

SCREEN TRANSVERSAL LIGHTLIKE SUBMERSIONS

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ABSTRACT. In this article, we introduce a new class of lightlike submersions, namely, screen transversal lightlike submersions from an indefinite Kaehler manifold onto a lightlike manifold. This class includes radical screen transversal and screen transversal anti-invariant lightlike submersions as particular cases. We also study some properties of these submersions.

1. INTRODUCTION

Semi-Riemannian geometry differs from Riemannian geometry in that the bilinear form equipped on each tangent space can be indefinite in the case of semi-Riemannian geometry. Classical examples include Lorentzian spaces and De Sitter spaces in general relativity [11]. An n -dimensional semi-Riemannian manifold is an n -dimensional smooth manifold $\mathcal{M}$ equipped with a symmetric, non-degenerate 2-tensor field g , called the semi-Riemannian metric on $\mathcal{M}$. A semi-Riemannian manifold is called Riemannian, if metric tensor g has signature $(+, \dots, +)$. The simplest example of Riemannian manifold is Euclidean space, which is $\mathbb{R}^n$ with the Riemannian metric $g = dx^1 \otimes dx^1 + \dots + dx^n \otimes dx^n$ while the simplest example of a semi-Riemannian manifold is the Minkowski space, which is $\mathbb{R}^n$ with the Lorentzian metric $g = -dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + \dots + dx^n \otimes dx^n$. If $\mathcal{M}$ is a semi-Riemannian manifold, there are three categories of curves, namely, spacelike, timelike and lightlike (null), depending upon their causal character.

The theory of lightlike manifolds has been developed and studied by Bejancu-Duggal [3] and Kupeli [9]. The major difference between the theory of semi-Riemannian manifolds and lightlike submanifolds arises because in the first case $\mathcal{TM} \cap \mathcal{TM}^\perp = \{0\}$, while in the second case, a part of normal vector bundle $\mathcal{TM}^\perp$ lies in the tangent bundle $\mathcal{TM}$ of submanifold $\mathcal{M}$ of the manifold $\bar{\mathcal{M}}$. In [3], Duggal and Bejancu introduced CR-lightlike manifolds as a generalization of lightlike real hypersurfaces of indefinite Kaehler manifolds. Further, Sahin studied transversal lightlike submanifolds [13], such that $\bar{\mathcal{J}}(\text{Rad}(\mathcal{TM}))$ is never tangent to $\mathcal{M}$, and screen transversal lightlike submanifolds [12] of indefinite Kaehler manifolds, which contains real lightlike curves.

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On the other hand, the basic idea of Riemannian submersions between Riemannian manifolds was initiated by O'Neill [10] and Gray [6]. A smooth map $\varphi : \mathcal{M} \rightarrow N$ between Riemannian manifolds $\mathcal{M}$ and N is called a Riemannian submersion if it has maximal rank and the derivative map φ_* preserves the lengths of horizontal vectors. Sahin and Gündüzalp [14] introduced lightlike submersions from a semi-Riemannian manifold onto a lightlike manifold. In our recent research published in [15]-[19], we conducted a rigorous analysis of various classes of lightlike submersions. Our investigation includes a comprehensive examination of their geometric properties.

It is important to note that radical distribution Δ is invariant with respect to the complex structure of a Kaehler manifold in screen Cauchy Riemann [15], screen Slant [17], and screen pseudo-slant lightlike submersions [16]. In [18], Δ admits two orthogonal complementary distributions $\mathcal{D}_1$ and $\mathcal{D}_1^l$ such that $\Delta = \mathcal{D}_1 \oplus_{orth} \mathcal{D}_1^l$, where $\bar{\mathcal{J}}\mathcal{D}_1 = \mathcal{D}_1$ and $\bar{\mathcal{J}}\mathcal{D}_1^l \subseteq ltr(\mathcal{Ker}\phi_*)$. However, in [19], $\bar{\mathcal{J}}\Delta = ltr(\mathcal{Ker}\varphi_*)$. Therefore, we ask the following: Are there any lightlike submersions for which $\bar{\mathcal{J}}\Delta \subset \mathcal{S}(\mathcal{Ker}\varphi_*)^\perp$? To answer this question, in the affirmative, we introduce a new class called screen transversal lightlike submersions.

We arranged the work as follows: In section 2, we briefly give a basic idea related to this paper. In section 3, we introduce the notions of screen transversal lightlike and screen transversal isotropic submersions, providing an example of each class. In section 4, we define radical screen transversal lightlike submersions with a non-trivial example and investigate the integrability of distributions involved in the definition of such submersions. We have obtained the necessary and sufficient conditions for foliations derived by the above distributions to be totally geodesic. Additionally, in this section, we derive a necessary and sufficient condition for the induced connection $\hat{\nabla}$ on $\mathcal{S}(\mathcal{Ker}\varphi_*)$ to be a metric connection. In the last section, we define screen transversal anti-invariant lightlike submersions with an example and prove that the distribution $\mathcal{D}'$ involved in the definition of such submersions is invariant under the almost complex structure $\bar{\mathcal{J}}$. Furthermore, we give a characterization for such submersions and derive a necessary and sufficient condition for the induced connection $\hat{\nabla}$ on $\mathcal{Ker}\varphi_*$ to be a metric connection.

2. PRELIMINARIES

An almost complex manifold $\mathcal{M}$ equipped with an almost complex structure $\bar{\mathcal{J}}$ and a semi-Riemannian metric g of index $r > 0$ is referred to as an indefinite almost Hermitian manifold if the metric g is compatible with the almost complex structure $\bar{\mathcal{J}}$, meaning that for all vector fields $\mathcal{U}$ and $\mathcal{V}$ on $\mathcal{M}$, the following condition holds:

$$g(\bar{\mathcal{J}}\mathcal{U}, \bar{\mathcal{J}}\mathcal{V}) = g(\mathcal{U}, \mathcal{V}). \quad (2.1)$$

An indefinite almost Hermitian manifold $(\mathcal{M}, \bar{\mathcal{J}}, g)$ with the Levi-Civita connection ∇ is referred to as an indefinite Kaehler manifold if for all vector fields $\mathcal{U}$ and $\mathcal{V}$ on $\mathcal{M}$, we have

$$(\nabla_{\mathcal{U}}\bar{\mathcal{J}})\mathcal{V} = 0. \quad (2.2)$$

An indefinite complex space form $\mathcal{M}(c)$ refers to a connected indefinite Kaehler manifold $\mathcal{M}$ characterized by a constant holomorphic sectional curvature c . The

curvature tensor field of $\mathcal{M}(c)$ is expressed as:

$$\begin{aligned} \mathcal{R}(\mathcal{U}, \mathcal{V})\mathcal{W} = & \frac{c}{4} \{g(\mathcal{Y}, \mathcal{Z})\mathcal{X} - g(\mathcal{U}, \mathcal{Z})\mathcal{V} + g(\bar{\mathcal{J}}\mathcal{V}, \mathcal{W})\bar{\mathcal{J}}\mathcal{U} \\ & - g(\bar{\mathcal{J}}\mathcal{U}, \mathcal{W})\bar{\mathcal{J}}\mathcal{V} + 2g(\mathcal{U}, \bar{\mathcal{J}}\mathcal{V})\bar{\mathcal{J}}\mathcal{W}\}, \quad \forall \mathcal{U}, \mathcal{V}, \mathcal{W} \in \Gamma(\mathcal{TM}). \end{aligned} \quad (2.3)$$

Let $(\bar{\mathcal{M}}, \bar{g})$ be an $(m+n)$ -dimensional $(m, n > 1)$ semi-Riemannian manifold, which is not a Riemannian manifold and $\bar{g}$ be a semi-Riemannian metric on $\bar{\mathcal{M}}$. Also, let $\mathcal{M}$ be a submanifold of $\bar{\mathcal{M}}$. Then the radical or null space of $\mathcal{M}$ is defined as $\mathcal{Rad}(\mathcal{T}_p\mathcal{M}) = \{\xi \in \mathcal{T}_p\mathcal{M} : g(\xi, \mathcal{U}) = 0, \forall \mathcal{U} \in \mathcal{T}_p\mathcal{M}\}$. If $\mathcal{Rad}(\mathcal{TM}) : p \in \mathcal{M} \rightarrow \mathcal{Rad}(\mathcal{T}_p\mathcal{M})$ defines a smooth distribution of rank $0 < r \leq m$ on $\mathcal{M}$, then $\mathcal{Rad}(\mathcal{TM})$ is called radical distribution on $\mathcal{M}$ and in this case the manifold $\mathcal{M}$ is called an r-lightlike submanifold.

On the other hand, if $\varphi : (\mathcal{M}, g) \rightarrow (N, g')$ is a smooth submersion from an m -dimensional semi-Riemannian manifold $\mathcal{M}$ onto an n -dimensional r-lightlike manifold N , then the pre image $f^{-1}(y)$ is an $(m-n)$ dimensional submanifold of $\mathcal{M}$, for any $y \in N$, called fibre. The kernel of derivative map $\varphi_* : \mathcal{T}_p\mathcal{M} \rightarrow \mathcal{T}_{f(p)}N$ at $p \in \mathcal{M}$ is given by $\mathcal{Ker}\varphi_{*p} = \{\mathcal{U} \in \mathcal{T}_p\mathcal{M} : \varphi_{*p}\mathcal{U} = 0\}$, and $(\mathcal{Ker}\varphi_{*p})^\perp$ is defined as $(\mathcal{Ker}\varphi_{*p})^\perp = \{\mathcal{V} \in \mathcal{T}_p\mathcal{M} : g(\mathcal{V}, \mathcal{U}) = 0, \forall \mathcal{U} \in \mathcal{Ker}\varphi_{*p}\}$. Since $\mathcal{T}_p\mathcal{M}$ is a semi-Riemannian vector space, $(\mathcal{Ker}\varphi_{*p})^\perp$ may not be a complementary space to $\mathcal{Ker}\varphi_{*p}$. Therefore, we consider $\Delta_p = (\mathcal{Ker}\varphi_{*p}) \cap (\mathcal{Ker}\varphi_{*p})^\perp \neq \{0\}$. In this case $\Delta : p \rightarrow \Delta_p$ is a distribution on $\mathcal{M}$, called the radical or null distribution. Since the radical distribution Δ is degenerate, its orthogonal complementary distribution $\mathcal{S}(\mathcal{Ker}\varphi_*)$ in $\mathcal{Ker}\varphi_*$ is non-degenerate. Thus $\mathcal{Ker}\varphi_* = \Delta \perp \mathcal{S}(\mathcal{Ker}\varphi_*)$. Similarly $(\mathcal{Ker}\varphi_*)^\perp = \Delta \perp \mathcal{S}(\mathcal{Ker}\varphi_*)^\perp$, where $\mathcal{S}(\mathcal{Ker}\varphi_*)^\perp$ is the complementary distribution to Δ in $(\mathcal{Ker}\varphi_*)^\perp$. Now, since $\mathcal{S}(\mathcal{Ker}\varphi_*)$ is non-degenerate in $\mathcal{TM}$, we have

$$\mathcal{TM} = \mathcal{S}(\mathcal{Ker}\varphi_*) \perp (\mathcal{S}(\mathcal{Ker}\varphi_*)^\perp), \quad (2.4)$$

where $(\mathcal{S}(\mathcal{Ker}\varphi_*))^\perp$ is the orthogonal complementary distribution to $\mathcal{S}(\mathcal{Ker}\varphi_*)$ in $\mathcal{TM}$. Since $\mathcal{S}(\mathcal{Ker}\varphi_*)^\perp$ is non-degenerate in $(\mathcal{S}(\mathcal{Ker}\varphi_*))^\perp$, we have $(\mathcal{S}(\mathcal{Ker}\varphi_*))^\perp = \mathcal{S}(\mathcal{Ker}\varphi_*)^\perp \perp (\mathcal{S}(\mathcal{Ker}\varphi_*)^\perp)^\perp$. Also, as $\Delta \subset (\mathcal{S}(\mathcal{Ker}\varphi_*)^\perp)^\perp$ and $(\mathcal{S}(\mathcal{Ker}\varphi_*)^\perp)^\perp$ is non-degenerate, there exists null vectors $\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_r$, such that $g(\xi_i, \xi_j) = g(\mathcal{N}_i, \mathcal{N}_j) = 0$, $g(\xi_i, \mathcal{N}_j) = \delta_{ij}$ and $g(\mathcal{Z}_\alpha, \xi_j) = g(\mathcal{Z}_\alpha, \mathcal{N}_j) = 0$, for $i, j \in \{1, \dots, r\}$ and $\alpha \in \{1, \dots, t\}$. Here $\{\xi_i\}$ is a basis of Δ , while $\{\mathcal{Z}_\alpha\}$ is a basis $\mathcal{S}(\mathcal{Ker}\varphi_*)^\perp$. The distribution produced by vector fields $\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_r$ is denoted by $\text{ltr}(\mathcal{Ker}\varphi_*)$ such that $\text{tr}(\mathcal{Ker}\varphi_*) = \text{ltr}(\mathcal{Ker}\varphi_*) \perp \mathcal{S}(\mathcal{Ker}\varphi_*)^\perp$. In this case $\text{ltr}(\mathcal{Ker}\varphi_*)$ and $\mathcal{Ker}\varphi_*$ are not orthogonal to each other. Moreover $\mathcal{TM} = \mathcal{Ker}\varphi_* \oplus \text{tr}(\mathcal{Ker}\varphi_*)$, where $\mathcal{Ker}\varphi_*$ and $\text{tr}(\mathcal{Ker}\varphi_*)$ are not orthogonal to each other.

Let $\dim(\Delta) = r$. Then, a smooth submersion $\varphi : \mathcal{M} \rightarrow N$ is called r-lightlike submersion if the length of horizontal vectors preserved under the derivative map φ_* and $r = \dim\{(\mathcal{Ker}\varphi_*) \cap (\mathcal{Ker}\varphi_*)^\perp\} = r, 0 < r < \min\{\dim(\mathcal{Ker}\varphi_*), \dim(\mathcal{Ker}\varphi_*)^\perp\}$. Moreover, φ is called co-isotropic submersion if $r = \dim(\mathcal{Ker}\varphi_*)^\perp < \dim(\mathcal{Ker}\varphi_*)$, isotropic submersion if $r = \dim(\mathcal{Ker}\varphi_*) < \dim(\mathcal{Ker}\varphi_*)^\perp$ and totally lightlike submersion if $r = \dim(\mathcal{Ker}\varphi_*)^\perp = \dim(\mathcal{Ker}\varphi_*)$. The geometry of semi-Riemannian

submersions can be characterized by O' Neill tensors $\mathcal{T}$ and $\mathcal{A}$, given as

$$\mathcal{T}_U \mathcal{V} = \hbar \nabla_{\nu U} \nu \mathcal{V} + \nu \nabla_{\nu U} \hbar \mathcal{V}, \quad (2.5)$$

$$\mathcal{A}_U \mathcal{V} = \nu \nabla_{\hbar U} \hbar \mathcal{V} + \hbar \nabla_{\hbar U} \nu \mathcal{V}, \quad (2.6)$$

for any $U, \mathcal{V} \in \Gamma(\text{Ker} \varphi_*)$. Here tensor $\mathcal{T}$ has the following symmetric property

$$\mathcal{T}_U \mathcal{V} = \mathcal{T}_V U, \quad \forall U, \mathcal{V} \in \Gamma(\text{Ker} \varphi_*) \quad (2.7)$$

Next, we assume $\varphi : (\mathcal{M}, g) \rightarrow (N, g')$ is a lightlike submersion from a real $(m+n)$ -dimensional $(m, n > 1)$ semi-Riemannian manifold $\mathcal{M}$ onto a lightlike manifold N . Further, we consider that $\text{Ker} \varphi_*$ is an m -dimensional lightlike distribution on $\mathcal{M}$ such that $\text{tr}(\text{Ker} \varphi_*)$ is the complementary distribution to $\text{Ker} \varphi_*$ in $\mathcal{M}$ in context of the pair $\{\mathcal{S}(\text{Ker} \varphi_*), \mathcal{S}(\text{Ker} \varphi_*)^\perp\}$. Additionally, let $\hat{g}$ and $\hat{\nabla}$ represent the induced metric to g and linear connection respectively on $\text{Ker} \varphi_*$. Then (2.5) gives

$$\nabla_U \mathcal{V} = \hat{\nabla}_U \mathcal{V} + \mathcal{T}_U \mathcal{V}, \quad (2.8)$$

$$\nabla_U X = \mathcal{T}_U X + \nabla_U^\perp X, \quad (2.9)$$

$\forall U, \mathcal{V} \in \Gamma(\text{Ker} \varphi_*), X \in \Gamma(\text{Ker} \varphi_*)^\perp$, where $\hat{\nabla}_U \mathcal{V} = \nu \nabla_U \mathcal{V}$ and $\nabla_U^\perp X = \hbar \nabla_U X$. Here $\{\hat{\nabla}_U \mathcal{V}, \mathcal{T}_U X\}$ and $\{\mathcal{T}_U V, \nabla_U^\perp X\}$ belong to $\Gamma(\text{Ker} \varphi_*)$ and $\Gamma(\text{tr}(\text{Ker} \varphi_*))$, respectively. Let $\mathcal{S}(\text{Ker} \varphi_*)^\perp \neq \{0\}$, and assume that L and S represent the projections of $\text{tr}(\text{Ker} \varphi_*)$ on $\text{ltr}(\text{Ker} \varphi_*)$ and $\mathcal{S}(\text{Ker} \varphi_*)^\perp$, respectively. Then, from (2.8) and (2.9), we obtain

$$\nabla_U \mathcal{V} = \hat{\nabla}_U \mathcal{V} + \mathcal{T}_U^l \mathcal{V} + \mathcal{T}_U^s \mathcal{V}, \quad (2.10)$$

$$\nabla_U \mathcal{N} = \mathcal{T}_U \mathcal{N} + \nabla_U^{\perp l} \mathcal{N} + D^{\perp s}(\mathcal{U}, \mathcal{N}), \quad (2.11)$$

$$\nabla_U \mathcal{W} = \mathcal{T}_U \mathcal{W} + D^{\perp l}(\mathcal{U}, \mathcal{W}) + \nabla_U^{\perp s} \mathcal{W}, \quad (2.12)$$

$\forall U, \mathcal{V} \in \Gamma(\text{Ker} \varphi_*), \mathcal{N} \in \Gamma(\text{ltr}(\text{Ker} \varphi_*))$ and $\mathcal{W} \in \Gamma(\mathcal{S}(\text{Ker} \varphi_*)^\perp)$. Now using (2.10) and (2.12) and taking into account that ∇ is a metric connection, we get

$$g(\mathcal{T}_U^s \mathcal{V}, \mathcal{W}) + g(\mathcal{V}, D^{\perp l} \mathcal{U}, \mathcal{W}) = -\hat{g}(\mathcal{T}_U \mathcal{W}, \mathcal{V}). \quad (2.13)$$

3. SCREEN TRANSVERSAL LIGHTLIKE SUBMERSIONS

In this section we define screen transversal lightlike submersions, giving a non-trivial example by using the following lemma.

Lemma 3.1. *Let $\varphi : (\mathcal{M}, g) \rightarrow (N, g')$ be an r -lightlike submersion from an indefinite Kaehler manifold $\mathcal{M}$ onto a lightlike manifold N . If $\bar{\mathcal{J}}(\Delta)$ is a vector subbundle of $\mathcal{S}(\text{Ker} \varphi_*)^\perp$, then $\bar{\mathcal{J}}(\text{ltr}(\text{Ker} \varphi_*))$ is also a vector subbundle of $\mathcal{S}(\text{Ker} \varphi_*)^\perp$, and $\bar{\mathcal{J}}(\Delta) \cap \bar{\mathcal{J}}(\text{ltr}(\text{Ker} \varphi_*)) = \{0\}$.*

Proof. Assume that $\bar{\mathcal{J}}(\text{ltr}(\text{Ker} \varphi_*)) = \text{ltr}(\text{Ker} \varphi_*)$. Since φ is an r -lightlike submersion, for any $\xi \in \Gamma(\Delta)$ and $\mathcal{N} \in \Gamma(\text{ltr}(\text{Ker} \varphi_*))$, by using (2.1), we have $g(\bar{\mathcal{J}}\xi, \bar{\mathcal{J}}\mathcal{N}) = g(\xi, \mathcal{N}) = 1$. Now, if $\bar{\mathcal{J}}\mathcal{N} \in \Gamma(\text{ltr}(\text{Ker} \varphi_*))$, by the hypothesis of the lemma, we would have $g(\bar{\mathcal{J}}\xi, \bar{\mathcal{J}}\mathcal{N}) = 0$. This leads to a contradiction. It follows that $\bar{\mathcal{J}}\mathcal{N} \notin \Gamma(\text{ltr}(\text{Ker} \varphi_*))$. Next, suppose $\bar{\mathcal{J}}\mathcal{N} \in \Gamma(\mathcal{S}(\text{Ker} \varphi_*)^\perp)$. Then,

we get $0 = g(\bar{\mathcal{J}}\xi, \bar{\mathcal{J}}\mathcal{N}) = g(\xi, \mathcal{N}) = 1$. It implies that $\bar{\mathcal{J}}\mathcal{N} \notin \Gamma(\mathcal{S}(\text{Ker}\varphi_*))$. Similarly, it can be shown $\bar{\mathcal{J}}\mathcal{N} \notin \Gamma(\Delta)$. Thus, we conclude that $\bar{\mathcal{J}}\mathcal{N} \in \Gamma(\mathcal{S}(\text{Ker}\varphi_*)^\perp)$. Finally, let $\mathcal{V} \in \Gamma(\bar{\mathcal{J}}(\Delta) \cap \bar{\mathcal{J}}(\text{ltr}(\text{Ker}\varphi_*)))$. This implies $\mathcal{V} \in \Gamma(\bar{\mathcal{J}}(\Delta))$, and thus $g(\mathcal{V}, \bar{\mathcal{J}}\mathcal{N}) = 0$, as $\mathcal{V} \in \Gamma(\bar{\mathcal{J}}(\text{ltr}(\text{Ker}\varphi_*)))$. However, for an r-lightlike submersion, there exists $\bar{\mathcal{J}}\mathcal{V} \in \Gamma(\Delta)$ such that $g(\bar{\mathcal{J}}\mathcal{V}, \mathcal{N}) \neq 0$. Therefore, using (2.1), we have $0 \neq g(\bar{\mathcal{J}}\mathcal{V}, \mathcal{N}) = -g(\mathcal{V}, \bar{\mathcal{J}}\mathcal{N}) = 0$, which is contradiction. Thus the proof is complete. $\square$

Definition 3.2. An r-lightlike submersion $\varphi : (\mathcal{M}, g, \bar{\mathcal{J}}) \rightarrow (N, g')$ from an indefinite Kaehler manifold $\mathcal{M}$ onto a lightlike manifold N is called a screen transversal lightlike submersion if $\bar{\mathcal{J}}(\Delta) \subset \mathcal{S}(\text{Ker}\varphi_*)^\perp$.

From definition (3.2), we have the following decomposition:

$$\mathcal{S}(\text{Ker}\varphi_*)^\perp = [\bar{\mathcal{J}}(\Delta) \oplus \bar{\mathcal{J}}(\text{ltr}(\text{Ker}\varphi_*))] \perp \mathcal{D},$$

where $\mathcal{D}$ is a non-null orthogonal complementary distribution to $\bar{\mathcal{J}}(\Delta) \oplus \bar{\mathcal{J}}(\text{ltr}(\text{Ker}\varphi_*))$ in $\mathcal{S}(\text{Ker}\varphi_*)^\perp$. From definition (3.2), it is also clear that:

- (i) $\bar{\mathcal{J}}(\text{ltr}(\text{Ker}\varphi_*)) \subset \mathcal{S}(\text{Ker}\varphi_*)^\perp$.
- (ii) There does not exist any screen transversal co-isotropic or totally lightlike submersions, since in these cases $\mathcal{S}(\text{Ker}\varphi_*)^\perp = \{0\}$.

Proposition 3.3. Let $\varphi : (\mathcal{M}, g, \bar{\mathcal{J}}) \rightarrow (N, g')$ be a screen transversal lightlike submersion from an indefinite Kaehler manifold $\mathcal{M}$ onto a lightlike manifold N . Then $\bar{\mathcal{J}}(\Delta)$ and $\bar{\mathcal{J}}(\text{ltr}(\text{Ker}\varphi_*))$ are not orthogonal.

Proof. Assume, for the sake of contradiction, that $\bar{\mathcal{J}}(\Delta)$ and $\bar{\mathcal{J}}(\text{ltr}(\text{Ker}\varphi_*))$ are orthogonal. Then we have

$$0 = g(\bar{\mathcal{J}}\xi, \bar{\mathcal{J}}\mathcal{N}) = g(\xi, \mathcal{N}) = 1,$$

where $\xi \in \Gamma(\Delta)$ and $\mathcal{N} \in \Gamma(\text{ltr}(\text{Ker}\varphi_*))$. which leads to a contradiction. Therefore, $\bar{\mathcal{J}}(\Delta)$ and $\bar{\mathcal{J}}(\text{ltr}(\text{Ker}\varphi_*))$ are indeed not orthogonal. This completes the proof. $\square$

Example 3.4. Assume that $\mathbb{R}_{0,1,7}^8$ and $\mathbb{R}_{1,0,4}^5$ are equipped with the semi-Riemannian metric

$$g = (du_1)^2 + (du_2)^2 + (du_3)^2 + (du_4)^2 - (du_5)^2 + (du_6)^2 + (du_7)^2 + (du_8)^2$$

and degenerate metric $g' = (dv_2)^2 + (dv_3)^2 + (dv_4)^2 + (dv_5)^2$, where $u_1, \dots, u_8$ and $v_1, \dots, v_5$ are the canonical coordinates on $\mathbb{R}^8$ and $\mathbb{R}^5$, respectively. Define the map

$$\varphi : (\mathbb{R}_{0,1,7}^8, g) \rightarrow (\mathbb{R}_{1,0,4}^5, g'); \quad (u_1, \dots, u_8) \mapsto \left(u_1 - u_5, u_2, u_6, \frac{u_3 + u_7}{\sqrt{2}}, \frac{u_4 + u_8}{\sqrt{2}} \right).$$

Then, we obtain

$$\text{Ker}\varphi_* = \text{Span} \left\{ \xi = \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_5}, \mathcal{S}_1 = \frac{\partial}{\partial u_3} - \frac{\partial}{\partial u_7}, \mathcal{S}_2 = \frac{\partial}{\partial u_4} - \frac{\partial}{\partial u_8} \right\}$$

and

$$(\text{Ker}\varphi_*)^\perp = \text{Span} \left\{ \xi, \bar{\mathcal{S}}_1 = \frac{\partial}{\partial u_2}, \bar{\mathcal{S}}_2 = \frac{\partial}{\partial u_6}, \bar{\mathcal{S}}_3 = \frac{\partial}{\partial u_3} + \frac{\partial}{\partial u_7}, \bar{\mathcal{S}}_4 = \frac{\partial}{\partial u_4} + \frac{\partial}{\partial u_8} \right\}.$$

Thus $\Delta = \text{Span}\{\xi\}$, where $\bar{\mathcal{J}}\Delta = \text{Span}\{\bar{S}_1 + \bar{S}_2\} \subset \mathcal{S}(\text{Ker}\varphi_*)^\perp$ and $\bar{\mathcal{J}}(\text{ltr}(\text{Ker}\varphi_*)) = \text{Span}\left\{\frac{-\bar{S}_1 + \bar{S}_2}{2}\right\} \subset \mathcal{S}(\text{Ker}\varphi_*)^\perp$. Finally, since the length of horizontal vectors is preserved by the derivative map φ_* , therefore φ is a screen transversal lightlike submersion.

Definition 3.5. Let $\varphi : (\mathcal{M}, g, \bar{\mathcal{J}}) \rightarrow (N, g')$ be an isotropic submersion from an indefinite Kaehler manifold $\mathcal{M}$ onto a lightlike manifold N . Then, we say that φ is a screen transversal isotropic submersion if $\bar{\mathcal{J}}(\Delta) \subset \mathcal{S}(\text{Ker}\varphi_*)^\perp$.

Theorem 3.6. Let $\varphi : (\mathcal{M}, g, \bar{\mathcal{J}}) \rightarrow (N, g')$ be an isotropic submersion from an indefinite Kaehler manifold $\mathcal{M}$ onto a lightlike manifold N . If $\text{Ker}\varphi_*$ is a real null curve of $\mathcal{M}$ [3], then φ is a screen transversal isotropic submersion.

Proof. Since φ is an isotropic submersion and $\text{Ker}\varphi_*$ is a real null curve, we have $\text{Ker}\varphi_* = \Delta = \text{Span}\{\xi\}$, for any $\xi \in \Gamma(\text{Ker}\varphi_*)$. In view of (2.1), we get $g(\bar{\mathcal{J}}\xi, \xi) = 0$, which implies that $\bar{\mathcal{J}}\xi \notin \Gamma(\text{ltr}(\text{Ker}\varphi_*))$. Further, since ξ and $\bar{\mathcal{J}}\xi$ are linearly independent, $\dim(\text{Ker}\varphi_*) = 1$, and $\bar{\mathcal{J}}\xi \notin \Gamma(\text{Ker}\varphi_*)$, we conclude that $\bar{\mathcal{J}}\xi \in \Gamma(\mathcal{S}(\text{Ker}\varphi_*)^\perp)$. Furthermore, for any $\mathcal{N} \in \Gamma(\text{ltr}(\text{Ker}\varphi_*))$, we have $g(\bar{\mathcal{J}}\mathcal{N}, \mathcal{N}) = 0$, which implies $\bar{\mathcal{J}}\mathcal{N} \notin \Gamma(\Delta)$. Now, since $\bar{\mathcal{J}}\xi \in \Gamma(\mathcal{S}(\text{Ker}\varphi_*)^\perp)$, we arrived $g(\bar{\mathcal{J}}\mathcal{N}, \xi) = g(\mathcal{N}, \bar{\mathcal{J}}\xi) = 0$. It follows that $\bar{\mathcal{J}}\mathcal{N} \notin \Gamma(\text{ltr}(\text{Ker}\varphi_*))$. Thus, we conclude that $\bar{\mathcal{J}}\mathcal{N} \in \Gamma(\mathcal{S}(\text{Ker}\varphi_*)^\perp)$. Moreover, we also have $g(\bar{\mathcal{J}}\mathcal{N}, \bar{\mathcal{J}}\xi) = g(\mathcal{N}, \xi) = 1$. Therefore, the screen transversal distribution $\mathcal{S}(\text{Ker}\varphi_*)^\perp$ can be expressed as $\mathcal{S}(\text{Ker}\varphi_*)^\perp = \{\bar{\mathcal{J}}\Delta \oplus \bar{\mathcal{J}}(\text{ltr}(\text{Ker}\varphi_*))\} \perp \mathcal{D}$, where $\mathcal{D}$ is complementary subspace of $\bar{\mathcal{J}}\Delta \oplus \bar{\mathcal{J}}(\text{ltr}(\text{Ker}\varphi_*))$ in $\mathcal{S}(\text{Ker}\varphi_*)^\perp$. Thus, by definition 3.5, we get the required proof. $\square$

Example 3.7. Assume that $\mathbb{R}_{0,1,3}^4$ and $\mathbb{R}_{1,0,2}^3$ are equipped with the semi-Riemannian metric $g = -(du_1)^2 + (du_2)^2 + (du_3)^2 + (du_4)^2$ and lightlike metric $g' = (dv_2)^2 + (dv_3)^2$, where $u_1, \dots, u_4$ and v_1, v_2, v_3 are the canonical coordinates on $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively. Define the map $\varphi : (\mathbb{R}_{0,1,3}^4, g) \rightarrow (\mathbb{R}_{1,0,2}^3, g')$ as $(u_1, \dots, u_4) \mapsto (u_1 - u_3, u_2, u_4)$. Then $\text{Ker}\varphi_* = \text{Span}\left\{\xi = \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_3}\right\}$ and $(\text{Ker}\varphi_*)^\perp = \text{Span}\left\{\xi, \bar{S}_1 = \frac{\partial}{\partial u_2}, \bar{S}_2 = \frac{\partial}{\partial u_4}\right\}$. Then $\Delta = \mathcal{S}(\text{Ker}\varphi_*) = \text{Span}\{\xi\}$ has the property that $\bar{\mathcal{J}}\Delta = \text{Span}\{\bar{S}_1 + \bar{S}_2\} \subset \mathcal{S}(\text{Ker}\varphi_*)^\perp$. Further, we get $\text{ltr}(\text{Ker}\varphi_*) = \text{Span}\left\{\mathcal{N} = -\frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_3}\right\}$, where $\bar{\mathcal{J}}(\mathcal{N}) = \text{Span}\{-\bar{S}_1 - \bar{S}_2\} \subset \mathcal{S}(\text{Ker}\varphi_*)^\perp$. Finally, we see that the length of horizontal vectors $\bar{S}_1, \bar{S}_2$ and $\mathcal{N}$ is preserved by the derivative map φ_* . Hence, φ is a screen transversal isotropic submersion.

4. RADICAL SCREEN TRANSVERSAL LIGHTLIKE SUBMERSIONS

In this section, we study radical screen transversal lightlike submersions from indefinite Kaehler manifolds onto lightlike manifolds.

Definition 4.1. Let $\varphi : (\mathcal{M}, g, \bar{\mathcal{J}}) \rightarrow (N, g')$ be a screen transversal lightlike submersion. Then φ is called a radical screen transversal lightlike submersion if $\bar{\mathcal{J}}(\mathcal{S}(\text{Ker}\varphi_*) = \mathcal{S}(\text{Ker}\varphi_*))$.

Note: In the remaining part of this section we assume that $\varphi : (\mathcal{M}, g, \bar{\mathcal{J}}) \rightarrow (N, g')$ be a radical screen transversal lightlike submersion from an indefinite Kaehler manifold $\mathcal{M}$ onto a lightlike manifold N .

Theorem 4.2. *Let $\varphi : (\mathcal{M}, g, \bar{\mathcal{J}}) \rightarrow (N, g')$ be a radical screen transversal lightlike submersion. Then $\mathcal{S}(\text{Ker}\varphi_*)$ is integrable if and only if $g(\mathcal{T}_\mathcal{U}^s \bar{\mathcal{J}}\mathcal{V}, \bar{\mathcal{J}}\mathcal{N}) = g(\mathcal{T}_\mathcal{V}^s \bar{\mathcal{J}}\mathcal{U}, \bar{\mathcal{J}}\mathcal{N})$, for any $\mathcal{U}, \mathcal{V} \in \Gamma(\mathcal{S}(\text{Ker}\varphi_*))$ and $\mathcal{N} \in \Gamma(\text{ltr}(\text{Ker}\varphi_*))$.*

Proof. Using (2.1), (2.2) and (2.10), we obtain

$$\begin{aligned} g([\mathcal{U}, \mathcal{V}], \mathcal{N}) &= g(\nabla_\mathcal{U} \mathcal{V} - \nabla_\mathcal{V} \mathcal{U}, \mathcal{N}) \\ &= g(\nabla_\mathcal{U} \bar{\mathcal{J}}\mathcal{V}, \bar{\mathcal{J}}\mathcal{N}) - g(\nabla_\mathcal{V} \bar{\mathcal{J}}\mathcal{U}, \bar{\mathcal{J}}\mathcal{N}) \\ &= g(\mathcal{T}_\mathcal{U}^s \bar{\mathcal{J}}\mathcal{V}, \bar{\mathcal{J}}\mathcal{N}) - g(\mathcal{T}_\mathcal{V}^s \bar{\mathcal{J}}\mathcal{U}, \bar{\mathcal{J}}\mathcal{N}), \end{aligned}$$

which completes the proof. $\square$

Theorem 4.3. *Let $\varphi : (\mathcal{M}, g, \bar{\mathcal{J}}) \rightarrow (N, g')$ be a radical screen transversal lightlike submersion. Then Δ is integrable if and only if for any $\mathcal{U} \in \Gamma(\mathcal{S}(\text{Ker}\varphi_*))$ and $\xi_1, \xi_2 \in \Gamma(\Delta)$ we have $g(\mathcal{T}_{\xi_1}^s \bar{\mathcal{J}}\mathcal{U}, \bar{\mathcal{J}}\xi_2) = g(\mathcal{T}_{\xi_2}^s \bar{\mathcal{J}}\mathcal{U}, \bar{\mathcal{J}}\xi_1)$.*

Proof. From (2.1), (2.2), and (2.10), we get

$$g([\xi_1, \xi_2], \mathcal{U}) = g(\nabla_{\xi_1} \bar{\mathcal{J}}\xi_2, \bar{\mathcal{J}}\mathcal{U}) - g(\nabla_{\xi_2} \bar{\mathcal{J}}\xi_1, \bar{\mathcal{J}}\mathcal{U}).$$

Then using (2.12) and (2.13), we derive

$$\begin{aligned} g([\xi_1, \xi_2], \mathcal{U}) &= g(\mathcal{T}_{\xi_1} \bar{\mathcal{J}}\xi_2, \bar{\mathcal{J}}\mathcal{U}) - g(\mathcal{T}_{\xi_2} \bar{\mathcal{J}}\xi_1, \bar{\mathcal{J}}\mathcal{U}), \\ &= g(\mathcal{T}_{\xi_1}^s \bar{\mathcal{J}}\mathcal{U}, \bar{\mathcal{J}}\xi_2) - g(\mathcal{T}_{\xi_2}^s \bar{\mathcal{J}}\mathcal{U}, \bar{\mathcal{J}}\xi_1). \end{aligned}$$

It completes the proof. $\square$

Theorem 4.4. *Let $\varphi : (\mathcal{M}, g, \bar{\mathcal{J}}) \rightarrow (N, g')$ be a radical screen transversal lightlike submersion. Then Δ defines a totally geodesic foliation on $\text{Ker}\varphi_*$ if and only if $\mathcal{T}_{\xi_1}^s \bar{\mathcal{J}}\mathcal{V}$ has no components in $\bar{\mathcal{J}}(\text{ltr}(\text{Ker}\varphi_*))$, for any $\mathcal{V} \in \Gamma(\mathcal{S}(\text{Ker}\varphi_*))$ and $\xi_1 \in \Gamma(\Delta)$.*

Proof. The distribution Δ defines a totally geodesic foliation on $\text{Ker}\varphi_*$ if and only if $\hat{\nabla}_{\xi_1} \xi_2 \in \Gamma(\Delta)$, for any $\xi_1, \xi_2 \in \Gamma(\Delta)$. Using (2.1), (2.2), (2.10), and (2.13), we obtain

$$g(\hat{\nabla}_{\xi_1} \xi_2, \mathcal{V}) = g(\nabla_{\xi_1} \xi_2, \mathcal{V}) = g(\nabla_{\xi_1} \bar{\mathcal{J}}\xi_2, \bar{\mathcal{J}}\mathcal{V}) = g(\mathcal{T}_{\xi_1}^s \bar{\mathcal{J}}\mathcal{V}, \bar{\mathcal{J}}\xi_2),$$

which completes the proof. $\square$

Theorem 4.5. *Let $\varphi : (\mathcal{M}, g, \bar{\mathcal{J}}) \rightarrow (N, g')$ be a radical screen transversal lightlike submersion. Then $\mathcal{S}(\text{Ker}\varphi_*)$ defines a totally geodesic foliation on $\text{Ker}\varphi_*$ if and only if $\mathcal{T}_\mathcal{U}^s \bar{\mathcal{J}}\mathcal{V}$ has no components in $\bar{\mathcal{J}}(\Delta)$, for any $\mathcal{U}, \mathcal{V} \in \Gamma(\mathcal{S}(\text{Ker}\varphi_*))$.*

Proof. Using (2.1), (2.2), (2.10), and (2.12), we obtain

$$g(\hat{\nabla}_\mathcal{U} \mathcal{V}, \mathcal{N}) = g(\nabla_\mathcal{U} \bar{\mathcal{J}}\mathcal{V}, \bar{\mathcal{J}}\mathcal{N}) = g(\mathcal{T}_\mathcal{U}^s \bar{\mathcal{J}}\mathcal{V}, \bar{\mathcal{J}}\mathcal{N}).$$

Thus, the proof follows. $\square$

Theorem 4.6. *Let $\varphi : (\mathcal{M}, g, \bar{\mathcal{J}}) \rightarrow (N, g')$ be a radical screen transversal lightlike submersion. Then the induced connection $\hat{\nabla}$ on $\text{Ker}\varphi_*$ is a metric connection if and only if $\mathcal{T}_U^s \bar{\mathcal{J}}\mathcal{V}$ has no components in $\bar{\mathcal{J}}(\text{ltr}(\text{Ker}\varphi_*))$, for any $U, \mathcal{V} \in \Gamma(\mathcal{S}(\text{Ker}\varphi_*))$.*

Proof. Let $U \in \Gamma(\text{Ker}\varphi_*)$ and $\xi \in \Gamma(\Delta)$. Using (2.2), we get $\nabla_U \xi = -\bar{\mathcal{J}}\nabla_U \bar{\mathcal{J}}\xi$. Further, using (2.10) and (2.12), we obtain

$$\hat{\nabla}_U \xi + \mathcal{T}_U^l \xi + \mathcal{T}_U^s \xi = -\bar{\mathcal{J}}\mathcal{T}_U \bar{\mathcal{J}}\xi - \bar{\mathcal{J}}\mathcal{D}^{\perp l}(U, \bar{\mathcal{J}}\xi) - \bar{\mathcal{J}}\nabla_U^{\perp s} \bar{\mathcal{J}}\xi.$$

So, for any $\mathcal{V} \in \Gamma(\mathcal{S}(\text{Ker}\varphi_*))$, we get

$$\hat{g}(\hat{\nabla}_U \xi, \mathcal{V}) = -\hat{g}(\bar{\mathcal{J}}\mathcal{T}_U \bar{\mathcal{J}}\xi, \mathcal{V}) = \hat{g}(\mathcal{T}_U \bar{\mathcal{J}}\xi, \bar{\mathcal{J}}\mathcal{V}).$$

Finally, using (2.13), we have $\hat{g}(\hat{\nabla}_U \xi, \mathcal{V}) = -\hat{g}(\mathcal{T}_U^s \bar{\mathcal{J}}\mathcal{V}, \bar{\mathcal{J}}\xi)$. It completes the proof. $\square$

Example 4.7. Let $\mathbb{R}_{0,2,10}^{12}$ and $\mathbb{R}_{2,0,6}^8$ be equipped with the semi-Riemannian metric

$$g = (du_1)^2 + (du_2)^2 + (du_3)^2 + (du_4)^2 - (du_5)^2 + (du_6)^2 \\ - (du_7)^2 + (du_8)^2 + (du_9)^2 + (du_{10})^2 + (du_{11})^2 + (du_{12})^2$$

and lightlike metric $g' = (dv_1)^2 + (dv_2)^2 + (dv_4)^2 + (dv_5)^2 + (dv_7)^2 + (dv_8)^2$, where $u_1, \dots, u_{12}$ and $v_1, \dots, v_8$ are the canonical coordinates on $\mathbb{R}^{12}$ and $\mathbb{R}^8$, respectively. Assume that the mapping $\varphi : (\mathbb{R}_{0,2,10}^{12}, g) \rightarrow (\mathbb{R}_{2,0,6}^8, g')$ is defined as

$$(u_1, \dots, u_{12}) \mapsto \left(\frac{u_1 - u_3}{\sqrt{2}}, \frac{u_2 - u_4}{\sqrt{2}}, u_5 + u_{11}, u_6, u_{12}, u_7 + u_9, u_8, u_{10} \right).$$

Then

$$\text{Ker}\varphi_* = \text{Span} \left\{ \mathcal{S}_1 = \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_3}, \mathcal{S}_2 = \frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_4}, \xi_1 = \frac{\partial}{\partial u_5} - \frac{\partial}{\partial u_{11}}, \xi_2 = \frac{\partial}{\partial u_7} - \frac{\partial}{\partial u_9} \right\}$$

and

$$(\text{Ker}\varphi_*)^\perp = \text{Span} \left\{ \xi_1, \xi_2, \bar{\mathcal{S}}_1 = \frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_3}, \bar{\mathcal{S}}_2 = \frac{\partial}{\partial u_2} - \frac{\partial}{\partial u_4}, \right. \\ \left. \bar{\mathcal{S}}_3 = \frac{\partial}{\partial u_6}, \bar{\mathcal{S}}_4 = \frac{\partial}{\partial u_8}, \bar{\mathcal{S}}_5 = \frac{\partial}{\partial u_{10}}, \bar{\mathcal{S}}_6 = \frac{\partial}{\partial u_{12}} \right\}.$$

Then, we have $\Delta = \text{Span}\{\xi_1, \xi_2\}$ and $\text{ltr}(\text{Ker}\varphi_*) = \text{Span}\left\{ \mathcal{N}_1 = -\frac{1}{2}\left(\frac{\partial}{\partial u_5} + \frac{\partial}{\partial u_{11}}\right), \mathcal{N}_2 = -\frac{1}{2}\left(\frac{\partial}{\partial u_7} + \frac{\partial}{\partial u_9}\right) \right\}$, where $\bar{\mathcal{J}}\xi_1 = \bar{\mathcal{S}}_3 - \bar{\mathcal{S}}_6$, $\bar{\mathcal{J}}\xi_2 = \bar{\mathcal{S}}_4 - \bar{\mathcal{S}}_5$, $\bar{\mathcal{J}}\mathcal{N}_1 = -\frac{\bar{\mathcal{S}}_3 + \bar{\mathcal{S}}_6}{2}$ and $\bar{\mathcal{J}}\mathcal{N}_2 = -\frac{\bar{\mathcal{S}}_4 + \bar{\mathcal{S}}_5}{2}$. Thus $\bar{\mathcal{J}}(\Delta) \subset \mathcal{S}(\text{Ker}\varphi_*)^\perp$ and $\bar{\mathcal{J}}(\text{ltr}(\text{Ker}\varphi_*)) \subset \mathcal{S}(\text{Ker}\varphi_*)^\perp$. Moreover, as $\bar{\mathcal{J}}\mathcal{S}_1 = \mathcal{S}_2$, $\mathcal{S}(\text{Ker}\varphi_*)$ is invariant with respect to $\bar{\mathcal{J}}$. Finally, since the lengths of horizontal vectors are preserved under the derivative map φ_* , therefore φ is a radical screen transversal lightlike submersion.

5. SCREEN TRANSVERSAL ANTI-INVARIANT LIGHTLIKE SUBMERSIONS

In this section, we study screen transversal lightlike submersions from indefinite Kaehler manifolds onto lightlike manifolds.

Definition 5.1. Let $\varphi : (\mathcal{M}, g, \bar{\mathcal{J}}) \rightarrow (N, g')$ be a screen transversal lightlike submersion from an indefinite Kaehler manifold $\mathcal{M}$ onto a lightlike manifold N . Then φ is called a screen transversal anti-invariant lightlike submersion if $\bar{\mathcal{J}}(\mathcal{S}(\text{Ker}\varphi_*) \subset \mathcal{S}(\text{Ker}\varphi_*)^\perp$.

From definition (5.1) the following decomposition is clear:

$$\mathcal{S}(\text{Ker}\varphi_*)^\perp = [\bar{\mathcal{J}}(\Delta) \oplus \bar{\mathcal{J}}(\text{ltr}(\text{Ker}\varphi_*))] \perp \bar{\mathcal{J}}(\mathcal{S}(\text{Ker}\varphi_*)) \perp \mathcal{D}', \quad (5.1)$$

where $\mathcal{D}'$ is a non-null orthogonal complementary distribution to $[\bar{\mathcal{J}}(\Delta) \oplus \bar{\mathcal{J}}(\text{ltr}(\text{Ker}\varphi_*))] \perp \bar{\mathcal{J}}(\mathcal{S}(\text{Ker}\varphi_*))$ in $\mathcal{S}(\text{Ker}\varphi_*)^\perp$.

Proposition 5.2. Let $\varphi : (\mathcal{M}, g, \bar{\mathcal{J}}) \rightarrow (N, g')$ be a screen transversal anti-invariant lightlike submersion. Then $\mathcal{D}'$ is invariant with respect to $\bar{\mathcal{J}}$, that is $\bar{\mathcal{J}}\mathcal{D}' = \mathcal{D}'$.

Proof. Using (2.1), for any $\mathcal{U} \in \Gamma(\mathcal{D}')$, $\xi \in \Gamma(\Delta)$ and $\mathcal{N} \in \Gamma(\text{ltr}(\text{Ker}\varphi_*))$, we have

$$\begin{aligned} g(\bar{\mathcal{J}}\mathcal{U}, \xi) &= -g(\mathcal{U}, \bar{\mathcal{J}}\xi) = 0, & g(\bar{\mathcal{J}}\mathcal{U}, \mathcal{N}) &= -g(\mathcal{U}, \bar{\mathcal{J}}\mathcal{N}) = 0, \\ g(\bar{\mathcal{J}}\mathcal{U}, \bar{\mathcal{J}}\xi) &= g(\mathcal{U}, \xi) = 0, & g(\bar{\mathcal{J}}\mathcal{U}, \bar{\mathcal{J}}\mathcal{N}) &= g(\mathcal{U}, \mathcal{N}) = 0. \end{aligned}$$

It follows that $\bar{\mathcal{J}}(\mathcal{D}') \cap \Delta = \{0\}$, $\bar{\mathcal{J}}(\mathcal{D}') \cap \text{ltr}(\text{Ker}\varphi_*) = \{0\}$, $\bar{\mathcal{J}}(\mathcal{D}') \cap \bar{\mathcal{J}}(\Delta) = \{0\}$ and $\bar{\mathcal{J}}(\mathcal{D}') \cap \bar{\mathcal{J}}(\text{ltr}(\text{Ker}\varphi_*)) = \{0\}$. Also for any $\mathcal{W} \in \Gamma(\mathcal{S}(\text{Ker}\varphi_*))$, we have $\bar{\mathcal{J}}(\mathcal{W}) \in \Gamma(\bar{\mathcal{J}}(\mathcal{S}(\text{Ker}\varphi_*)))$, and since $\bar{\mathcal{J}}(\mathcal{S}(\text{Ker}\varphi_*))$ and $\mathcal{D}'$ are orthogonal, we get $g(\bar{\mathcal{J}}\mathcal{U}, \mathcal{W}) = -g(\mathcal{U}, \bar{\mathcal{J}}\mathcal{W}) = 0$. Thus, we arrived at $\mathcal{S}(\text{Ker}\varphi_*) \cap \bar{\mathcal{J}}(\mathcal{D}') = \{0\}$ and $\bar{\mathcal{J}}(\mathcal{S}(\text{Ker}\varphi_*)) \cap \bar{\mathcal{J}}(\mathcal{D}') = \{0\}$. Therefore, we derive

$$\text{Ker}\varphi_* \cap \bar{\mathcal{J}}(\mathcal{D}') = \{0\}, \quad \text{ltr}(\text{Ker}\varphi_*) \cap \bar{\mathcal{J}}(\mathcal{D}') = \{0\}$$

and

$$\{\bar{\mathcal{J}}(\mathcal{S}(\text{Ker}\varphi_*)) \perp \bar{\mathcal{J}}(\text{ltr}(\text{Ker}\varphi_*)) \cap \bar{\mathcal{J}}(\Delta)\} \cap \bar{\mathcal{J}}(\mathcal{D}') = \{0\}.$$

It shows that $\mathcal{D}'$ is invariant with respect to $\bar{\mathcal{J}}$. $\square$

Theorem 5.3. Let $\varphi : (\mathcal{M}, g, \bar{\mathcal{J}}) \rightarrow (N, g')$ be a screen transversal lightlike submersion from an indefinite complex space from $\mathcal{M}(c)$ onto a lightlike manifold N . Then φ is a screen transversal anti-invariant lightlike submersion if and only if, for any $\mathcal{U}, \mathcal{V} \in \Gamma(\mathcal{S}(\text{Ker}\varphi_*))$, $\xi \in \Gamma(\Delta)$ and $\mathcal{N} \in \Gamma(\text{ltr}(\text{Ker}\varphi_*))$, we have $g(\mathcal{R}(\mathcal{U}, \mathcal{V})\xi, \bar{\mathcal{J}}\mathcal{N}) = 0$.

Proof. Taking into account that $\bar{\mathcal{J}}(\Delta) \in \mathcal{S}(\text{Ker}\varphi_*)^\perp$ and using Lemma 3.1, it is clear that $\bar{\mathcal{J}}(\text{ltr}(\text{Ker}\varphi_*)) \in \mathcal{S}(\text{Ker}\varphi_*)^\perp$. Now, from (2.1), we get $g(\bar{\mathcal{J}}\mathcal{U}, \mathcal{N}) = -g(\mathcal{U}, \bar{\mathcal{J}}\mathcal{N}) = 0$, which gives $\bar{\mathcal{J}}(\mathcal{S}(\text{Ker}\varphi_*)) \cap \Delta = \{0\}$. Moreover, we also have $g(\bar{\mathcal{J}}\mathcal{U}, \bar{\mathcal{J}}\xi) = 0$ and $g(\bar{\mathcal{J}}\mathcal{U}, \bar{\mathcal{J}}\mathcal{N}) = 0$, which leads to $\bar{\mathcal{J}}(\mathcal{S}(\text{Ker}\varphi_*)) \cap \bar{\mathcal{J}}(\Delta) = \{0\}$ and $\bar{\mathcal{J}}(\mathcal{S}(\text{Ker}\varphi_*)) \cap \bar{\mathcal{J}}(\text{ltr}(\text{Ker}\varphi_*)) = \{0\}$. In a similar way, we derive that $\bar{\mathcal{J}}(\mathcal{S}(\text{Ker}\varphi_*)) \cap \text{ltr}(\text{Ker}\varphi_*) = \{0\}$. Moreover, since for any $\xi \in \Gamma(\Delta)$, we have $\bar{\mathcal{J}}\xi \in \Gamma(\mathcal{S}(\text{Ker}\varphi_*)^\perp)$, using (2.3), we arrived at $g(\mathcal{R}(\mathcal{U}, \mathcal{V})\xi, \bar{\mathcal{J}}\mathcal{N}) =$

$\frac{c}{2}g(\mathcal{U}, \bar{\mathcal{J}}\mathcal{V})g(\xi, \mathcal{N})$. Since, there exists $\xi \in \Gamma(\Delta)$ and $\mathcal{N} \in \Gamma(\text{ltr}(\text{Ker}\varphi_*))$, such that $g(\xi, \mathcal{N}) \neq 0$, it follows that $g(\mathcal{R}(\mathcal{U}, \mathcal{V})\xi, \bar{\mathcal{J}}\mathcal{N}) = 0$ if and only if $\bar{\mathcal{J}}(\mathcal{S}(\text{Ker}\varphi_*)) \perp \mathcal{S}(\text{Ker}\varphi_*)$. Finally since $\bar{\mathcal{J}}(\mathcal{S}(\text{Ker}\varphi_*)) \cap \text{ltr}(\text{Ker}\varphi_*) = \{0\}$, we have $g(\mathcal{R}(\mathcal{U}, \mathcal{V})\xi, \bar{\mathcal{J}}\mathcal{N}) = 0$ if and only if $\bar{\mathcal{J}}(\mathcal{S}(\text{Ker}\varphi_*)) \subset \mathcal{S}(\text{Ker}\varphi_*)^\perp$. It completes the proof. $\square$

Consider the projection morphism of $\mathcal{S}(\text{Ker}\varphi_*)^\perp$ on $\bar{\mathcal{J}}(\Delta)$, $\bar{\mathcal{J}}(\mathcal{S}(\text{Ker}\varphi_*))$, $\bar{\mathcal{J}}(\text{ltr}(\text{Ker}\varphi_*))$ and $\mathcal{D}'$ by ψ_1 , ψ_2 , ψ_3 and ψ_4 respectively. So, for any $\mathcal{W} \in \bar{\mathcal{J}}(\mathcal{S}(\text{Ker}\varphi_*))^\perp$, we have

$$\mathcal{W} = \psi_1\mathcal{W} + \psi_2\mathcal{W} + \psi_3\mathcal{W} + \psi_4\mathcal{W}. \quad (5.2)$$

Also, for any $\mathcal{W} \in \bar{\mathcal{J}}(\mathcal{S}(\text{Ker}\varphi_*))^\perp$, we put

$$\bar{\mathcal{J}}\mathcal{W} = \alpha\mathcal{W} + \beta\mathcal{W}, \quad (5.3)$$

where $\alpha\mathcal{W}$ and $\beta\mathcal{W}$ are the tangential and transversal parts of $\bar{\mathcal{J}}\mathcal{W}$. Applying $\bar{\mathcal{J}}$ to (5.2), we get

$$\bar{\mathcal{J}}\mathcal{W} = \bar{\mathcal{J}}\psi_1\mathcal{W} + \bar{\mathcal{J}}\psi_2\mathcal{W} + \bar{\mathcal{J}}\psi_3\mathcal{W} + \bar{\mathcal{J}}\psi_4\mathcal{W}, \quad (5.4)$$

which implies

$$\alpha\mathcal{W} = \bar{\mathcal{J}}\psi_1\mathcal{W} + \bar{\mathcal{J}}\psi_2\mathcal{W}, \quad \beta\mathcal{W} = \bar{\mathcal{J}}\psi_3\mathcal{W} + \bar{\mathcal{J}}\psi_4\mathcal{W}, \quad (5.5)$$

Now if $\bar{\mathcal{J}}\psi_1 = \alpha_1$, $\bar{\mathcal{J}}\psi_2 = \alpha_2$, $\bar{\mathcal{J}}\psi_3 = \beta_1$ and $\bar{\mathcal{J}}\psi_4 = \beta_2$, then (5.3) becomes

$$\bar{\mathcal{J}}\mathcal{W} = \alpha_1\mathcal{W} + \alpha_2\mathcal{W} + \beta_1\mathcal{W} + \beta_2\mathcal{W}, \quad (5.6)$$

where $\alpha_1\mathcal{W} \in \Gamma(\Delta)$, $\alpha_2\mathcal{W} \in \Gamma(\mathcal{S}(\text{Ker}\varphi_*))$, $\beta_1\mathcal{W} \in \Gamma(\text{ltr}(\text{Ker}\varphi_*))$ and $\beta_2\mathcal{W} \in \Gamma(\mathcal{D}')$.

Theorem 5.4. *Let $\varphi : (\mathcal{M}, g, \bar{\mathcal{J}}) \rightarrow (N, g')$ be a screen transversal anti-invariant lightlike submersion from an indefinite Kaehler manifold $\mathcal{M}$ onto a lightlike manifold N . Then the induced connection $\hat{\nabla}$ on $\text{Ker}\varphi_*$ is a Levi-Civita connection if and only if $\nabla_{\mathcal{U}}^{\perp s} \bar{\mathcal{J}}\xi$ has no component in $\bar{\mathcal{J}}(\mathcal{S}(\text{Ker}\varphi_*))$, for any $\mathcal{V} \in \Gamma(\text{Ker}\varphi_*)$ and $\xi \in \Gamma(\Delta)$.*

Proof. The connection $\hat{\nabla}$ is a Levi-Civita connection on $\text{Ker}\varphi_*$ if and only if Δ is parallel concerning to $\hat{\nabla}$ [3]. Using (2.2), we get

$$\nabla_{\mathcal{U}}\xi = -\bar{\mathcal{J}}\nabla_{\mathcal{U}}\bar{\mathcal{J}}\xi. \quad (5.7)$$

Now, from (2.10) and (2.12), equation (5.7) becomes

$$\begin{aligned} & \hat{\nabla}_{\mathcal{U}}\xi + \mathcal{T}_{\mathcal{U}}^l\xi + \mathcal{T}_{\mathcal{U}}^s\xi \\ &= -\bar{\mathcal{J}}\mathcal{T}_{\mathcal{U}}\bar{\mathcal{J}}\xi - \bar{\mathcal{J}}\mathcal{D}^{\perp l}(\mathcal{U}, \bar{\mathcal{J}}\xi) - \alpha_1\nabla_{\mathcal{U}}^{\perp s}\bar{\mathcal{J}}\xi - \alpha_2\nabla_{\mathcal{U}}^{\perp s}\bar{\mathcal{J}}\xi - \beta_1\nabla_{\mathcal{U}}^{\perp s}\bar{\mathcal{J}}\xi - \beta_2\nabla_{\mathcal{U}}^{\perp s}\bar{\mathcal{J}}\xi. \end{aligned}$$

Taking the tangential components of above equation, we have

$$\hat{\nabla}_{\mathcal{U}}\xi = -\alpha_1\nabla_{\mathcal{U}}^{\perp s}\bar{\mathcal{J}}\xi - \alpha_2\nabla_{\mathcal{U}}^{\perp s}\bar{\mathcal{J}}\xi,$$

which completes the proof. $\square$

Example 5.5. Let $\mathbb{R}_{0,1,5}^6$ and $\mathbb{R}_{1,0,3}^4$ are equipped with the semi-Riemannian metric $g = -(du_1)^2 + (du_2)^2 + (du_3)^2 + (du_4)^2 + (du_5)^2 + (du_6)^2$ and null metric $g' = (dv_2)^2 + (dv_3)^2 + (dv_4)^2$, where $u_1, \dots, u_6$ and $v_1, \dots, v_4$ are the canonical coordinates on $\mathbb{R}^6$ and $\mathbb{R}^4$, respectively. Let us define the map $\varphi : (\mathbb{R}_{0,1,5}^6, g) \rightarrow (\mathbb{R}_{1,0,3}^4, g')$ as $(u_1, \dots, u_6) \mapsto (u_1 - u_3, u_2, u_4, u_5)$. Then, we get $\text{Ker}\varphi_* = \text{Span}\left\{\xi = \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_3}, \mathcal{S} = \frac{\partial}{\partial u_6}\right\}$ and $(\text{Ker}\varphi_*)^\perp = \text{Span}\left\{\xi, \bar{\mathcal{S}}_1 = \frac{\partial}{\partial u_2}, \bar{\mathcal{S}}_2 = \frac{\partial}{\partial u_4}, \bar{\mathcal{S}}_3 = \frac{\partial}{\partial u_5}\right\}$. Thus $\Delta = \text{Span}\{\xi\}$ has the property that $\bar{\mathcal{J}}\Delta = \text{Span}\{\bar{\mathcal{S}}_1 + \bar{\mathcal{S}}_2\} \subset \mathcal{S}(\text{Ker}\varphi_*)^\perp$. Further, we obtain $\text{ltr}(\text{Ker}\varphi_*) = \text{Span}\left\{\mathcal{N} = \frac{1}{2}\left(-\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_3}\right)\right\}$, where $\bar{\mathcal{J}}(\mathcal{N}) = \text{Span}\left\{\frac{-\bar{\mathcal{S}}_1 + \bar{\mathcal{S}}_2}{2}\right\} \subset \mathcal{S}(\text{Ker}\varphi_*)^\perp$. Finally, we have $\mathcal{S}(\text{Ker}\varphi_*) = \text{Span}\{\mathcal{S}\}$, where $\bar{\mathcal{J}}(\mathcal{S}) \subset \mathcal{S}(\text{Ker}\varphi_*)^\perp$, and by easy calculation, we also see that the length of horizontal vectors is preserved by the derivative map φ_* . Thus, φ is a screen transversal anti-invariant lightlike submersion.

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