamma(R[X])) = 2$.*
(3) *$diam(\Gamma(R[X_1, X_2, \ldots, X_n])) = 2$ for all $n > 0$.*
(4) *$diam(\Gamma(R[[X]])) = 2$.*
(5) *Either $Z(R)$ is the union of two primes with intersection (0) or $Z(R)$ is prime and $(Z(R))^2 \neq (0)$.*
(6) *$diam(\mathbb{PA}(R)) = 2$.*

Proof. Note that (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) are clear from [4, Theorem 3.11]. Moreover, (5) ⇒ (6) is clear from the converse part of Theorem 3.5. Finally, we prove (6) ⇒ (1). Suppose that $diam(\mathbb{PA}(R)) = 2$ and $diam(\Gamma(R)) \neq 2$. We know from [2, Theorem 2.3] that $\Gamma(R)$ is connected and $diam(\Gamma(R)) \leq 3$. Therefore, we obtain that $diam(\Gamma(R)) \in \{0, 1, 3\}$. As $diam(\mathbb{PA}(R)) = 2$, it follows that $diam(\Gamma(R)) \neq 0$. If $diam(\Gamma(R)) = 1$, then we obtain from Theorem 3.4 that $\mathbb{PA}(R)$ is complete. This is a contradiction. Finally, suppose that $diam(\Gamma(R)) = 3$. As $\Gamma(R)$ is a subgraph of $\mathbb{PA}(R)$, it follows that $diam(\Gamma(R)) \leq diam(\mathbb{PA}(R))$. This implies that $diam(\mathbb{PA}(R)) \geq 3$. This is again a contradiction. This proves that $diam(\Gamma(R)) = 2$. □

Theorem 3.7. *Let R be a Noetherian ring such that $R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $diam(\mathbb{PA}(R)) = 3$ if and only if $diam(\Gamma(R[[X]])) = 3$.*

Proof. The proof is clear from the proof of Theorem 3.4, Theorem 3.5 and the fact that $\mathbb{PA}(R)$ is a spanning subgraph of $\Gamma(R[[X]])$. □

Theorem 3.8. *Let R be a Noetherian ring such that $R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $diam(\mathbb{PA}(R)) = diam(\Gamma(R[[X]])) = diam(\Gamma(R))$.*

Proof. Note that $V(\mathbb{PA}(R)) = V(\Gamma(R[[X]]))$. As $\mathbb{PA}(R)$ is a spanning subgraph of $\Gamma(R[[X]])$, it follows that $diam(\mathbb{PA}(R)) \geq diam(\Gamma(R[[X]]))$. We know from [2, Theorem 2.3] that $\Gamma(R)$ is connected and $diam(\Gamma(R)) \leq 3$. If $diam(\Gamma(R)) = 1$, then we obtain from Theorem 3.4 that $diam(\mathbb{PA}(R)) = diam(\Gamma(R[[X]])) = diam(\Gamma(R)) = 1$. If $diam(\Gamma(R)) = 2$, then we obtain from Theorem 3.6 that $diam(\mathbb{PA}(R)) = diam(\Gamma(R[[X]])) = diam(\Gamma(R)) = 2$. Finally, if $diam(\Gamma(R)) = 3$, then we obtain from $diam(\Gamma(R)) \leq diam(\Gamma(R[[X]])) \leq$

$diam(\mathbb{PA}(R)) \leq 3$ together with Theorem 3.7 that $diam(\mathbb{PA}(R)) = diam(\Gamma(R[[X]])) = diam(\Gamma(R)) = 3$. □

Proposition 3.9. *Let R be a Noetherian ring such that $R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $|Min(R)| = 2$. Then $diam(\mathbb{PA}(R)) = diam(\Gamma(R[[X]])) = 2$ if and only if R is reduced.*

Proof. By hypothesis, R is a Noetherian ring such that $R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $|Min(R)| = 2$. Let $Min(R) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$. Suppose that $diam(\mathbb{PA}(R)) = diam(\Gamma(R[[X]])) = 2$. Hence, we obtain from (4) $\Rightarrow$ (5) of Theorem 3.6 that either $Z(R)$ is the union of two primes with intersection (0) or $Z(R)$ is prime and $(Z(R))^2 \neq (0)$. As $|Min(R)| = 2$, it follows that $Z(R) = \mathfrak{p}_1 \cup \mathfrak{p}_2$ and $\mathfrak{p}_1 \cap \mathfrak{p}_2 = (0)$. Moreover, we obtain from [3, Proposition 1.8] and [11, Theorem 10] that $nil(R) = \mathfrak{p}_1 \cap \mathfrak{p}_2 = (0)$. Hence, R is reduced.

Conversely, suppose that R is reduced with $|Min(R)| = 2$. Therefore, $(0) = nil(R) = \mathfrak{p}_1 \cap \mathfrak{p}_2$ and $Z(R) = \mathfrak{p}_1 \cup \mathfrak{p}_2$. Hence, we obtain from (5) $\Rightarrow$ (6) and (5) $\Rightarrow$ (2) of Theorem 3.6 that $diam(\mathbb{PA}(R)) = diam(\Gamma(R[[X]])) = 2$. □

Lemma 3.10. *Let R be a Noetherian ring. If $Z(R)$ is an annihilator ideal, then $\mathbb{PA}(R)$ is connected and $diam(\mathbb{PA}(R)) \leq 2$.*

Proof. Let $f(X), g(X) \in Z(R[[X]])^*$ be distinct. Let $f(X) = \sum_{i=0}^{\infty} a_i X^i$ and let $g(X) = \sum_{j=0}^{\infty} b_j X^j$. If $f(X)$ and $g(X)$ are adjacent in $\mathbb{PA}(R)$, then $d(f(X), g(X)) = 1$. Suppose that $f(X)$ and $g(X)$ are not adjacent in $\mathbb{PA}(R)$. We claim that $d(f(X), g(X)) = 2$. By hypothesis, $Z(R)$ is an annihilator ideal. Suppose that $Z(R) = ann(r)$ for some $r \in R \backslash \{0\}$. Therefore, we obtain from the converse part of Theorem 2.7 that r is a universal vertex of $\mathbb{PA}(R)$. Note that $r \in Z(R)^* \subseteq Z(R[[X]])^*$. Let $h(X) = r$. This implies that $f(X) - h(X) - g(X)$ is a path of length two from $f(X)$ to $g(X)$ in $\mathbb{PA}(R)$. This proves that $\mathbb{PA}(R)$ is connected and $diam(\mathbb{PA}(R)) \leq 2$. □

We know that $\mathbb{Z}_n$ is a Noetherian ring for all integers $n \geq 2$. In [14, Theorem 2.2], Park et al. determined $diam(\Gamma(\mathbb{Z}_n[X]])$, where $\mathbb{Z}_n[X]]$ is either $\mathbb{Z}_n[X]$ or $\mathbb{Z}_n[[X]]$. So, we use [14, Theorem 2.2] to determine $diam(\mathbb{PA}(\mathbb{Z}_n))$ in the following example.

Example 3.11. Let $R = \mathbb{Z}_n$ for some integer $n \geq 2$. Then the following statements hold:

(1) $diam(\mathbb{PA}(\mathbb{Z}_n)) = 1$ if and only if $n = p^2$ for some prime p.
(2) $diam(\mathbb{PA}(\mathbb{Z}_n)) = 2$ if and only if $n = p^r$ for some prime p and some integer $r \geq 3$ or $n = pq$ for some distinct primes p and q.
(3) $diam(\mathbb{PA}(\mathbb{Z}_n)) = 3$ if and only if $n = pqr$ for some distinct primes p, q, and some integer $r \geq 2$.

Proof. By hypothesis, $R = \mathbb{Z}_n$ for some integer $n \geq 2$. Note that $Z(\mathbb{Z}_n)^* \neq \emptyset$. As $\mathbb{Z}_n$ is Noetherian and $\mathbb{Z}_n \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$, it follows from Theorem 3.8 that $diam(\mathbb{PA}(\mathbb{Z}_n)) = diam(\Gamma(\mathbb{Z}_n[[X]])) = diam(\Gamma(\mathbb{Z}_n))$. Hence, the proof is clear from [14, Theorem 2.2]. □

Remark 3.12. Let $R = \mathbb{Z}_{pq}$ for some distinct primes p and q. We know that $\mathbb{Z}_n$ is reduced if and only if $n = 0$ or n is a square-free integer. This implies that $\mathbb{Z}_{pq}$ is a Noetherian reduced ring. Hence, we obtain from Lemma 2.6 that $\mathbb{PA}(\mathbb{Z}_{pq}) = \Gamma(\mathbb{Z}_{pq}[[X]])$. Moreover, we know from [14, Theorem 2.2(2)] that $\Gamma(\mathbb{Z}_{pq}[[X]])$ is complete bipartite. Therefore, $\mathbb{PA}(\mathbb{Z}_{pq})$ is complete bipartite.

4. On the clique of $\mathbb{PA}(R)$

The aim of this section is to prove some interesting results on the clique of $\mathbb{PA}(R)$. In particular, we provide some sufficient conditions under which $\mathbb{PA}(R)$ contains an infinite clique. Moreover, we find $\omega(\mathbb{PA}(R))$ and $\chi(\mathbb{PA}(R))$ when $\omega(\Gamma(R[[X]]))$ is finite. We start this section with the following lemma.

Lemma 4.1. *Let R be a ring. If there exists $a \in Z(R)^*$ such that $a^2 = 0$, then $\mathbb{PA}(R)$ contains an infinite clique and so, $\omega(\mathbb{PA}(R)) = \infty$.*

Proof. Note that for any $n \in \mathbb{N}$, $aX^n \in Z(R[X])^* \subseteq Z(R[[X]])^*$ and for any distinct $m, n \in \mathbb{N}$, $aX^m \neq aX^n$. From $a^2 = 0$ and $X \notin Z(R[[X]])^*$, it follows that $(aX^m)(aX^n) = a^2X^{m+n} = 0$. Therefore, the subgraph of $\mathbb{PA}(R)$ induced by $\{aX^n \mid n \in \mathbb{N}\}$ is an infinite clique. Hence, $\omega(\mathbb{PA}(R)) = \infty$. □

Corollary 4.2. *Let R be a ring such that $Z(R)$ is an annihilator ideal. Then $\mathbb{PA}(R)$ contains an infinite clique and so, $\omega(\mathbb{PA}(R)) = \infty$.*

Proof. By hypothesis, $Z(R)$ is an annihilator ideal. Suppose that $Z(R) = ann(a)$ for some $a \in R\backslash\{0\}$. Note that $a \in Z(R)^*$. Therefore, $a^2 = 0$. Thus, we obtain from Lemma 4.1 that $\mathbb{PA}(R)$ contains an infinite clique and so, $\omega(\mathbb{PA}(R)) = \infty$. □

Corollary 4.3. *Let R be a ring which is not reduced. Then $\mathbb{PA}(R)$ contains an infinite clique and so, $\omega(\mathbb{PA}(R)) = \infty$.*

Proof. By hypothesis, R is not reduced. Therefore, it is possible to find $a \in R\backslash\{0\}$ such that $a^2 = 0$. Note that $a \in Z(R)^*$. Thus, we obtain from Lemma 4.1 that $\mathbb{PA}(R)$ contains an infinite clique and so, $\omega(\mathbb{PA}(R)) = \infty$. □

Example 4.4. Let $R = \mathbb{Z}_{p^2}$ for some prime p. Then $\mathbb{PA}(\mathbb{Z}_{p^2})$ contains an infinite clique and so, $\omega(\mathbb{PA}(\mathbb{Z}_{p^2})) = \infty$.

Proof. By hypothesis, $R = \mathbb{Z}_{p^2}$. Note that there exists $p \in Z(\mathbb{Z}_{p^2})$ such that $p^2 = 0$. Therefore, we obtain from Lemma 4.1 that $\mathbb{PA}(\mathbb{Z}_{p^2})$ contains an infinite clique and so, $\omega(\mathbb{PA}(\mathbb{Z}_{p^2})) = \infty$. □

Proposition 4.5. *Let R be a ring. Then the following statements are equivalent:*
(1) $\omega(\Gamma(R[[X]])) < \infty$.
(2) $\omega(\mathbb{PA}(R)) < \infty$.
(3) $\mathbb{PA}(R)$ *does not contain any infinite clique.*
(4) R *is reduced and* $\Gamma(R)$ *does not contain any infinite clique.*

Moreover, if any of the above equivalent statements hold, then $\omega(\mathbb{PA}(R)) = \chi(\mathbb{PA}(R)) = n$, where n is the number of minimal prime ideals of R.

Proof. (1) $\Rightarrow$ (2): This is clear, since $\mathbb{PA}(R)$ is a spanning subgraph of $\Gamma(R[[X]])$.
(2) $\Rightarrow$ (3): This is clear.
(3) $\Rightarrow$ (4): By assumption, $\mathbb{PA}(R)$ does not contain any infinite clique. First, we prove that R is reduced. Suppose that R is not reduced. Then we obtain from Corollary 4.3 that $\mathbb{PA}(R)$ contains an infinite clique. This is a contradiction. Therefore, R is reduced. Now, we claim that $\Gamma(R)$ does not contain any infinite clique. Suppose that $\Gamma(R)$ contains an infinite clique. As $\Gamma(R)$ is a subgraph of $\mathbb{PA}(R)$, it follows that $\mathbb{PA}(R)$ contains an infinite clique. This is again a contradiction. Therefore, $\Gamma(R)$ does not contain any infinite clique.
(4) $\Rightarrow$ (1): Assume that R is reduced and $\Gamma(R)$ does not contain any infinite clique. It is convenient to denote Beck's zero-divisor graph of R by $\Gamma_0(R)$. Hence, $\Gamma_0(R)$ does not contain any infinite clique. Therefore, we obtain from (4) $\Rightarrow$ (3) of [7, Theorem 3.7] that (0) is the intersection of a finite number of prime ideals of R. As any prime ideal of R contains a minimal prime ideal of R by [11, Theorem 10], we obtain that there exist minimal prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$ of R such that $(0) = \bigcap_{i=1}^{n} \mathfrak{p}_i$. Note that $n \geq 2$, since R is not an integral domain. Observe that for each $i \in \{1, 2, \ldots, n\}$, $\mathfrak{p}_i[[X]] \in Spec(R[[X]])$ and $\bigcap_{i=1}^{n} \mathfrak{p}_i[[X]] = (0)$. Hence, we get that $\{\mathfrak{p}_i[[X]] \mid i \in \{1, 2, \ldots, n\}\}$ is the set of all minimal prime ideals of $R[[X]]$. Observe that $R[[X]]$ is reduced. By applying (3) $\Rightarrow$ (2) of [7, Theorem 3.7] to $R[[X]]$, it follows that $\omega(\Gamma_0(R[[X]])) < \infty$ and so, $\omega(\Gamma(R[[X]])) < \infty$. Next, we prove the moreover part.

Assume that (4) holds. Thus, R is reduced and $\Gamma(R)$ does not contain any infinite clique. It is shown in the proof of (4) $\Rightarrow$ (1) of this proposition that $(0) = \bigcap_{i=1}^{n} \mathfrak{p}_i$, where $\{\mathfrak{p}_i \mid i \in \{1, 2, \ldots, n\}\}$ is the set of all minimal prime ideals of R, $(0) = \bigcap_{i=1}^{n} \mathfrak{p}_i[[X]]$, and $\{\mathfrak{p}_i[[X]] \mid i \in \{1, 2, \ldots, n\}\}$ is the set of all minimal prime ideals of $R[[X]]$. By applying (3) $\Rightarrow$ (1) of [7, Theorem 3.7] to $R[[X]]$, it follows that $\chi(\Gamma_0(R[[X]])) < \infty$. By applying [7, Theorem 3.8] to $R[[X]]$, we get that $\omega(\Gamma_0(R[[X]])) = \chi(\Gamma_0(R[[X]])) = n+1$ and so, $\omega(\Gamma(R[[X]])) = \chi(\Gamma(R[[X]])) = n+1-1 = n$. Since R is reduced, $\mathbb{PA}(R) = \Gamma(R[[X]])$. Therefore, we obtain that $\omega(\mathbb{PA}(R)) = \chi(\mathbb{PA}(R)) = n$, where n is the number of minimal prime ideals of R. □

Example 4.6. For any fixed integer $r \geq 2$, let $n = p_1 p_2 \cdots p_r$ for distinct primes $p_1, p_2, \ldots, p_r$. Then $\omega(\mathbb{PA}(\mathbb{Z}_n)) = \chi(\mathbb{PA}(\mathbb{Z}_n)) = r$.

Proof. We know from [14, Lemma 3.1] that $\omega(\Gamma(\mathbb{Z}_n[[X]])) < \infty$. Note that $Min(\mathbb{Z}_n) = \{p_i\mathbb{Z}_n \mid i \in \{1, 2, \ldots, r\}\}$. Therefore, $|Min(\mathbb{Z}_n)| = r$. Hence, we obtain from moreover part of Proposition 4.5 that $\omega(\mathbb{PA}(\mathbb{Z}_n)) = \chi(\mathbb{PA}(\mathbb{Z}_n)) = r$, where r is the number of minimal prime ideals of $\mathbb{Z}_n$. □

5. On the girth of $\mathbb{PA}(R)$

In this section, we discuss some results on the girth of $\mathbb{PA}(R)$. If R is non-reduced, then we show that $gr(\mathbb{PA}(R)) = gr(\Gamma(R[[X]])) = 3$. For a reduced ring R, we prove that $gr(\mathbb{PA}(R)) \leq 4$ (respectively, $gr(\Gamma(R[[X]])) \leq 4$). We start this section with the following proposition.

Proposition 5.1. *Let R be a ring which is not reduced. Then $gr(\mathbb{PA}(R)) = gr(\Gamma(R[[X]])) = 3$.*

Proof. By hypothesis, R is not reduced. Hence, it is possible to find $a \in R \backslash \{0\}$ such that $a^2 = 0$. Therefore, we obtain from Lemma 4.1 that $\mathbb{PA}(R)$ contains an infinite clique. Hence, $gr(\mathbb{PA}(R)) = 3$. As $\mathbb{PA}(R)$ is a spanning subgraph of $\Gamma(R[[X]])$, it follows that $gr(\Gamma(R[[X]])) = 3$. □

Proposition 5.2. *Let R be a reduced ring. Then $gr(\mathbb{PA}(R)) \leq 4$ (respectively, $gr(\Gamma(R[[X]])) \leq 4$).*

Proof. By hypothesis, R is a reduced ring. Therefore, we obtain from Lemma 2.6 that $\mathbb{PA}(R) = \Gamma(R[[X]])$. If $\Gamma(R)$ contains an infinite clique, then it is clear that $gr(\mathbb{PA}(R)) = 3$. Assume that $\Gamma(R)$ does not contain any infinite clique. Then we know from the proof of (4) $\Rightarrow$ (1) of Proposition 4.5 that $\{\mathfrak{p}_i[[X]] \mid i \in \{1, 2, \ldots, n\}\}$ is the set of all minimal prime ideals of $R[[X]]$. Now, either $n \geq 3$ or $n = 2$. If $n \geq 3$, then it is clear that $gr(\mathbb{PA}(R)) = 3$. If $n = 2$, then $\Gamma(R[[X]])$ is complete bipartite with infinite parts say, $V_1 = \mathfrak{p}_1[[X]] \backslash \mathfrak{p}_2[[X]]$ and $V_2 = \mathfrak{p}_2[[X]] \backslash \mathfrak{p}_1[[X]]$. Therefore, $gr(\mathbb{PA}(R)) = 4$.

Thus, we obtain from above discussion that $gr(\mathbb{PA}(R)) \leq 4$. As $\mathbb{PA}(R)$ is a spanning subgraph of $\Gamma(R[[X]])$, it follows that $gr(\Gamma(R[[X]])) \leq 4$. □

Example 5.3. Let $R = \mathbb{Z}_n$ for some integer $n \geq 2$. Then the following statements hold:

(1) $gr(\mathbb{PA}(\mathbb{Z}_n)) = 3$ if and only if $n = p^r$ for some prime p and some integer $r \geq 2$ or $n = pqr$ for some distinct primes p, q, and some integer $r \geq 2$.
(2) $gr(\mathbb{PA}(\mathbb{Z}_n)) = 4$ if and only if $n = pq$ for some distinct primes p and q.

Proof. By hypothesis, $R = \mathbb{Z}_n$ for some integer $n \geq 2$. As $\mathbb{PA}(\mathbb{Z}_n)$ is a spanning subgraph of $\Gamma(\mathbb{Z}_n[[X]])$, it follows that $gr(\mathbb{PA}(\mathbb{Z}_n)) \geq gr(\Gamma(\mathbb{Z}_n[[X]]))$. Hence, the proof of one side is clear from [14, Theorem 2.8]. We next prove the other side of the proof.

(1) Suppose that $n = p^r$ for some prime p and some integer $r \geq 2$ or $n = pqr$ for some distinct primes p, q, and some integer $r \geq 2$. Note that $\mathbb{Z}_{p^r}$ (respectively, $\mathbb{Z}_{pqr}$) is a non-reduced ring. Therefore, we obtain from Proposition 5.1 that $gr(\mathbb{PA}(\mathbb{Z}_{p^r})) = 3$ (respectively, $gr(\mathbb{PA}(\mathbb{Z}_{pqr})) = 3$).

(2) Suppose that $n = pq$ for some distinct primes p and q. We know from Remark 3.12 that $\mathbb{PA}(\mathbb{Z}_{pq})$ is complete bipartite. Therefore, $gr(\mathbb{PA}(\mathbb{Z}_{pq})) = 4$. □

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