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WELL-POSEDNESS FOR THE STOCHASTIC INCOMPRESSIBLE HALL-MAGNETOHYDRODYNAMIC SYSTEM

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ABSTRACT. In this paper, we first establish the existence and uniqueness of solutions of the stochastic incompressible Hall-magnetohydrodynamic system in the deterministic case in Fourier-Besov-Morrey critical spaces. In the second part, we show the existence and uniqueness of solutions in the stochastic case.

1. INTRODUCTION

In this article, we study the following three-dimensional stochastic incompressible Hall-magnetohydrodynamic system:

$$\begin{cases} \partial_t u - v\Delta u + (u\cdot\nabla)u + \nabla p - (\nabla\times b)\times b = g_1\dot{W}, & t>0, x\in\mathbb{R}^3, \\ \partial_t b - \sigma_D\Delta b + \sigma_H\nabla\times((\nabla\times b)\times b) - \nabla\times(u\times b) = g_2\dot{W}, & t>0, x\in\mathbb{R}^3, \\ \nabla\cdot u = \nabla\cdot b = 0, & t>0, x\in\mathbb{R}^3, \\ (u,b)|_{t=0} = (u_0,b_0), & x\in\mathbb{R}^3. \end{cases} \tag{1.1}$$

In this context, the variables u, b, and p correspond to the velocity, magnetic, and scalar pressure fields, respectively. Additionally, we use $\sigma_H > 0$ and $\nu > 0$ to represent the Hall coefficient and kinematic viscosity, respectively. The variables g_i $(i = 1, 2)$ and W are random external force and a standard infinite dimensional Wiener process, respectively. The diffusive coefficient σ_D can be expressed as $\sigma_D = \frac{1}{\sigma\mu}$, where the constants $\sigma > 0$ refer to the electrical conductivity and $\mu > 0$ refer to magnetic permeability. The given initial data u_0 signify the initial velocity and b_0 represent the initial magnetic field.

If $g_i = 0$, numerous studies have explored the global-in-time well-posedness and stability of equation (1.1) across various functional settings. Acheritogary et al. [3] obtained the global existence of weak solutions for the system (1.1). Ferreira and Benvenutti [7] established the local-in-time well-posedness of strong solutions in the H^2 space. In [8], the authors established the local existence of smooth solutions for large data and global smooth solutions for small data

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within the incompressible viscous or inviscid Hall-MHD model. More recently, Raphael and Jin [10] delved into the existence and uniqueness issues of the three-dimensional incompressible Hall-magnetohydrodynamic equations within critical regularity spaces. In [23], Renhui Wan and Yong Zhou obtained two Fujita-Kato type results applicable to the 3D Hall-magnetohydrodynamic equations. In 2021, Nakasato [1] proved the existence of a global-in-time solution in critical Fourier-Besov spaces. Motivated by the aforementioned research, the objective of this paper is to establish the well-posedness of the system (1.1) in critical Fourier-Besov-Morrey spaces, which are larger than Fourier-Besov spaces. It is noteworthy that when $\sigma_H = 0$, the problem (1.1) simplifies to the well-known incompressible magnetohydrodynamic system,

$$\begin{cases} \partial_t u - v\Delta u + (u\cdot\nabla)u + \nabla p = (\nabla\times b)\times b, & t>0, x\in\mathbb{R}^3,\\ \partial_t b - \sigma_D\Delta b = \nabla\times(u\times b), & t>0, x\in\mathbb{R}^3,\\ \nabla\cdot u = \nabla\cdot b = 0, & t>0, x\in\mathbb{R}^3,\\ (u,b)|_{t=0} = (u_0,b_0), & x\in\mathbb{R}^3. \end{cases} \tag{1.2}$$

Li and Zheng [14] established the well-posedness and blow-up criterion for the mild solution of the problem (1.2) within the framework of Fourier-Herz space, involving highly oscillating functions. In a different direction, the authors in [17, 18] proved the global well-posedness and a singular limit of problem (1.2) in Fourier-Sobolev spaces. Examining the Cauchy problem (1.2), Miao and Yuan [15] obtained the global well-posedness for small data and local well-posedness for large data in the Besov space $\dot{B}^{\frac{n}{p}-1}_{p,q}(\mathbb{R}^n)$ with $1\le p<\infty$ and $1\le q\le\infty$. Awati and al. in [4] established the concept of weak-strong uniqueness of solutions with initial data in $\dot{B}^{\frac{n}{p}-1}_{p,q}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ for cases where $\frac{n}{2p}+\frac{2}{q}>1$. Moreover, our focus narrows down to a magnetic field solution that closely approximates the constant equilibrium state B at spatial infinity, where $\bar{B}\in\mathbb{R}^3$ represents a constant magnetic field. To reformulate the problem (1.1) under the assumption that $\bar{B}\neq 0$, we introduce a new vector-valued function $B := b-\bar{B}$, leading to the following initial-value problem:

$$\begin{cases} \partial_t u - v\Delta u + \nabla\left(p+\frac{|B|^2}{2}\right) - (\nabla\times B)\times\bar{B} = h, & t>0, x\in\mathbb{R}^3,\\ \partial_t B - \sigma_D\Delta B + \sigma_H\nabla\times((\nabla\times B)\times\bar{B}) - \nabla\times(u\times\bar{B}) = \nabla\times g, & t>0, x\in\mathbb{R}^3,\\ \nabla\cdot u = \nabla\cdot B = 0, & t>0, x\in\mathbb{R}^3,\\ (u,B)|_{t=0} = (u_0,B_0), & x\in\mathbb{R}^3, \end{cases} \tag{1.3}$$

where h and g are the nonlinear terms, defined respectively by:

$$h := -(u\cdot\nabla)u + (B\cdot\nabla)B, \quad g := u\times B - \sigma_H(\nabla\times B)\times B.$$

Numerous researchers have explored the global existence result for the problem (1.3) (see references [9, 11, 25, 13]). In order to determine the appropriate function class for a solution (u,p,b) to the problem (1.3), we examine the following linear system:

$$\begin{cases} \partial_t B - \sigma_D \Delta B + \sigma_H \nabla \times ((\nabla \times B) \times \bar{B}) = f, & t > 0, x \in \mathbb{R}^3, \\ B|_{t=0} = B_0, & x \in \mathbb{R}^3, \end{cases} \tag{1.4}$$

where $\sigma_D, \sigma_H > 0$, $\bar{B} = \left(\bar{B}_1, \bar{B}_2, \bar{B}_3\right)^T \in \mathbb{R}^3$. Notice that $e^{tH} B_0$ is the solution of the linear problem (1.4) and we have,

$$e^{tH} B_0 \simeq e^{i|\bar{B}|t\Delta} e^{t\sigma_D \Delta} B_0$$

$$\left|\widehat{e^{tH} B_0}\right| \simeq e^{-\sigma_D t|\xi|^2} \left|e^{-i|\bar{B}|t|\xi|^2} \widehat{B_0}\right| = \left|\widehat{e^{\sigma_D t \Delta B_0}}\right|.$$

Let us present the scaling property of the system (1.1). If (u, b, p) solves (1.1) with initial data (u_0, b_0, p_0) then $(u_\lambda, b_\lambda, p_\lambda)$ also solves (1.1) with the initial data $(u_{0,\lambda}, b_{0,\lambda}, p_{0,\lambda})$, where

$$\begin{cases} u_\lambda(t, x) = \lambda u\left(\lambda^2 t, \lambda x\right), \\ p_\lambda(t, x) = \lambda^2 p\left(\lambda^2 t, \lambda x\right), \quad \lambda > 0 \\ b_\lambda(t, x) = \lambda b\left(\lambda^2 t, \lambda x\right). \end{cases}$$

The critical space for the system (1.1) is naturally defined as a result of this scaling invariant characteristic.

To facilitate clarity in our description, we introduce the following symbols. The symbol $\mathcal{S}\left(\mathbb{R}^n\right)$ is the usual Schwartz space of infinitely differentiable rapidly decreasing complex-valued functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ be the space of tempered distributions. For $1 \leq p \leq \infty$, we use $L^p = L^p\left(\mathbb{R}^n\right)$ to denote the Lebesgue space and $\widehat{L}^p\left(\mathbb{R}^n\right) = \widehat{L}^p$ to denote the Fourier-Lebesgue space defined by

$$\widehat{L}^p = \{u \in \mathcal{S}';\ \|u\|_{\widehat{L}^p} < \infty\}, \quad \text{where } \|u\|_{\widehat{L}^p} = \|\widehat{u}\|_{L^{p'}},$$

where p' is the conjugate of p satisfying $\frac{1}{p} + \frac{1}{p'} = 1$. We use $\mathcal{P}(\mathbb{R}^n)$ to denote the set of polynomials.

Regarding the stochastic cas (if $g_i \neq 0$), we can see [20] for the regularity result for stochastic magnetohydrodynamic equations in probabilistic evolution spaces of Besov type. Moreover [2] provide a number of results relating to the identification of the regularity of the driving noises and conditions under which the fractional stochastic magnetohydrodynamic system has a martingale solution with $(-\Delta)^{\beta_i}, \quad \beta_i > 0$ in $\mathbb{R}^d$, $d = 2, 3$. Furthermore [19] establish the global well-posedness result in $\tilde{L}^4(0, T; \mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q})$ for small initial data (u_0, b_0) belonging in the critical Fourier-Besov-Morrey spaces $\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}(\mathbb{R}^3)$.

The work is organized as follows. In Section 2, we recall the fundamental framework, defining the functional spaces involved. In Section 3, we present some technical lemmas. In Section 3, we prove our main result in Fourier-Besov-Morrey critical spaces in the deterministic case, and in Section 4, we prove our main result in the stochastic case.

2. Preliminaries and main results

In this section, we will present the Littlewood-Paley theory and Fourier-Besov-Morrey spaces.
Consider the Littlewood-Paley dyadic decomposition of unity denoted by $\{\phi_j\}_{j\in\mathbb{Z}}$, where ϕ is a non-negative radially symmetric function belonging to the space $\mathcal{S}'$ such that

$$\operatorname{supp}(\hat{\phi}) \subset \left\{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\right\}$$

and

$$\hat{\phi}_j(\xi) = \hat{\phi}\left(2^{-j}\xi\right), \quad \sum_{j\in\mathbb{Z}} \hat{\phi}_j(\xi) = 1.$$

Then, we present the homogeneous Fourier-Besov spaces.

Definition 2.1. [6] Let $1 \leq p, q \leq +\infty$ and $s \in \mathbb{R}$, the homogeneous Fourier-Besov space is defined as

$$\mathcal{F}\dot{\mathcal{B}}^s_{p,q} = \left\{ u \in \mathcal{S}' : \ \hat{u} \in L^1_{loc}, \ \|u\|_{\mathcal{F}\dot{\mathcal{B}}^s_{p,q}(\mathbb{R}^n)} = \left\{ \sum_{j\in\mathbb{Z}} 2^{jqs} \left\| \hat{\phi}_j \hat{u} \right\|^q_{L^{p'}} \right\}^{1/q} < +\infty \right\},$$

with appropriate modifications made when $q = \infty$.

Now, we define the Morrey spaces $\mathcal{M}^\lambda_p\left(\mathbb{R}^n\right)$.

Definition 2.2. ([22]) For $1 \leq p < \infty, 0 \leq \lambda < n$, the Morrey space $\mathcal{M}^\lambda_p = \mathcal{M}^\lambda_p\left(\mathbb{R}^n\right)$ is given by $\mathcal{M}^\lambda_p\left(\mathbb{R}^n\right) = \left\{ f \in L^p_{loc}\left(\mathbb{R}^n\right); \|f\|_{\mathcal{M}^\lambda_p} < \infty \right\}$, where

$$\|f\|_{\mathcal{M}^\lambda_p} = \sup_{x_0\in\mathbb{R}^n} \sup_{r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x_0,r))}.$$

Remark 2.3. [12] **1)** The space $\mathcal{M}^\lambda_p$ equipped with the norm $\|\cdot\|_{\mathcal{M}^\lambda_p}$ is a Banach space.
2) If $1 \leq p_1, p_2, p_3 < \infty$, $0 \leq \lambda_1, \lambda_2, \lambda_3 < n$ with $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$, and $\frac{\lambda_3}{p_3} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}$, then we obtain the Hölder inequality

$$\|fg\|_{\mathcal{M}^{\lambda_3}_{p_3}} \leq \|f\|_{\mathcal{M}^{\lambda_1}_{p_1}} \|g\|_{\mathcal{M}^{\lambda_2}_{p_2}}.$$

3)For $1 \leq p < \infty$ and $0 \leq \lambda < n$,

$$\|\varphi * f\|_{\mathcal{M}^\lambda_p} \leq \|\varphi\|_{L^1} \|g\|_{\mathcal{M}^\lambda_p}, \tag{2.1}$$

for all $\varphi \in L^1$ and $f \in \mathcal{M}^\lambda_p$.

Lemma 2.4. [12] *Let $1 \leq p_2 \leq p_1 < \infty, 0 \leq \lambda_1, \lambda_2 < n$, $\frac{n-\lambda_1}{p_1} \leq \frac{n-\lambda_2}{p_2}$ and let γ be a multi-index. If $\sup p(\widehat{f}) \subset \{|\xi| \leq A2^j\}$, then the following inequality holds for any constant $C > 0$ independent of f and j.*

$$\left\| (i\xi)^\gamma \widehat{f} \right\|_{\mathcal{M}^{\lambda_2}_{p_2}} \leq C 2^{j|\gamma| + j\left(\frac{n-\lambda_2}{p_2} - \frac{n-\lambda_1}{p_1}\right)} \|\widehat{f}\|_{\mathcal{M}^{\lambda_1}_{p_1}}. \tag{2.2}$$

Now, we define the function spaces $\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}\left(\mathbb{R}^n\right)$.

Definition 2.5. [8] (Homogeneous Fourier-Besov-Morrey spaces) Let $s \in \mathbb{R}, 0 \leq \lambda < n, 1 \leq p < +\infty$, and $1 \leq q \leq +\infty$. The space $\mathcal{F}\dot{\mathcal{N}}^{s}_{p,\lambda,r}(\mathbb{R}^n)$ denotes the set of all $u \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ such that

$$\|u\|_{\mathcal{F}\dot{\mathcal{N}}^{s}_{p,\lambda,q}(\mathbb{R}^n)} = \left\{ \sum_{j \in \mathbb{Z}} 2^{jqs} \left\| \hat{\phi}_j \hat{u} \right\|^q_{\mathcal{M}^{\lambda}_{p'}} \right\}^{1/q} < +\infty, \tag{2.3}$$

with appropriate modifications made when $q = \infty$.

Now, we define the Chemin-Lerner-type spaces and we present the definition of the mixed space-time spaces.

Definition 2.6. [8] Let $s \in \mathbb{R}, 1 \leq p < \infty, 1 \leq q, \rho \leq \infty, 0 \leq \lambda < n$ and $T \in (0, \infty]$. The space-time norm is defined on $u(t,x)$ by

$$\|u(t,x)\|_{\mathcal{L}^{\rho}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{s}_{p,\lambda,q}\right)} = \left\{ \sum_{j \in \mathbb{Z}} 2^{jqs} \left\| \hat{\phi}_j \hat{u} \right\|^q_{L^{\rho}\left(0,T;\mathrm{M}^{\lambda}_{p'}\right)} \right\}^{1/q},$$

where $\mathcal{L}^{\rho}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{s}_{p,\lambda,q}\right)$ is the set of distributions in $\mathcal{S}'\left(\mathbb{R} \times \mathbb{R}^n\right)/\mathcal{P}(\mathbb{R}^n)$ with finite $\|\cdot\|_{\mathcal{L}^{\rho}(0,T;\mathcal{F}\dot{\mathcal{N}}^{s}_{p,\lambda,q})}$ norm.

As a consequence of the Minkowski inequality, we can establish the following continuous embedding connecting the Chemin-Lerner spaces with the Bochner spaces and the Fourier-Besov-Morrey spaces,

$$\begin{aligned} \mathcal{L}^{r}(0,T;\mathcal{F}\dot{\mathcal{N}}^{s}_{p,\lambda,q}\Big) &\subseteq L^{r}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{s}_{p,\lambda,q}\right), \quad q \leq r, \\ L^{r}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{s}_{p,\lambda,q}\right) &\subseteq \mathcal{L}^{r}(0,T;\mathcal{F}\dot{\mathcal{N}}^{s}_{p,\lambda,q}\Big), \quad r \leq q. \end{aligned}$$

Now, we define the filtered probability space

Definition 2.7. Let $\left(\Omega, \mathbf{F}, \{\mathbf{F}_t\}_{t \in [0,T]}, \mathbf{P}\right)$ be a filtered probability space with the expectation $\mathbb{E}$ and $T > 0$. We designate by $\mathcal{M}_T$ the smallest σ-algebra of $\mathbf{F}_t$-adapted distribution processes g defined on $\Omega \times [0,T] \times \mathbb{R}^d$ which are progressively measurable; more precisely $g(\omega, t, \cdot) \in \mathbf{F}_t \times \mathcal{B}([0,t])$, for all $t \in [0,T]$.

Now, let us define the Chemin-Lerner type space of Bochner.

Definition 2.8. Let $T > 0$, $1 \leq r, p < \infty$ and $0 \leq \lambda < n$. The Chemin-Lerner type space of Bochner $\tilde{L}^r(0,T;\mathcal{F}\mathcal{M}^{s}_{p,\lambda,q})$ is defined as the space of distribution process $f \in \mathcal{D}_T$ such that $f(\omega,t) \in \mathcal{S}'_h\left(\mathbb{R}^d\right)$ and the quasinorm

$$\|f\|_{\tilde{L}^r(0,T;\mathcal{F}\mathcal{M}^{s}_{p,\lambda,q})} = \left(\mathbb{E}\Big(\int_0^T \|f\|_{\mathcal{F}\mathcal{M}^{s}_{p,\lambda,q}} ds \Big)^r \right)^{\frac{1}{r}}.$$

At the end of this section, we recall the Burkholder-Davis-Gundy inequality.

Lemma 2.9. *Let $\{W_t, t \geq 0\}$ be an n-dimensional Wiener motion. For each $p > 0$ and $T > 0$, there exists a constant $c_p \geq 1$ such that $\forall \varphi \in \Lambda^2$*

$$\frac{1}{c_p}\mathbb{E}\left[\left(\int_0^T |\varphi_t|^2 \, dt \right)^{p/2} \right] \leq \mathbb{E}\left[\sup_{t \leq T} \left| \int_0^t \varphi_s dW_s \right|^p \right] \leq c_p \mathbb{E}\left[\left(\int_0^T |\varphi_t|^2 \, dt \right)^{p/2} \right].$$

Definition 2.10. For a Banach space X and $\alpha \in \mathbb{R}$, we define the stochastic Banach spaces$\mathcal{L}_\alpha^r\big(\Omega \times (0,T), \mathcal{P}; X\big)$ to be the space of X-valued processes with the norm

$$\|f\|_{\mathcal{L}_\alpha^r\big(\Omega\times(0,T),\mathcal{P};X\big)} = \left(\mathbb{E}\int_0^T \rho^{\alpha r}\|f(\rho,\cdot)\|_X^r d\rho\right)^{\frac{1}{r}}.$$

Remark 2.11. According to [19, Lemme 4.4], we conclude that there are functionsΦ such that

$$\Phi(\omega,\cdot,\cdot) \in \mathcal{L}_\alpha^r\big(\Omega \times (0,T), \mathcal{P}; \mathcal{FM}_{p,\lambda,q}^s\big) \cap \mathcal{FM}_{r,\lambda,l}^t.$$

We now proceed to state our main result which establishes the local existence.

Theorem 2.12. *(Deterministic case) Let $1 \le p \le \infty$, $0 \le \lambda < 3$ and*

$$(u_0, B_0) \in \mathcal{F}\dot{\mathcal{N}}_{p,\lambda,1}^{-1+\frac{3}{p}}\left(\mathbb{R}^3\right) \times \left(\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,1}^{-1+\frac{3}{p}} \cap \mathcal{F}\dot{\mathcal{N}}_{p,\lambda,1}^{\frac{3}{p}}\right)\left(\mathbb{R}^3\right)$$

where $\nabla\cdot u_0 = \nabla\cdot B_0 = 0$. There exists a positive constant $\varepsilon_0 \ll 1$ such that if

$$\|B_0\|_{\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,1}^{\frac{3}{p}}} \le \varepsilon_0,$$

then there exists $T > 0$ such that the problem (1.3) has a unique solution (u, B) satisfying

$$u \in C\left([0,T); \mathcal{F}\dot{\mathcal{N}}_{p,\lambda,1}^{-1+\frac{3}{p}}\right) \cap L^1\left(0,T; \mathcal{F}\dot{\mathcal{N}}_{p,\lambda,1}^{1+\frac{3}{p}}\right)$$

$$B \in C\left([0,T); \mathcal{F}\dot{\mathcal{N}}_{p,\lambda,1}^{-1+\frac{3}{p}} \cap \mathcal{F}\dot{\mathcal{N}}_{p,\lambda,1}^{\frac{3}{p}}\right) \cap L^1\left(0,T; \mathcal{F}\dot{\mathcal{N}}_{p,\lambda,1}^{1+\frac{3}{p}} \cap \mathcal{F}\dot{\mathcal{N}}_{p,\lambda,1}^{2+\frac{3}{p}}\right).$$

Theorem 2.13. *(Stochastic case) Let λ, p satisfy $0 \le \lambda < 3$ and $1 \le p \le \infty$, Let $\left(\Omega, \mathbf{F}, \{\mathbf{F}_t\}_{t\in[0,T]}, \mathbf{P}\right)$ be a filtered probability basis. Assume that u_0, b_0 are $\mathbf{F}_0$ measurable, small enough with $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$, and $g_1, g_2 \in \mathcal{D}_T$. Suppose that for any positive T,*

$$\|(u_0,b_0)\|_{\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)\times\left(\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,1}^{-1+\frac{3}{p}}\cap\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,1}^{\frac{3}{p}}\right)(\mathbb{R}^3)} + \|(g_1,g_2)\|_{\mathcal{L}_0^1\big(\Omega\times(0,T),\mathcal{P};\mathcal{M}\big)} \le K.$$

Where K is a positiv constant. Then there exist a unique global mild solution of problem 1.1 in $L^1\left(0,T; \mathcal{F}\dot{\mathcal{N}}_{p,\lambda,1}^{1+\frac{3}{p}}\right) \cap L^1\left(0,T; \mathcal{F}\dot{\mathcal{N}}_{p,\lambda,1}^{1+\frac{3}{p}} \cap \mathcal{F}\dot{\mathcal{N}}_{p,\lambda,1}^{2+\frac{3}{p}}\right) \cap \mathcal{L}_0^1\big(\Omega \times (0,T), \mathcal{P}; \mathcal{M}\big)$.

3. A priory estimates

Let us proceed to illustrate the different types of product estimations within the Chemin-Lerner spaces of Fourier-Besov-Morrey spaces. The classical bilinear estimates are provided in the following lemma.

Lemma 3.1. *(Bilinear estimates) Let $s > 0$, $0 \le \lambda,\ \lambda_1,\ \lambda_2 < n$, $1 \le r_1,\ r_2 \le \infty$, and $1 \le p,\ p_1,\ p_2,\ q \le \infty$, such that $\frac{\lambda}{p} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2} = \frac{\lambda_1}{r_1} + \frac{\lambda_2}{r_1}$. Then there exists a positive constant C such that the following estimate holds:*

$$\|fg\|_{\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^s} \le C\left(\|f\|_{\hat{\mathcal{M}}_{p_1}^{\lambda_1}}\|g\|_{\mathcal{F}\dot{\mathcal{N}}_{p_2,\lambda_2,q}^s} + \|g\|_{\hat{\mathcal{M}}_{r_1}^{\lambda_1}}\|f\|_{\mathcal{F}\dot{\mathcal{N}}_{r_2,\lambda_2,q}^s}\right).$$

Proof. We put, $f_j = \phi_j * f$. Thanks to Bony's para-product formula, we decompose fg as follows:

$$\begin{aligned} fg &= \sum_{k\in\mathbb{Z}} P_{k-1}fg_k + \sum_{k\in\mathbf{Z}} f_k P_{k-1}g + \sum_{k\in\mathbb{Z}} f_k\tilde{g}_k \\ &:= I_1 + I_2 + I_3, \end{aligned}$$

with $P_k f := \sum_{l\le k-1} f_l$. We first determine the estimation of I_1. We note that $\phi_j * (P_{j'-1}f\phi_{j'}*\ g) = 0\,(|j-j'|\ge 4)$. Then we have

$$\begin{aligned} \left\|\widehat{I_1}\right\|_{\mathcal{M}^{\lambda}_{p'}} &\le \sum_{|j-j'|\le 3} \left\|\widehat{\phi}_j \mathcal{F}\left(P_{j'-1}fg_{j'}\right)\right\|_{\mathcal{M}^{\lambda}_{p'}} \\ &\le C\sum_{|j-j'|\le 3} \left\|\mathcal{F}\left(P_{j'-1}fg_{j'}\right)\right\|_{\mathcal{M}^{\lambda}_{p'}} \\ &\le C\sum_{|j-j'|\le 3} \left\|\mathcal{F}\left(P_{j'-1}f\right)\right\|_{\mathcal{M}^{\lambda_1}_{p_1'}} \left\|\widehat{\phi} j'\widehat{g}\right\|_{\mathcal{M}^{\lambda_2}_{p_2'}} \\ &\le C\|\widehat{f}\|_{\mathcal{M}^{\lambda_1}_{p_1'}} \sum_{|j-j'|\le 3} \left\|\ \widehat{\phi}_{j'}\hat{g}\right\|_{\mathcal{M}^{\lambda_2}_{p_2'}}. \end{aligned} \tag{3.1}$$

In a similar way, we get the estimate of I_2 as follows:

$$\|\widehat{I}_2\|_{\mathcal{M}^{\lambda}_{p'}} \le C\|\widehat{g}\|_{\mathcal{M}^{\lambda_1}_{r_1'}} \sum_{|j-j'|\le 3} \|\widehat{\phi} j'\widehat{f}\|_{\mathcal{M}^{\lambda_2}_{r_2'}}. \tag{3.2}$$

Whereas we observe that $\phi_j * (f_{j'}\tilde{g}_{j'}) = 0\,(j'\le j-4)$. Then we obtain

$$\begin{aligned} \|\widehat{I}_3\|_{\mathcal{M}^{\lambda}_{p'}} &\le \sum_{j'\ge j-3} \|\widehat{\phi}_j \mathcal{F}\left(f_{j'}\tilde{g}_{j'}\right)\|_{\mathcal{M}^{\lambda}_{p'}} \\ &\le C\sum_{j'\ge j-3} \|\widehat{\phi}_{j'}\hat{f} * \widehat{\tilde{\phi}}_{j'}\widehat{g}\|_{\mathcal{M}^{\lambda}_{p'}} \\ &\le C\sum_{j'\ge j-3} \left\|\widehat{\phi}_{j'}\widehat{f}\right\|_{\mathcal{M}^{\lambda_1}_{p_1'}} \|\widehat{\tilde{\phi}}_{j'}\widehat{g}\|_{\mathcal{M}^{\lambda_2}_{p_2'}} \\ &\le C\|\widehat{f}\|_{\mathcal{M}^{\lambda_1}_{p_1'}} \sum_{j'\ge j-3} \left\|\widehat{\tilde{\phi}} j'\widehat{g}\right\|_{\mathcal{M}^{\lambda_2}_{p_2'}}. \end{aligned} \tag{3.3}$$

Combining (3.1)-(3.3), we have

$$\begin{aligned} \left\|\widehat{\phi}_j\mathcal{F}(fg)\right\|_{\mathcal{M}^{\lambda}_{p'}} &\le C\|\widehat{f}\|_{\mathcal{M}^{\lambda_1}_{p_1'}} \left(\sum_{|j-j'|\le 3}\left\|\widehat{\phi}_{j'}\widehat{g}\right\|_{\mathcal{M}^{\lambda_2}_{p_2'}} + \sum_{j'\ge j-3}\left\|\widehat{\tilde{\phi}}_{j'}\hat{g}\right\|_{\mathcal{M}^{\lambda_2}_{p_2'}}\right) \\ &+ C\|\widehat{g}\|_{\mathcal{M}^{\lambda_1}_{r_1'}} \sum_{|j-j'|\le 3}\left\|\widehat{\phi}_{j'}\widehat{f}\right\|_{\mathcal{M}^{\lambda_2}_{r_2'}}. \end{aligned}$$

By Minkowski's inequality, we get

$$
\begin{aligned}
\|fg\|_{\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}} \leq & C\|\widehat{f}\|_{\mathcal{M}^{\lambda_1}_{p'_1}} \left(\sum_{j\in\mathbb{Z}} 2^{sjq} \left(\sum_{|j-j'|\leq 3} \left\|\widehat{\phi}_{j'}\widehat{g}\right\|_{\mathcal{M}^{\lambda_2}_{p'_2}}\right)^q\right)^{\frac{1}{q}} \\
& + C\|\widehat{g}\|_{\mathcal{M}^{\lambda_1}_{r'_1}} \left(\sum_{j\in\mathbb{Z}} 2^{sjq} \left(\sum_{|j-j'|\leq 3} \|\widehat{\phi}_{j'}\hat{f}\|_{\mathcal{M}^{\lambda_2}_{r'_2}}\right)^q\right)^{\frac{1}{q}} \\
& + C\|\widehat{f}\|_{\mathcal{M}^{\lambda_1}_{p'_1}} \left(\sum_{j\in\mathbb{Z}} 2^{sjq} \left(\sum_{j-j'\leq 3} \left\|\widehat{\tilde{\phi}}_{j'}\widehat{g}\right\|_{\mathcal{M}^{\lambda_2}_{p'_2}}\right)^q\right)^{\frac{1}{q}} \\
\leq & C\|\widehat{f}\|_{\mathcal{M}^{\lambda_1}_{p'_1}} \sum_{|l|\leq 3} \left(\sum_{j\in\mathbb{Z}} 2^{sjq} \left\| \widehat{\phi}_{j-l}\widehat{g}\right\|^q_{\mathcal{M}^{\lambda_2}_{p'_2}}\right)^{\frac{1}{q}} \\
& + C\|\widehat{g}\|_{\mathcal{M}^{\lambda_1}_{r'_1}} \sum_{|I|\leq 3} \left(\sum_{j\in\mathbf{Z}} 2^{sjq} \left\| \widehat{\phi}_{j-l}\widehat{f}\right\|^q_{\mathcal{M}^{\lambda_2}_{r'_2}}\right)^{\frac{1}{q}} \\
& + C\|\widehat{f}\|_{\mathcal{M}^{\lambda_1}_{p'_1}} \sum_{l\leq 3} \left(\sum_{j\in\mathbb{Z}} 2^{sj\sigma} \left\|\widehat{\tilde{\phi}}_{j-l}\widehat{g}\right\|^q_{\mathcal{M}^{\lambda_2}_{p'_2}}\right)^{\frac{1}{q}} \\
:= & J_1 + J_2 + J_3.
\end{aligned}
$$

For J_1 and J_2, we obtain

$$
\begin{aligned}
J_1 + J_2 = & C\|\widehat{f}\|_{\mathcal{M}^{\lambda_1}_{p'_1}} \sum_{|l|\leq 3} 2^{sl} \left(\sum_{j\in\mathbb{Z}} 2^{s(j-l)q} \left\| \widehat{\phi}_{j-l}\widehat{g}\right\|^q_{\mathcal{M}^{\lambda_2}_{p'_2}}\right)^{\frac{1}{q}} \\
& + C\|\widehat{g}\|_{\mathcal{M}^{\lambda_1}_{r'_1}} \sum_{|l|\leq 3} 2^{sl} \left(\sum_{j\in\mathbb{Z}} 2^{s(j-l)q} \|\widehat{\phi}_{j-l}\widehat{f}\|_{\mathcal{M}^{\lambda_2}_{r'_2}}\right)^{\frac{1}{q}} \\
\leq & C\left(\|f\|_{\hat{\mathcal{M}}^{\lambda_1}_{p_1}} \|g\|_{\mathcal{F}\dot{\mathcal{N}}^s_{p_2,\lambda_2,q}} + \|g\|_{\hat{\mathcal{M}}^{\lambda_1}_{r_1}} \|f\|_{\mathcal{F}\dot{\mathcal{N}}^s_{r_2,\lambda_2,q}}\right).
\end{aligned}
$$

As for J_3, we get

$$
\begin{aligned}
h'_3 = & C\|\widehat{f}\|_{\mathcal{M}^{\lambda_1}_{p'_1}} \sum_{l\leq 3} \left(\sum_{j\in\mathbb{Z}} 2^{s(j-l)q} 2^{slq} \left\|\widehat{\tilde{\phi}}_{j-l}\widehat{g}\right\|^q_{\mathcal{M}^{\lambda_2}_{p'_2}}\right)^{\frac{1}{q}} \\
\leq & C\|f\|_{\hat{\mathcal{M}}^{\lambda_1}_{p_1}} \|g\|_{\mathcal{F}\dot{\mathcal{N}}^s_{p_2,\lambda_2,q}}.
\end{aligned}
$$

Thus, the proof of Lemma 3.1 is complete. $\square$

Additionally, we need to show the smoothing properties in the Chemin-Lerner spaces. The following inhomogeneous heat equation with a diffusive coefficient

$(\nu > 0)$ is first taken into consideration:

$$\begin{cases} \partial_t u - v\Delta u = f, & t > 0, x \in \mathbb{R}^3, \\ u|_{t=0} = u_0, & x \in \mathbb{R}^3. \end{cases} \tag{3.4}$$

Lemma 3.2. *(Smoothing estimates for (3.4)) Let $s \in \mathbb{R}$, $0 \leq \lambda < 3$ and $1 \leq p, q \leq \infty$. The inhomogeneous heat equation for (3.4) has a unique solution u and there exists some constant $C = C(r) > 0$ such that*

$$v^{\frac{1}{r}} \|u\|_{L^r\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{s+\frac{2}{r}}_{p,\lambda,q}\right)} \leq C \left(\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{s}_{p,\lambda,q}} + v^{-1+\frac{1}{r_1}} \|f\|_{\mathcal{L}^{r_1}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{s-2+\frac{2}{r_1}}_{p,\lambda,q}\right)} \right).$$

Proof. Suppose that u satisfies (3.4), then for any $j \in \mathbb{Z}$, we have

$$u_j(t) = e^{t\nu\Delta}\phi_j * u_0 + \int_0^t e^{(t-s)\nu\Delta} f_j(s) \mathrm{d}s. \tag{3.5}$$

To estimate the first term of the right hand side in (3.5), we obtain by using $\operatorname{supp}\widehat{\phi}_j \subset \{\xi \in \mathbb{R}^n;\ 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$ that

$$\begin{aligned} &\left\| \widehat{\phi}_j e^{-t\nu|\xi|^2} \widehat{u}_0 \right\|_{L^r\left(0,T;\mathcal{M}^{\lambda}_{p'}\right)} \leq \left(\int_0^\infty e^{-\frac{v}{4}2^{2j}tr} z \right)^{\frac{1}{r}} \left\| \widehat{\phi}_j \widehat{u}_0 \right\|_{\mathcal{M}^{\lambda}_{p'}} \\ &\leq C \left(v2^{2j} \right)^{-\frac{1}{r}} \left\| \widehat{\phi}_j \widehat{u}_0 \right\|_{\mathcal{M}^{\lambda}_{p'}}. \end{aligned} \tag{3.6}$$

Using Young's inequality, we obtain

$$\begin{aligned} \left\| \int_0^t e^{(t-s)\nu|\xi|^2} \widehat{\phi}_j \widehat{f}(s) \mathrm{d}s \right\|_{L^r\left(0,T;\mathcal{M}^{\lambda}_{p'}\right)} &\leq \left(\int_0^T \left(\int_0^t e^{-\frac{v}{4}2^{2j}(t-s)} \left\| \widehat{\phi}_j \widehat{f} \right\|_{\mathcal{M}^{\lambda}_{p'}} \mathrm{d}s \right)^r \mathrm{d}t \right)^{\frac{1}{r}} \\ &\leq \left\| e^{-\frac{v}{4}2^{2j}\cdot} \right\|_{L^\theta(0,\infty)} \left\| \widehat{\phi}_j \widehat{f} \right\|_{L^{r_1}\left(0,T;\mathcal{M}^{\lambda}_{p'}\right)} \\ &\leq C \left(v2^{2j} \right)^{-1-\frac{1}{r}+\frac{1}{r_1}} \left\| \widehat{\phi}_j \widehat{f} \right\|_{L^{r_1}\left(0,T;\mathcal{M}^{\lambda}_{p'}\right)}, \end{aligned} \tag{3.7}$$

where $\frac{1}{\theta} = 1 + \frac{1}{r} - \frac{1}{r_1}$. Combining (3.6) with (3.7) and multiplying by 2^{sj}, we get

$$v^{\frac{1}{r}} 2^{\left(s+\frac{2}{r}\right)j} \|u_j\|_{L^r\left(0,T;\mathcal{M}^{\lambda}_{p}\right)} \leq C2^{sj} \|\phi_j * u_0\|_{\widehat{\mathcal{M}^{\lambda}_{p}}} + Cv^{-1+\frac{1}{r_1}} 2^{\left(s-2+\frac{2}{r_1}\right)j} \|f_j\|_{L^{r_1}\left(0,T;\widehat{\mathcal{M}^{\lambda}_{p}}\right)}.$$

Therefore, we take the $\ell^q(\mathbb{Z})$-norm and we get the result. □

We also show the following estimate for the problem (1.4).

Lemma 3.3. *(Smoothing estimates for (1.4)) Let $s \in \mathbb{R}, 1 \leq p, q \leq \infty$, $1 \leq r_1 \leq r \leq \infty$, $T \in \mathbb{R}+$ and $0 \leq \lambda < 3$. Suppose that $B_0 \in \mathcal{F}\dot{\mathcal{N}}^{s}_{p,\lambda,q}$ and $f \in$*

$\mathcal{L}^{r_1}(0,T;\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{s-2+\frac{2}{r_1}})$. *Then* (1.4) *has a unique solution B and there exists some constant $C = C(r) > 0$ such that*

$$\sigma_D^{\frac{1}{r}}\|B\|_{L^r(0,T;\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{s+\frac{2}{r}})} \leq C\left(\|B_0\|_{\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{s}} + \sigma_D^{-1+\frac{1}{r_1}}\|f\|_{L^{r_1}\left(0,T;\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{s-2+\frac{2}{r_1}}\right)}\right).$$

Proof. Firstly, we derive the integral forms of the solution to the linear induction equations (1.4). The Fourier transform of (1.4) with regard to $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, if $f \equiv 0$, yields

$$\partial_t\widehat{B} + \sigma_D|\xi|^2\widehat{B} + \sigma_H\Omega_\xi\Omega_{\bar{B}}\Omega_\xi\widehat{B} = 0,$$

where $\Omega_\xi, \Omega_{\bar{B}}$ are defined by

$$\Omega_\xi = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}, \quad \Omega_{\bar{B}} = \begin{pmatrix} 0 & -\bar{B}_3 & \bar{B}_2 \\ \bar{B}_3 & 0 & -\bar{B}_1 \\ -\bar{B}_2 & \bar{B}_1 & 0 \end{pmatrix},$$

respectively. Put $\widehat{H}(\xi) := -\sigma_D|\xi|^2 I - \sigma_H\Omega_\xi\Omega_{\bar{B}}\Omega_\xi$, As can be observed, the characteristic polynomial associated with $\widehat{H}(\xi)$ is determined by

$$\left(\lambda + \sigma_D|\xi|^2\right)\left(\lambda^2 + 2\sigma_D|\xi|^2 + \sigma_D^2|\xi|^4 + \sigma_H|\xi||\xi\cdot\bar{B}|^2\right). \tag{3.8}$$

The following three expressions are present in the above equation (3.8)

$$\lambda_0(\xi) := -\sigma_D|\xi|^2 \text{ and } \lambda_\pm(\xi) := -\sigma_D|\xi|^2 \pm i\sigma_H|\xi||\xi\cdot\bar{B}|.$$

Suppose that the matrices $\widehat{H}(\xi)$ can be diagonalizable, we find that for any $v \in \mathcal{S}(\mathbb{R}^3)$,

$$e^{t\hat{H}(\xi)}v = e^{t\lambda_0(\xi)}\widehat{P_0}(\xi)v + e^{t\lambda_+(\xi)}\widehat{P_+}(\xi)v + e^{t\lambda_-(\xi)}\widehat{P_-}(\xi)v$$

where the Hermitian matrices $\widehat{P}_i(\xi)(i = 0, \pm)$ are defined by the followings

$$\widehat{P_0}(\xi) := \frac{1}{|\xi|^2}\begin{pmatrix} \xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & \xi_2^2 & \xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & \xi_3^2 \end{pmatrix}$$

$$\widehat{P_+}(\xi) := \frac{1}{2|\xi|^2|\xi\cdot\bar{B}|^2}$$

$$\begin{pmatrix} R_2^2 + R_3^2 & -R_1R_2 + i|\xi||\xi\cdot\bar{B}|R_3 & -R_1R_3 - i|\xi||\xi\cdot\bar{B}|R_2 \\ -R_1R_2 - i|\xi||\xi\cdot\bar{B}|R_3 & R_1^2 + R_3^2 & -R_2R_3 + i|\xi||\xi\cdot\bar{B}|R_1 \\ -R_1R_3 + i|\xi||\xi\cdot\bar{B}|R_2 & -R_2R_3 - i|\xi||\xi\cdot\bar{B}|R_1 & R_1^2 + R_2^2 \end{pmatrix},$$

$$\widehat{P_-}(\xi) := \widehat{P_+}(\xi)^T,$$

with $R_i = R_i(\xi) := \xi_i\xi\cdot\bar{B}(i = 1, 2, 3)$. Therefore, $e^{tH}B_0$ given by $\mathcal{F}^{-1}\left[e^{t\hat{H}(\xi)}\widehat{B}_0\right]$ is the solution to the problem (1.4) with $f \equiv 0$. Using the Duhamel principle, we get the following integral form of (1.4),

$$B(t) = e^{tH}B_0 + \int_0^t e^{(t-s)H}f(s)\mathrm{d}s.$$

Using the definition of e^{tH}, we can prove that

$$\left\|\widehat{\phi}je^{t\widehat{H}(\xi)}\widehat{v}\right\|_{\mathcal{M}^{\lambda}_{p'}} \leq C\left\|e^{-\sigma_D|\xi|^2t}\widehat{\phi}_j\widehat{v}\right\|_{\mathcal{M}^{\lambda}_{p'}}, \tag{3.9}$$

for all $j\in\mathbb{Z}$. Similarly to the proof of Lemma 3.2, and by using (3.9), we have

$$\begin{aligned}
\|B_j\|_{L^r\left(0,T;\widehat{\mathcal{M}}^{\lambda}_p\right)} &\leq \left\|e^{tH}\phi_j * B_0\right\|_{L^r\left(0,T;\widehat{\mathcal{M}}^{\lambda}_p\right)} + \left\|\int_0^t e^{(t-s)H}f_j(s)\mathrm{d}s\right\|_{L^r\left(0,T;\widehat{\mathcal{M}}^{\lambda}_p\right)}\\
&\leq C\left\|e^{-\sigma_D|\xi|^2t}\widehat{\phi}j\widehat{B}_0\right\|_{L^r\left(0,T;\mathcal{M}^{\lambda}_{p'}\right)} + C\left(\int_0^T\left(\int_0^t\left\|e^{-(t-s)\sigma_D|\xi|^2}\widehat{\phi}_j\widehat{f}\right\|_{\mathcal{M}^{\lambda}_{p'}}\mathrm{d}s\right)^r\mathrm{d}t\right)^{\frac{1}{r}}\\
&\leq C\left(\sigma_D2^{2j}\right)^{-\frac{1}{r}}\left\|\widehat{\phi}j\widehat{B}_0\right\|_{\mathcal{M}^{\lambda}_{p'}} + C\left(\sigma_D2^{2j}\right)^{-1-\frac{1}{r}+\frac{1}{r_1}}\left\|\widehat{\phi}j\widehat{f}\right\|_{L^{r_1}(0,T;\mathcal{M}^{\lambda}_{p'})}.
\end{aligned}$$

Then, the proof of Lemma (3.3) is completed. □

4. Proof of Theorem 2.13

In this section, we show our result concerning the existence and uniqueness of the solution of 1.1 given by Theorem 2.13, for which we consider the Lemma 4.1 that we'll use to show the Theorem 2.13

Lemma 4.1. *Let v_0 be F_0 measurable, g progressively measurable on $\omega\times[0,T]\times\mathbb{R}^3$ and $0\leq\lambda<3$.*
Then, ther exists a positive constant σ such that for $g\in\tilde{L}^1(0,T;\mathcal{FM}^s_{p,\lambda,q})$

$$\left\|\int_0^t \mathcal{S}_\beta(t-\rho)\mathbb{P}gdW\right\|_{\mathcal{L}^1_0\left(\Omega\times(0,T),\mathcal{P};M^{\lambda}_p\right)} < \sigma.$$

where $\mathcal{S}_\beta := e^{t\Delta}$.

Proof : Since $\mathbb{P}$ is bounded in $\mathcal{FM}^s_{p,\lambda,q}(\mathbb{R}^3)$[24], it follows from Lemma 2.9 that

$$\begin{aligned}
\left\|\int_0^t \mathcal{S}_\beta(t-\rho)\mathbb{P}gdW\right\|_{\mathcal{L}^1_0\left(\Omega\times(0,T),\mathcal{P}\mathcal{FM}^s_{p,\lambda,q}\right)} &= \mathbb{E}\Big(\int_0^T\left\|\int_0^t\mathcal{S}_\beta(t-\rho)\mathbb{P}gdW\right\|_{\mathcal{FM}^s_{p,\lambda,q}}\Big)\\
&\lesssim \mathbb{E}\Big(\int_0^T\left\|\int_0^t gdW\right\|_{\mathcal{FM}^s_{p,\lambda,q}}\Big)\\
&\lesssim \int_0^T\mathbb{E}\Big(\sup_{t\in[0,T]}\int_0^t\|g\|_{\mathcal{FM}^s_{p,\lambda,q}}dW\Big)\\
&\lesssim \int_0^T\mathbb{E}\Big(\int_0^T\|g\|_{\mathcal{FM}^s_{p,\lambda,q}}d\rho\Big).
\end{aligned}$$

□

In order to solve the problem (1.1), we consider the following equivalent integral equation coming from Duhamel's principle

$$\begin{cases} u = \mathcal{S}_\beta(t)u_0 + \int_0^t \mathcal{S}_\beta(t-\rho)\mathbb{P}g_1 dW - \int_0^t \mathcal{S}_\beta(t-\rho)\mathbb{P}\nabla\cdot(u\otimes u - b\otimes b)(\cdot,\rho)d\rho, \\ b = \mathcal{S}_\beta(t)b_0 + \int_0^t \mathcal{S}_\beta(t-\rho)\mathbb{P}g_2 dW - \int_0^t \mathcal{S}_\beta(t-\rho)\mathbb{P}\nabla\cdot(b\otimes u - u\otimes u)(\cdot,\rho)d\rho \\ \qquad\qquad - \int_0^t \mathcal{S}_\beta(t-\rho)\nabla\times((\nabla\times b)\times b). \end{cases} \tag{4.1}$$

Put

$$X = L^1\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)\cap L^1\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\cap\mathcal{F}\dot{\mathcal{N}}^{2+\frac{3}{p}}_{p,\lambda,1}\right)\cap\mathcal{L}^1_0\left(\Omega\times(0,T),\mathcal{P};\mathcal{M}\right).$$

To prove Theorem 2.13, it is sufficient to show that

$$\left\|\left(\mathcal{S}_\beta(t)u_0 + \int_0^t \mathcal{S}_\beta(t-\rho)\mathbb{P}g_1 dW, \mathcal{S}_\beta(t)b_0 + \int_0^t \mathcal{S}_\beta(t-\rho)\mathbb{P}g_2 dW\right)\right\|_X \le C_0,$$

Lemma 4.1 allows us to obtain

$$\begin{aligned}\left\|\mathcal{S}_\beta(t)u_0 + \textstyle\int_0^t \mathcal{S}_\beta(t-\rho)\mathbb{P}g_1 dW\right\|_X &\le C\left(\|u_0\|_{\mathrm{N}^s_{p,\lambda}} + \|g_i\|_{\mathcal{L}^1_0\left(\Omega\times(0,T),\mathcal{P};\mathcal{M}\right)}\right)\\ &\le C_0.\end{aligned}$$

Then we will follow the same steps in the proof of Theorem 2.12, we obtain the proof of Theorem 2.13.

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