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ON THE CONVERGENCE OF ISHIKAWA ITERATIVE PROCESS FOR A PAIR OF SINGLE-VALUED AND MULTI-VALUED NONEXPANSIVE MAPPINGS

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Abstract. Let K be a nonempty compact convex subset of a complete CAT(0) space X, and let $t : K \to K$ and $T : K \to CB(K)$ be a single-valued nonexpansive mapping and a multi-valued nonexpansive mapping, respectively. Assume, in addition, that $F(t) \cap F(T) \neq \emptyset$ and $Tp = \{p\}$ for all $p \in F(t) \cap F(T)$. We prove that the sequence of modified Ishikawa iteration method generated from an arbitrary $x_0 \in K$ by $y_n = (1-\beta_n)x_n \oplus \beta_n z_n, x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n t y_n$, where $z_n \in Tx_n$ and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of positive numbers satisfying $0 < a \leq \alpha_n$, $\beta_n \leq b < 1$ converges strongly to a common fixed point of t and T; that is, there exists $x \in K$ such that $x = tx \in Tx$. Our results extends and generalizes some theorems of T. Puttasontiphot [24].

1. Introduction

Let X be a Banach space, and let K be a nonempty subset of X. We will denote by $CB(K)$ the family of nonempty bounded closed subset of K and by $K(K)$ the family of compact convex subset of K. Let $H(.,.)$ be the Hausdorff distance on $CB(X)$, that is,

$$H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}, \qquad A,B \in CB(X)$$

where $d(a,B) = \inf\{\| a-b \| : b \in B\}$ is the distance from the point a to the subset B. A mapping $t : K \to K$ is said to be nonexpansive ,if $\| tx - ty \| \leq \| x - y \|$ for all $x, y \in K$. A point x is called a fixed point of t if $tx = x$. A multi-valued mapping $T : K \to CB(X)$ is said to be nonexpansive, if $H(Tx,Ty) \leq \| x - y \|$ for all $x, y \in K$. A point x is called a fixed point for a multi-valued mapping T, if $x \in Tx$. We use the notation $F(T)$ standing for the set of fixed points of a mapping T and $F(t) \cap F(T)$ standing for the set of common fixed points of t and T. Precisely, a point x is called a common fixed point of t and T, if $x = tx \in Tx$. In 1974, Ishikawa introduced the following well-known iteration.

Definition 1.1. [15] Let X be a Banach space, let K closed convex subset of X and let t be a self map on K. For $x_0 \in K$, the sequence $\{x_n\}$ of Ishikawa iterates

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of t is defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n t y_n \\ y_n &= (1-\beta_n)x_n + \beta_n t x_n, \qquad n \geq 0, \end{aligned} \tag{1.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences.

A nonempty subset K of X is said to be proximal if, for any $x \in X$, there exists an element $y \in K$ such that $\| x - y \| = d(x, K)$. We will denote $P(K)$ by the family of nonempty proximal bounded subsets of K. In 2005, Sastry and Babu [27] defined the Ishikawa iteration scheme for multi-valued mappings as follows. Let K be a compact convex subset of a Hilbert space X, and let $T : K \to P(K)$ be a multi-valued map and $p \in F(T)$. For $x_0 \in K$, the sequence $\{x_n\}$ of Ishikawa iterates of T is defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n \acute{z}_n \\ y_n &= (1-\beta_n)x_n + \beta_n z_n, \qquad n \geq 0, \end{aligned} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$ with $z_n \in Tx_n$ such that $\| z_n - p \| = d(p, Tx_n)$ and $\acute{z}_n \in Ty_n$ such that $\| \acute{z}_n - p \| = d(p, Ty_n)$. They also proved the strong convergence of the above Ishikawa iteration scheme for a multi-valued nonexpansive map T with a fixed point p under some control conditions in a Hilbert space. Recently, Panyanak [23] extended the result of Sastry and Babu [27] to a uniformly convex Banach space and also modified the above Ishikawa iteration scheme as follows. Let K be a nonempty convex subset of a uniformly convex Banach space X, and let $T : K \to P(K)$ be a multi-valued mapping. For $x_0 \in K$, the sequence $\{x_n\}$ of Ishikawa iterates T is defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n \acute{z}_n \\ y_n &= (1-\beta_n)x_n + \beta_n z_n, \qquad n \geq 0, \end{aligned} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$ with $z_n \in Tx_n$ and $u_n \in F(T)$ such that $\| z_n - u_n \| = d(u_n, Tx_n)$ and $\| x_n - u_n \| = d(x_n, F(T))$, respectively. Moreover, $\acute{z}_n \in Ty_n$ and $v_n \in F(T)$ such that $\| \acute{z}_n - v_n \| = d(v_n, Tx_n)$ and $\| y_n - v_n \| = d(y_n, F(T))$, respectively. Very recently, Song and Wang [32, 33] improved the result of [23, 27] by mean of the following Ishikawa iterative scheme. Let $T : K \to CB(K)$ be a multi-valued mapping, and $\alpha_n, \beta_n \in (0, 1)$. For $x_0 \in K$, the sequence $\{x_n\}$ of Ishikawa iterates T is defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n \acute{z}_n \\ y_n &= (1-\beta_n)x_n + \beta_n z_n, \qquad n \geq 0, \end{aligned} \tag{1.4}$$

where $z_n \in Tx_n$ and $\acute{z}_n \in Ty_n$ such that $\| z_n - \acute{z}_n \| \leq H(Tx_n, Ty_n) + \gamma_n$ and $\| z_{n+1} - \acute{z}_n \| \leq H(Tx_{n+1}, Ty_n) + \gamma_n$, respectively. Moreover, $\gamma \in (0, +\infty)$ such that $\lim\limits_{n\to\infty} \gamma_n = 0$. At the same period, Shahzad and Zegeye [31] modified the Ishikawa iterative scheme and extended the result of [6, Theorem 2] to a multi-valued quasi-nonexpansive mapping as follows. Let K be a nonempty convex subset of a Banach space X, and let $T : K \to CB(K)$ be a multi-valued map.

For $x_0 \in K$, the sequence $\{x_n\}$ of Ishikawa iterates T is defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n \acute{z}_n \\ y_n &= (1-\beta_n)x_n + \beta_n z_n, \qquad n \geq 0, \end{aligned} \tag{1.5}$$

where $\alpha_n, \beta_n \in [0,1]$, are real sequences $z_n \in Tx_n$ and $\acute{z}_n \in Ty_n$. In [24],T. Puttasontiphot studied (1.5) in the frame work of CAT(0) space and prove some strong convergence theorems for quasi-nonexpansive multi-valued mappings. Motivated and inspired by the above work ,in this paper, we introduce an iterative process in a new sense, called the modified Ishikawa iteration scheme with respect to a pair of single-valued and multi-valued nonexpansive mappings. We also establish the strong convergence theorem of a sequence from such process in a nonempty compact convex subset of a complete CAT(0) space. Our result extend and generalizes some theorems of T. Puttasontiphot [24].

2. Preliminaries

A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well-known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces, $\mathbb{R}$-trees (see [1]), Euclidean buildings (see [2]), the complex Hilbert ball with a hyperbolic metric (see [3]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry see Bridson and Haefliger [1]. Burago et al [4] contains a somewhat more elementary treatment and Gromov [14] a deeper study. Fixed point theory in CAT(0) space was first studied by Kirk (see [16], [17]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory for single-valued and multi-valued mappings in CAT(0) spaces has been rapidly developed and many of papers have appeared (see [18, 6, 5, 7, 12, 10, 8, 9, 29, 11, 20, 21, 30, 22, 25, 26]). It is worth mentioning that the results in CAT(0) spaces can be applied to any CAT(κ) space with $\kappa \leq a$ since any CAT(κ) space is a CAT($\acute{\kappa}$) space for every $\acute{\kappa} \geq \kappa$ (see [1], p.165). Let (X,d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or more briefly, a geodesic from x to y) is a map c from a closed interval $[0,l] \subseteq \mathbb{R}$ to X such that $c(0) = x$, $c(l)) = y$ and $d(c(t), c(\acute{t})) = |t - \acute{t}|$ for all $t, \acute{t} \in [0,l]$. In particular, c is an isometry and $d(x,y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique this geodesic segment is denoted by $[x,y]$. The space (X,d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X,d) consists of three points x_1, x_2, x_3 in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X,d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) = \Delta(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x_i}, \overline{x_j})$ for all $i, j \in \{1, 2, 3\}$. A geodesic space is said to be a CAT(0) space if all geodesic triangle satisfy the following comparison axiom. **CAT(0)**: Let Δ be a geodesic triangle in X and let $\overline{\Delta}$ be a comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) inequality if for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$

$$d(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y}).$$

Let $x, y \in X$, by Lemma 2.1(iv) of [9] for each $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = td(x, y), \ and, \ d(y, z) = (1 - t)d(x, y). \tag{2.1}$$

From now on, we will use notation $(1 - t)x \oplus ty$ for the unique point z satisfying (2.1). By using this notation Dhompongsa and Panyanak [9] obtained the following Lemma which will be used frequently in the proof of our results.

Lemma 2.1. *Let X be a CAT(0) space. Then*

$$d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z), \tag{2.2}$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

If x, y_1, y_2 are points in a CAT(0) space and if $y_0 = \frac{1}{2}y_1 \oplus \frac{1}{2}y_2$, then the CAT(0) inequality implies

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2. \tag{2.3}$$

This is the (**CN**) inequality of Bruhat and Tits [3]. In fact (cf. [1], p. 163), a geodesic space is a CAT(0) space if and only if it satisfy (2.3).

Definition 2.2. Let K be a nonempty subset of a CAT(0) space X and $t : K \to K$ be a mapping. t is called nonexpansive if for each $x, y \in K$

$$d(tx, ty) \leq d(x, y).$$

A point $x \in K$ is called a fixed point of t if $x = tx$. A multivalued mapping $T : K \to CB(X)$ is said to be nonexpansive if for each $x, y \in K$

$$H(Tx, Ty) \leq d(x, y).$$

A point $x \in K$ is called a fixed point for a multivalued mapping T if $x \in Tx$.

Definition 2.3. (Modified Ishikawa Iteration Process) Let K be a nonempty compact convex subset of a CAT(0) space X, let $t : K \to K$ be a single valued nonexpansive mapping and let $T : K \to CB(X)$ be a multi valued nonexpansive mapping. The sequence $\{x_n\}$ of the modified Ishikawa iteration is defined by

$$\begin{aligned} y_n &= (1 - \beta_n)x_n \oplus \beta_n z_n, \\ x_{n+1} &= (1 - \alpha_n)x_n \oplus \alpha_n t y_n, \end{aligned} \tag{2.4}$$

where $x_0 \in K$, $z_n \in Tx_n$ and $0 < a \leq \alpha_n$, $\beta_n \leq b < 1$.

Lemma 2.4. [21] *Let X be a complete CAT(0) space and let $x \in X$. Suppose $\{t_n\}$ be a sequence in $[0,1]$ such that $0 < \liminf_{n\to\infty} t_n \leq \limsup_{n\to\infty} t_n < 1$ and $\{x_n\}$, $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x) \leq r$, $\limsup_{n\to\infty} d(y_n, x) \leq r$ and $\lim_{n\to\infty} d((1-t)x_n \oplus ty_n, x) = r$ for some $r \geq 0$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

The following observation will be used in proving our results and the proof is straight forward.

Lemma 2.5. *Let X be a complete CAT(0) space and let K be a nonempty closed convex subset of X. Then*

$$d(y, Ty) \leq d(y, x) + d(x, Tx) + H(Tx, Ty)$$

where $x, y \in K$ and T is a multi valued nonexpansive mapping from K into $CB(K)$.

Definition 2.6. [28] A mapping $t : K \to K$ with nonempty fixed point set $F(t)$ in K is said to satisfy Condition (A) with respect to the sequence $\{x_n\}$ if there is a non-decreasing function $f : [0, 1) \to [0, 1)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $f(d(x_n, F(t))) \leq d(x_n, tx_n)$ for all $n \geq 1$.

3. Main results

We first prove the following lemmas, which plays very important roles in this section.

Lemma 3.1. *Let K be a nonempty compact convex subset of a complete CAT(0) space X and let $t : K \to K$ and $T : K \to CB(X)$ be a single-valued and multi-valued nonexpansive mapping, respectively, and $F(t) \cap F(T) \neq \emptyset$ satisfying $Tp = \{p\}$ for all $p \in F(t) \cap F(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (2.4). Then $\lim_{n\to\infty} x_n$ exists for all $p \in F(t) \cap F(T)$.*

Proof. Choosing $x_0 \in K$ and $p \in F(t) \cap F(T)$, we have by Lemma 2.1

$$\begin{aligned}
d(x_{n+1}, p) &\leq d(((1-\alpha_n)x_n \oplus \alpha_n t y_n), p) \\
&\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(y_n, p) \\
&\leq (1-\alpha_n)d(x_n, p) + \alpha_n \{d((1-\beta_n)x_n \oplus \beta_n z_n, p)\} \\
&\leq (1-\alpha_n)d(x_n, p) + \alpha_n \{(1-\beta_n)d(x_n, p) + \beta_n d(z_n, p)\} \\
&\leq (1-\alpha_n)d(x_n, p) + \alpha_n \{(1-\beta_n)d(x_n, p) + \beta_n H(Tx_n, Tp)\} \\
&\leq (1-\alpha_n)d(x_n, p) + \alpha_n(1-\beta_n)d(x_n, p) + \alpha_n\beta_n d(x_n, p) \\
&= d(x_n, p)
\end{aligned}$$

Since $\{d(x_n, p)\}$ is a decreasing and bounded sequence, we can conclude that the limit of the sequence $\{d(x_n, p)\}$ exists. $\square$

Lemma 3.2. *Let K be a nonempty compact convex subset of a complete CAT(0) space X and let $t : K \to K$ and $T : K \to CB(X)$ be a single-valued and multi-valued nonexpansive mapping, respectively, and $F(t) \cap F(T) \neq \emptyset$ satisfying $Tp = \{p\}$ for all $p \in F(t) \cap F(T)$. Let $\{x_n\}$ be the sequence of the modified*

Ishikawa iteration defined by (2.4)*. If* $0 < a \leq \alpha_n \leq b < 1$ *for some* $a, b \in \mathbb{R}$*, then* $\lim_{n\to\infty} d(ty_n, x_n) = 0$.

Proof. Let $p \in F(t) \cap F(T)$. By Lemma 3.1, we put $\lim_{n\to\infty} d(x_n, p) = c$ and consider

$$\begin{aligned} d(ty_n, p) &\leq d(y_n, p) \\ &= d((1-\beta_n)x_n \oplus \beta_n z_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(z_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n H(Tx_n, Tp) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(x_n, p) \\ &= d(x_n, p). \end{aligned}$$

Thus, we have

$$\begin{aligned} \limsup_{n\to\infty} d(ty_n, p) &\leq \limsup_{n\to\infty} d(y_n, p) \\ &\leq \limsup_{n\to\infty} d(x_n, p) \\ &= c. \end{aligned} \tag{3.1}$$

Furthermore,

$$\begin{aligned} c &= \lim_{n\to\infty} d(x_{n+1}, p) \\ &= \lim_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n ty_n, p) \\ &\leq \lim_{n\to\infty} \{(1-\alpha_n)d(x_n, p) + \alpha_n d(ty_n, p)\}. \end{aligned}$$

Hence from Lemma 2.4, we conclude that

$$\lim_{n\to\infty} d(ty_n, x_n) = 0$$

□

Lemma 3.3. *Let K be a nonempty compact convex subset of a complete CAT(0) space X and let $t : K \to K$ and $T : K \to CB(X)$ be a single-valued and multi-valued nonexpansive mapping, respectively, and $F(t) \cap F(T) \neq \emptyset$ satisfying $Tp = \{p\}$ for all $p \in F(t) \cap F(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by* (2.4)*. If $0 < a \leq \alpha_n,\ \beta_n \leq b < 1$ for some $a, b \in \mathbb{R}$, then* $\lim_{n\to\infty} d(x_n, z_n) = 0$.

Proof. Let $p \in F(t) \cap F(T)$, we put, as in Lemma 3.2, $\lim_{n\to\infty} d(x_n, p) = c$. For $n \geq 0$, we have

$$\begin{aligned} d(x_{n+1}, p) &\leq d(((1-\alpha_n)x_n \oplus \alpha_n ty_n), p) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(ty_n, p) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(y_n, p). \end{aligned}$$

and hence

$$\begin{aligned} d(x_{n+1}, p) - d(x_n, p) &\leq -\alpha_n d(x_n, p) + \alpha_n d(y_n, p) \\ d(x_{n+1}, p) - d(x_n, p) &\leq \alpha_n (d(y_n, p) - d(x_n, p)) \\ \frac{d(x_{n+1}, p) - d(x_n, p)}{\alpha_n} &\leq d(y_n, p) - d(x_n, p). \end{aligned}$$

Therefore as $0 < a \leq \alpha_n \leq b < 1$, we have

$$(\frac{d(x_{n+1}, p) - d(x_n, p)}{\alpha_n}) + d(x_n, p) \leq d(y_n, p) \tag{3.2}$$

Thus,

$$\liminf_{n\to\infty} \{(\frac{d(x_{n+1}, p) - d(x_n, p)}{\alpha_n}) + d(x_n, p)\} \leq \liminf_{n\to\infty} d(y_n, p) \tag{3.3}$$

It follows that

$$c \leq \liminf_{n\to\infty} d(y_n, p). \tag{3.4}$$

Since from (3.1), $\limsup_{n\to\infty} d(y_n, p) \leq c$, we have

$$\begin{aligned} c &= \lim_{n\to\infty} d(y_n, p) \\ &= \lim_{n\to\infty} d((1 - \beta_n)x_n \oplus \beta_n z_n, p) \\ &= \lim_{n\to\infty} \{(1 - \beta_n)d(x_n, p) + \beta_n d(x_n, p)\} \end{aligned} \tag{3.5}$$

Recall that

$$\begin{aligned} d(z_n, p) &= d(z_n, Tp) \\ &\leq H(Tx_n, Tp) \\ &\leq d(x_n, p). \end{aligned} \tag{3.6}$$

Hence

$$\limsup_{n\to\infty} d(z_n, p) \leq \lim_{n\to\infty} d(x_n, p) = c.$$

Using the fact that $0 < a \leq \beta_n \leq b < 1$ and by (3.5), we have $\lim_{n\to\infty} d(x_n, z_n) = 0$. □

Lemma 3.4. *Let K be a nonempty compact convex subset of a complete CAT(0) space X and let $t : K \to K$ and $T : K \to CB(X)$ be a single-valued and multi-valued nonexpansive mapping, respectively, and $F(t) \cap F(T) \neq \emptyset$ satisfying $Tp = \{p\}$ for all $p \in F(t) \cap F(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (2.4). If $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for some $a, b \in \mathbb{R}$, then $\lim_{n\to\infty} d(tx_n, x_n) = 0$.*

Proof. Consider

$$\begin{aligned} d(tx_n, x_n) &\leq d(tx_n, ty_n) + d(ty_n, x_n) \\ &\leq d(x_n, y_n) + d(ty_n, x_n) \\ &\leq d(x_n, (1 - \beta_n)x_n \oplus \beta_n z_n) + d(ty_n, x_n) \\ &\leq \beta_n d(x_n, z_n) + d(ty_n, x_n) \end{aligned} \tag{3.7}$$

Therefore, we have

$$\lim_{n\to\infty} d(tx_n, x_n) \le \lim_{n\to\infty} \beta_n d(x_n, z_n) + \lim_{n\to\infty} d(ty_n, x_n)$$

Hence, by Lemmas 3.2 and 3.3, we can conclude that $\lim_{n\to\infty} d(tx_n, x_n) = 0$. □

Next, we give the sufficient conditions which imply the existence of common fixed points for single valued and multi valued nonexpansive mappings, respectively, as follows.

Theorem 3.5. *Let K be a nonempty compact convex subset of a complete CAT(0) space X and let $t : K \to K$ and $T : K \to CB(X)$ be a single-valued and multi-valued nonexpansive mapping, respectively, and $F(t) \cap F(T) \neq \emptyset$ satisfying $Tp = \{p\}$ for all $p \in F(t) \cap F(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (2.4) and t satisfies Condition (A). If $0 < a \le \alpha_n$, $\beta_n \le b < 1$ for some $a, b \in \mathbb{R}$, then $y \in F(t) \cap F(T)$.*

Proof. From Lemma 3.4, we have

$$\lim_{n\to\infty} d(tx_n, x_n) = 0.$$

Since t satisfies Condition (A), we have

$$\lim_{n\to\infty} d(x_n, F(t)) = 0.$$

Thus, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $d(x_{n_i}, p_k) < \frac{1}{2^k}$ for some $\{p_k\} \subset F(t)$ and all k. Note that

$$d(x_{n_{k+1}}, p_k) \le d(x_{n_k}, p_k) < \frac{1}{2^k}.$$

We now show that $\{p_k\}$ is a Cauchy sequence in K. Notice that

$$\begin{aligned} d(p_{k+1}, p_k) &\le d(p_{k+1}, x_{n_{k+1}}) + d(x_{n_{k+1}}, p_k) \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}} \end{aligned}$$

This shows that $\{p_k\}$ is a Cauchy sequence in K and thus converges to $y \in K$. Since

$$d(p_k, ty) \le d(ty, tp_k) \le d(y, p_k)$$

and $p_k \to y$ as $k \to \infty$, it follows that $d(y, ty) = 0$ and thus $y \in F(t)$. Also, by Lemma 2.5 and Lemma 3.4, we have by using the fact that K is compact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to y$ as $i \to \infty$.

$$\begin{aligned} d(y, Ty) &\le d(y, x_{n_i}) + d(x_{n_i}, Tx_{n_i}) + H(Tx_{n_i}, Ty) \\ &\le d(y, x_{n_i}) + d(x_{n_i}, Tz_{n_i}) + d(x_{n_i}, y) \\ &\to 0 \quad \text{as} \quad i \to \infty. \end{aligned}$$

It follows that $y \in F(T)$. Therefore $y \in F(t) \cap F(T)$. □

Finally, we conclude this paper with the following convergence theorem.

Theorem 3.6. *Let K be a nonempty compact convex subset of a complete CAT(0) space X and let $t : K \to K$ and $T : K \to CB(X)$ be a single-valued and multi-valued nonexpansive mapping, respectively, and $F(t) \cap F(T) \neq \emptyset$ satisfying $Tp = \{p\}$ for all $p \in F(t) \cap F(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (2.4) with $0 < a \leq \alpha_n,\ \beta_n \leq b < 1$ for some $a, b \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to a common fixed point of t and T.*

Proof. Since $\{x_n\}$ is contained in K which is compact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some $y \in K$, that is,

$$\lim_{i\to\infty} d(x_{n_i}, y) = 0.$$

By Theorem 3.5, we have $y \in F(t) \cap F(T)$, and by Lemma 3.1, we have $\lim_{n\to\infty} d(x_n, y)$ exists. It must be the case in which $\lim_{n\to\infty} d(x_n, y) = \lim_{i\to\infty} d(x_{n_i}, y) = 0$. Therefore, $\{x_n\}$ converges strongly to a common fixed point y of t and T. □

References

1. M. Bridson and A. Haefliger, Metric Spaces of Non-Positive Curvature, Springer-Verlag, Berlin, Haedelberg, 1999.
2. K.S. Brown, Buildings, Springer-Verlag, New York, 1989.
3. F. Bruhat and J. Tits, Groupes reductifs sur un corps local. I. Donnees radicielles values, Inst. Hautes Etudes Sci. Publ. Math. **41** (1972), 5-251.
4. D. Burago, Y. Burago, and S. Ivanov, A Course in Metric Geometry, Graduate Studies in Math. Vol33, Amer. Math. Soc., Providence, RI, 2001.
5. P. Chaoha and A. Phon-on, *A note on fixed point sets in CAT(0) spaces*, J. Math. Anal. Appl. **320** (2006), 983-987.
6. S. Dhompongsa, A. Kaewkhao, and B. Panyanak, *Lim's theorems for multivalued mappings in CAT(0) spaces*, J. Math. Anal. Appl. **312** (2005), 478-487.
7. S. Dhompongsa, W. A. Kirk, and B. Sims, *Fixed points of uniformly lipschitzian mappings,* Nonlinear Anal. TMA, **65** (2006), 762-772.
8. S. Dhompongsa, W. A. Kirk, and B. Panyanak, *Nonexpansive set-valued mappings in metric and Banach spaces,* J. Nonlinear and Convex Anal. **8** (2007), 35-45.
9. S. Dhompongsa and B. Panyanak, *On Δ-convergence theorems in CAT(0) spaces,* Comput. Math. Appl. **56** (2008), 2572-2579.
10. S. Dhompongsa, W. Fupinwong and A. Kaewkhao, *Common fixed points of a nonexpansive semigroup and a convergence theorem for Mann iterations in geodesic metric spaces,* Nonlinear Anal. TMA, **70** (2009), 4268-4273.
11. R. Espinola and A. Fernandez-Leon, *CAT(k)-spaces, weak convergence and fixed points,* J. Math. Anal. Appl. **353** (2009), 410-427.
12. K. Fujiwara, K. Nagano, and T. Shioya, *Fixed point sets of parabolic isometries of CAT(0) spaces,* Comment. Math. Helv. **81** (2006), 305-335.
13. K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive mappings,* Marcel Dekker, Inc. New York, 1984.
14. M. Gromov, Metric Sructures for Riemannian and non-Riemannian Spaces, Progress in Mathematics 152, Birkhauser, Boston, 1999.
15. S. Ishikawa, *Fixed points by a new iteration method,* Proc. Amer. Math. Soc. **44** (1974), 147-150.
16. W. A. Kirk, *Geodesic geometry and fixed point theory,* In Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003), pp. 195-225, Colecc. Abierta, 64, Univ. Seville Secr. Publ., Seville, (2003).

17. W. A. Kirk, *Geodesic geometry and fixed point theory II,* In International Conference on Fixed Point Theory and Applications,pp. 113-142, Yokohama Publ. Yokohama, (2004).
18. W. A. Kirk, *Fixed point theorems in CAT(0) spaces and $\mathbb{R}$-trees,* Fixed Point Theory and Appl. **2004** (2004), 309-316.
19. W. A. Kirk and B. Panyanak, *A concept of convergence in geodesic spaces,* Nonlinear Anal., TMA, **68** (2008), 3689-3696.
20. T. Laokul and B. Panyanak, *Approximating fixed points of nonexpansive mappings in CAT(0) spaces,* Int. J. Math. Anal.**3** (2009), 1305-1315.
21. W. Laowang and B. Panyanak, *Strong and Δ-convergence theorems for multivalued mappings in CAT(0) spaces,* J. Inequal. Appl. **2009** Article ID 730132, 16 pages.
22. B. Nanjaras, B. Panyanak and W. Phuengrattana, *Fixed point theorems and convergence theorems for Suzuki-generalized nonexpansive mappings in CAT(0) spaces,* Nonlinear Anal. Hybrid Systems, **4** (2010), 25-31.
23. B. Panyanak, *Mann and Ishikawa iterative process for multivalued mappings in Banach spaces,* Computers and Mathematics with Applications,vol. **54** (6), (2007), 872-877.
24. T. Puttasontiphot, *Mann and Ishikawa Iteration Schemes for multivalued mappings in CAT(0) Spaces,* Applied Mathematical Sciences,**4**(61), (2010) ,3005-3018.
25. A. Razani and H. Salahifard, *Invariant approximation for CAT(0) spaces,* Nonlinear Anal.TMA, **72** (2010), 2421-2425.
26. S. Saejung, *Halpern's iteration in CAT(0) spaces,* Fixed Point Theory and Appl. **2010**, Article ID 471781, 13 pages.
27. K. P. R. Sastry and G. V. R. Babu, *Convergence of Ishikawa iterates for a multivalued mapping with a fixed point,* Czechoslovak Math. J. **55** (130), no. 4, (2005), 817-826,.
28. H. F. Senter and W. G. Dotson, *Approximating fixed points of nonexpansive mappings,* Proc. Amer. Math. Soc. **44** (1974), 375-380.
29. N. Shahzad and J. Markin, *Invariant approximation for commuting mappings in CAT(0) and hyperconvex spaces,* J. Math. Anal. Appl.**337** (2008), 1457-1464.
30. N. Shahzad, *Invarient approximation in CAT(0) spaces,* Nonlinear Anal. TMA, **70** (2009), 4338-4340.
31. N. Shahzad and H. Zegeye,*On Mann and Ishikawa iteration schemes for multivalued maps in Banach spaces,* Nonlinear Anal. TMA,vol. **71** (3-4), (2009), 838-844.
32. Y. Song and H. Wang, Erratum to: *"Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces, [Comput. Math. Appl. **54** (2007), 872-877]",* Comput. Math. Appl.vol. **55** (12) (2008), 2999-3002.
33. Y. Song and H. Wang, *Convergence of iterative algorithms for multivalued mappings in Banach spaces,* Nonlinear Anal. TMA, **70** (4) (2009), 1547-1556.

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