Gulf Journal of Mathematics ISSN: 2309-4966
Vol 17, Issue 1 (2024) 261-274 https://doi.org/10.56947/gjom.v17i1.2098

EXISTENCE OF WEAK SOLUTIONS FOR A CLASS OF $(p(u), q(u))$-LAPLACIAN PROBLEMS

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Abstract. This paper investigates the existence of weak solutions for (p, q)-Laplacian problems where p and q depend on the unknown solution. We focus on the case where p and q are local quantities. By means of a singular perturbation technique, we prove the existence of weak solutions for certain Dirichlet problem.

1. Introduction

The study of partial differential equations involving the (p, q)-Laplacian has broad applications in physics, biophysics, plasma physics, and the study of chemical reactions. For example, these equations occur in general reaction-diffusion problems:

$$u_t = -\operatorname{div}\left[\left(a_p|\nabla u|^{p-2} + b_q|\nabla u|^{q-2}\right)\nabla u\right] + f(x, u),$$

where $a_p, b_q \in \mathbb{R}^+$ are some positive constants, the function u is a general measure of concentration, the term $\operatorname{div}\left[\left(a_p|\nabla u|^{p-2} + b_q|\nabla u|^{q-2}\right)\nabla u\right]$ represents diffusion with the coefficient $D(u) = a_p|\nabla u|^{p-2} + b_q|\nabla u|^{q-2}$, and $f(x, u)$ is the reaction term related to the source and loss processes. The reaction term $f(x, u)$, in general, has a polynomial form.
Many authors have studied the existence and uniqueness of their various types of solutions [2, 5, 15, 20, 21, 24, 23, 25, 28] due to the importance of this type of problems.

Extending the above results to situations where p and q depend on both the spatial variable x and the unknown solution u is the main goal of this work. We focus on cases where the dependence of p and q on u is a local quantity. Specifically, we examine the following problem:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(u)-2}\nabla u) - \operatorname{div}(|\nabla u|^{q(u)-2}\nabla u) + |u|^{p(u)-2}u + |u|^{q(u)-2}u = f \text{ in } \Omega \\ u = 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

Date: Received: July 15, 2023; Accepted: Jul 9, 2024.
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2010 *Mathematics Subject Classification.* Primary 35J60; Secondary 35J05, 35D30.
Key words and phrases. $(p(u), q(u))$-Laplacian, elliptic problems, variable nonlinearity, generalised Sobolev spaces.

where Ω be a bounded domain of $\mathbb{R}^N$ with $N \geq 2$, $\partial\Omega$ Lipschitz-continuous, f is a given data and p, q are the nonlinear exponent functions $p, q : \mathbb{R} \to [1, +\infty)$ such that

$$p, q \text{ are continuous and } \quad 1 < \alpha \leq q \leq p \leq \beta < \infty \quad \text{for some constants } \alpha, \beta. \tag{1.2}$$

The situation in which the variable exponent p is dependent on the unknown solution u is a non-standard one. Such problems appear in the applications of some numerical techniques for total variation image restoration, which are used in some restoration problems of mathematical image processing and computer vision [6, 7, 26]. Türola, J. in [26] have illustrated some numerical examples which suggest that the consideration of the exponents $p = p(u)$ conserves the edges and decreases the noise of the restored images u. In [6], we present a numerical example suggesting that when the exponent of the regularisation term is $p = p(|\nabla u|)$, the noise in the restored images u is reduced. To the best of our knowledge, there are few important contributions on the well-posedness of the solutions of these $p(u)$-Laplacian problems. Andreianov et al. [3] have recently developed the study of these problems. They provided partial existence and uniqueness results for weak solutions in homogeneous Dirichlet boundary condition cases for the following system

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(u)-2}\nabla u) + u = f & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

S. Ouaro and N. Sawadogo in [17] and [18] considered the following nonlinear Fourier boundary value problem

$$\begin{cases} b(u) - \operatorname{div} a(x, u, \nabla u) = f & \text{in } \Omega \\ a(x, u, \nabla u) \cdot \eta + \lambda u = g & \text{on } \partial\Omega. \end{cases}$$

The existence and uniqueness results of entropy and weak solutions are established by an approximation method and convergent sequences in terms of Young measure.

Our inspiration came from the work of C. Allalou, K. Hilal and S. Ait Temghart in [1], the paper of S. A. Temghart, C. Allalou, and A. Abbassi in [22] and the work of M. Chipot and H. B. de Oliveira in [8], where the authors proved the existence of a $p(u)$ problem, the existence of [1] and [8] being based on the Schauder fixed point theorem. For the first time in the literature, a prototype of the nonlocal dependence between p and u was considered. The structure of this paper is as follows. In Section 2 we present the main assumptions and we give some definitions, basic properties of generalised Sobolev spaces. In Section 3, we implemented an approximation strategy based on some regularized problem which admits a solution based on theory of monotone operators and Schauder fixed-point theorem. Next, we prove that this sequence of solutions converges to a function u that solves our problem (1.1).

2. Preliminaries

We can search for the weak solutions in a Sobolev space with variable exponents because the functions p, q depend on the solution u and therefore on the space variable x.
Let Ω be a bounded domain of $\mathbb{R}^N$ with $N \geq 2$ and $\partial\Omega$ Lipschitz-continuous, we say that a real-valued continuous function $h(.)$ is log-Hölder continuous in Ω if

$$\exists C > 0 : \ |h(x) - h(y)| \leq \frac{C}{\ln\left(\frac{1}{|x-y|}\right)} \quad \forall x, y \in \Omega, \quad |x - y| < \frac{1}{2}. \tag{2.1}$$

For any Lebesgue-measurable function $h : \Omega \to [1, \ \infty)$, we define

$$h_- := \operatorname*{ess\,inf}_{x\in\Omega} h(x) \ , \ h_+ := \operatorname*{ess\,sup}_{x\in\Omega} h(x), \tag{2.2}$$

and we introduce the variable exponent Lebesgue space by:

$$L^{h(\cdot)}(\Omega) = \{ \ u : \Omega \to \mathbb{R} \ / \ \rho_{h(\cdot)}(u) \ := \int_\Omega |u(x)|^{h(x)} dx < \infty\}. \tag{2.3}$$

Equipped with the following norm

$$\|u\|_{h(\cdot)} \ := \inf\left\{\lambda > 0 \ : \ \rho_{h(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1\right\}, \tag{2.4}$$

$L^{h(\cdot)}(\Omega)$ will be a Banach space. If

$$1 < h_- \leq h_+ < \infty, \tag{2.5}$$

$L^{h(\cdot)}(\Omega)$ is separable and reflexive. The dual space of $L^{h(\cdot)}(\Omega)$ is $L^{h'(\cdot)}(\Omega)$, where $h'(x)$ is the generalised Hölder conjugate of $h(x)$,

$$\frac{1}{h(x)} + \frac{1}{h'(x)} = 1.$$

The next proposition states that there is a gap between the modular and the norm in $L^{q(\cdot)}(\Omega)$.

Proposition 2.1. *(See [12])*
If (2.5) holds, for $u \in L^{h(x)}(\Omega)$, then the following assertions hold

$$\min\left\{\|u\|_{h(\cdot)}^{h_-}, \ \|u\|_{h(\cdot)}^{h_+}\right\} \leq \rho_{h(\cdot)}(u) \leq \max\left\{\|u\|_{h(\cdot)}^{h_-}, \ \|u\|_{h(\cdot)}^{h_+}\right\},$$

$$\min\left\{\rho_{h(\cdot)}(u)^{\frac{1}{h_-}} \ , \ \rho_{h(\cdot)}(u)^{\frac{1}{h_+}}\right\} \leq \|u\|_{h(\cdot)} \leq \max\left\{\rho_{h(\cdot)}(u)^{\frac{1}{h_-}} \ , \ \rho_{h(\cdot)}(u)^{\frac{1}{h_+}}\right\}, \tag{2.6}$$

$$\|u\|_{h(\cdot)}^{h_-} - 1 \leq \rho_{h(\cdot)}(u) \leq \|u\|_{h(\cdot)}^{h_+} + 1. \tag{2.7}$$

Proposition 2.2. *(Generalised Hölder's inequality) (See [14])*
- For any functions $u \in L^{h(\cdot)}(\Omega)$ and $v \in L^{h'(\cdot)}(\Omega)$, we have:

$$\int_\Omega uv dx \leq (\frac{1}{h_-} + \frac{1}{h'_-})\|u\|_{h(\cdot)}\|v\|_{h'(\cdot)} \leq 2\|u\|_{h(\cdot)}\|v\|_{h'(\cdot)}.$$

- For all h satisfying to (2.5), we have the following continuous imbedding,

$$L^{h(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega) \ \textit{whenever} \ h(x) \geq r(x) \ \textit{for a.e.} \ x \in \Omega. \tag{2.8}$$

In generalised Lebesgue spaces, there holds a version of Young's inequality,

$$|uv| \leq \delta \frac{|u|^{h(x)}}{h(x)} + C(\delta) \frac{|v|^{h'(x)}}{h'(x)},$$

for some positive constant $C(\delta)$ and any $\delta > 0$.
We define also the generalized Sobolev space by

$$W^{1,h(\cdot)}(\Omega) := \{u \in L^{h(\cdot)}(\Omega) : \nabla u \in L^{h(\cdot)}(\Omega)\},$$

which is a Banach space with the norm

$$\|u\|_{1,h(\cdot)} := \|u\|_{h(\cdot)} + \|\nabla u\|_{h(\cdot)}. \tag{2.9}$$

The space $W^{1,h(\cdot)}(\Omega)$ is separable and is reflexive when (2.5) is satisfied. We also have

$$W^{1,h(\cdot)}(\Omega) \hookrightarrow W^{1,r(\cdot)}(\Omega) \text{ whenever } h(x) \geq r(x) \text{ for a.e. } x \in \Omega. \tag{2.10}$$

Now, we introduce the following function space

$$W_0^{1,h(\cdot)}(\Omega) := \{u \in \mathrm{W}_0^{1,1}(\Omega) : \nabla u \in L^{h(\cdot)}(\Omega)\},$$

endowed with the following norm

$$\|u\|_{W_0^{1,h(\cdot)}(\Omega)} := \|u\|_1 + \|\nabla u\|_{h(\cdot)}. \tag{2.11}$$

If $h \in C(\overline{\Omega})$, then the norm in $W_0^{1,h(\cdot)}(\Omega)$ is equivalent to $\|\nabla u\|_{h(\cdot)}$. When h is log-Hölder continuous, then $C_0^\infty(\Omega)$ is dense in $W_0^{1,h(\cdot)}(\Omega)$.
If h is a measurable function in Ω satisfying $1 \leq h_- \leq h_+ < N$ and the Log-Hölder continuity property (2.1), then

$$\|u\|_{h^*(\cdot)} \leq C\|\nabla u\|_{h(\cdot)} \quad \forall u \in W_0^{1,h(\cdot)}(\Omega),$$

for some positive constant C, where

$$h^*(x) := \begin{cases} \frac{Nh(x)}{N-h(x)} & \text{if } h(x) < N \\ \infty & \text{if } h(x) \geq N. \end{cases}$$

On the other hand, if h satisfies (2.1) and $h_- > N$, then

$$\|u\|_\infty \leq C\|\nabla u\|_{h(\cdot)} \quad \forall u \in W_0^{1,h(\cdot)}(\Omega) \ ,$$

where C is another positive constant.

In the reference [8], the authors used the following result which is a particular case of a more general one established by Zhikov [27] about the appropriate way to take the limit in a sequence of nonlinear elliptic equations. The proof of this lemma in [8] does not require all the hypotheses considered by Zhikov in [27] (see Lemmas 2.4 and 3.3).

Lemma 2.3. [8] *Assume that*

$$1 < \alpha \leq h_n(x) \leq \beta < \infty \quad \forall n \in \mathbb{N},$$

$$\textit{for a.e. } x \in \Omega, \textit{ for some constants } \alpha \textit{ and } \beta, \tag{2.12}$$

$$h_n \to h \quad \textit{a.e. in } \Omega, \textit{ as } n \to \infty, \tag{2.13}$$

$$\nabla u_n \rightharpoonup \nabla u \text{ in } L^1(\Omega)^N, \text{ as } n \to \infty, \tag{2.14}$$

$$\| |\nabla u_n|^{h_n(x)} \|_{L^1(\Omega)} \le C, \text{ for some positive constant } C \text{ not depending on } n. \tag{2.15}$$

Then $\nabla u \in \left(L^{h(\cdot)}(\Omega)\right)^N$ *and*

$$\liminf_{n \to \infty} \int_\Omega |\nabla u_n|^{h_n(x)} dx \ge \int_\Omega |\nabla u|^{h(x)} dx. \tag{2.16}$$

In this paper, we treat a $(p(u), q(u))$-problem, so we need the following similar argument to establish that $|\nabla u|^{p(u)}$ and $|\nabla u|^{q(u)}$ belong to $L^1(\Omega)$.

Lemma 2.4. [8] *Let* $p_n, q_n : \mathbb{R} \to [1, +\infty)$ *be two Lebesgue-measurable functions and* $\alpha, \beta \in (1, +\infty)$. *Assume that*

$1 < \alpha \le q_n(x), p_n(x) \le \beta < \infty \quad \forall n \in \mathbb{N},$

$q_n \to q, \ p_n \to p$ *a.e. in* Ω, *as* $n \to \infty$,

$\nabla u_n \rightharpoonup \nabla u$ *in* $L^1(\Omega)^N$, *as* $n \to \infty$,

$\| |\nabla u_n|^{q_n(x)} \|_{L^1(\Omega)}, |\nabla u_n|^{p_n(x)} \|_{L^1(\Omega)} \le C$, *for some positive constant* C *not depending on* n.

Then we deduce that $|\nabla u_n|^{q(x)}, |\nabla u_n|^{p(x)} \in L^1(\Omega)$ *and*

$$\liminf_{n \to \infty} \int_\Omega \left(|\nabla u_n|^{q_n(x)} + |\nabla u_n|^{p_n(x)}\right) dx \ge \int_\Omega \left(|\nabla u|^{q(x)} + |\nabla u|^{p(x)}\right) dx.$$

Proof. Since $1 < \alpha \le q_n(x), p_n(x) \le \beta < +\infty$, we can apply Lemma 2.3 separately for (p_n, p) and (q_n, q) to get

$$|\nabla u_n|^{p(x)} \in L^1(\Omega), \ \liminf_{n \to \infty} \int_\Omega |\nabla u_n|^{p_n(x)} dx \ge \int_\Omega |\nabla u|^{p(x)} dx,$$

$$|\nabla u_n|^{q(x)} \in L^1(\Omega), \ \liminf_{n \to \infty} \int_\Omega |\nabla u_n|^{q_n(x)} dx \ge \int_\Omega |\nabla u|^{q(x)} dx.$$

By summing the inequalities above, we deduce the result. □

3. Existence results

In this section, we prove the existence of weak solutions for the local problem (1.1). Firstly, we define the following space: for all $u \in \mathbb{R}$ such that $1 < p(u) < \infty$

$$W_0^{1,p(u)}(\Omega) := \{u \in W_0^{1,1}(\Omega) : \int_\Omega |\nabla u|^{p(u)} dx < \infty\}.$$

It is a Banach space for the norm $\|u\|_{W_0^{1,p(\cdot)}(\Omega)}$ defined at (2.11) which is equivalent to $\|\nabla u\|_{p(u)}$ when $p(u) \in C(\overline{\Omega})$. Since p is continuous then from the fact that $1 < \alpha \le p$, $W_0^{1,p(u)}(\Omega)$ is separable and reflexive.

Next, for each $\varepsilon > 0$, we consider the following auxiliary problem (namely, the regularized problem)

$$(\mathcal{P}_\varepsilon)\begin{cases} -div(|\nabla u|^{p(u)-2}\nabla u + |\nabla u|^{q(u)-2}\nabla u) + |u|^{q(u)-2}u + |u|^{p(u)-2}u \\ \qquad\qquad +\varepsilon\left(|u|^{\beta-2}u - div(|\nabla u|^{\beta-2}\nabla u)\right) = f \quad \text{in } \Omega, \\ u = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases}$$

where the exponent functions p, q are continuous and satisfy

$$N < \alpha \leq p(u), q(u) \leq \beta < +\infty \quad \forall u \in \mathbb{R}. \tag{3.1}$$

Proposition 3.1. *Assume that $f \in W_0^{-1,\alpha}(\Omega)$ and p, q satisfy (3.1). Then, for each $\varepsilon > 0$, the problem $(\mathcal{P}_\varepsilon)$ admits a weak solution u_ε, i.e.*

$$\int_\Omega |\nabla u_\varepsilon|^{p(u_\varepsilon)-2}\nabla u_\varepsilon\cdot\nabla v dx + \int_\Omega |\nabla u_\varepsilon|^{q(u_\varepsilon)-2}\nabla u_\varepsilon\cdot\nabla v dx + \int_\Omega |u_\varepsilon|^{p(u_\varepsilon)-2}u_\varepsilon v dx$$
$$+\int_\Omega |u_\varepsilon|^{q(u_\varepsilon)-2}u_\varepsilon v dx + \varepsilon\left(\int_\Omega |\nabla u_\varepsilon|^{\beta-2}\nabla u_\varepsilon\cdot\nabla v dx + \int_\Omega |u_\varepsilon|^{\beta-2}u_\varepsilon v dx\right) = \langle f, v\rangle, \tag{3.2}$$

$\forall v \in W_0^{1,\beta}(\Omega)$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $(W_0^{1,\alpha}(\Omega))'$ and $W_0^{1,\alpha}(\Omega)$.

Proof. Let $z \in L^2(\Omega)$, then

$$N < \alpha \leq p(z(x)), q(z(x)) \leq \beta < \infty \quad \text{for a.e.} \quad x \in \Omega. \tag{3.3}$$

We recall that $f \in W^{-1,\alpha'}(\Omega) \subset W^{-1,\beta'}(\Omega)$. Now, we focus on the operator $T_\varepsilon : W_0^{1,\beta}(\Omega) \to W^{-1,\beta'}(\Omega)$ defined by

$$\langle T_\varepsilon(u), v\rangle = \int_\Omega \left(|\nabla u|^{p(z)-2}\nabla u\cdot\nabla v + |\nabla u|^{q(z)-2}\nabla u\cdot\nabla v\right)dx$$
$$+\int_\Omega\left(|u|^{p(z)-2}uv + |u|^{q(z)-2}uv\right)dx + \varepsilon\left[\int_\Omega\left(|\nabla u|^{\beta-2}\nabla u\cdot\nabla v dx + |u|^{\beta-2}uv\right)dx\right],$$

for all $u, v \in W_0^{1,\beta}(\Omega)$. We can establish that:

(1) T_ε is continuous, bounded and strictly monotone;
(2) T_ε is coercive.

From (1) and (2), the operator T_ε is continuous, strictly monotone (hence, maximal monotone too), and coercive. It follows that T_ε is a strictly monotone surjective operator (see [19, Corollary 2.8.7, p. 135]). Hence, there exists a unique solution $u_z \in W_0^{1,\beta}(\Omega)$ such that

$$\begin{aligned}
&\int_\Omega |\nabla u_z|^{p(z)-2}\nabla u_z \cdot \nabla v dx + \int_\Omega |\nabla u_z|^{q(z)-2}\nabla u_z \cdot \nabla v dx + \int_\Omega |u_z|^{p(z)-2} u_z v dx \\
&+ \int_\Omega |u_z|^{q(z)-2} u_z v dx + \varepsilon\left(\int_\Omega |\nabla u_z|^{\beta-2}\nabla u_z \cdot \nabla v dx + \int_\Omega |u_z|^{\beta-2} u_z v dx\right) = \langle f, v\rangle,
\end{aligned} \tag{3.4}$$

$\forall v \in W_0^{1,\beta}(\Omega)$.
We take $v = u_z$ in (3.4) to derive that

$$\begin{aligned}
\int_\Omega \left(|\nabla u_z|^{p(z)} + |\nabla u_z|^{q(z)}\right) dx + \int_\Omega \left(|u_z|^{p(z)} + |u_z|^{q(z)}\right) dx + \varepsilon\left(\int_\Omega |u_z|^\beta dx + \int_\Omega |\nabla u_z|^\beta dx\right)& \\
\leq \|f\|_{-1,\alpha'} \|\nabla u_z\|_\alpha& \\
\leq C\|\nabla u_z\|_\beta,&
\end{aligned}$$

where $C = C(\alpha, \beta, \Omega, f)$, and $\|\cdot\|_{-1,\alpha'}$ is the operator norm associated to the norm $\|\nabla \cdot\|_\alpha$. Therefore

$$\begin{aligned}
\varepsilon\|u_z\|_{1,\beta}^\beta &\leq C\|\nabla u_z\|_\beta \\
&\leq C\|u_z\|_{1,\beta}.
\end{aligned}$$

Hence,

$$\|u_z\|_{1,\beta} \leq C, \tag{3.5}$$

where $C = C(\alpha, \beta, \Omega, \varepsilon, f)$ is a positive constant without z-dependence. From the fact that $\beta > N \geq 2$, we can deduce that

$$\|u_z\|_{L^2(\Omega)} \leq C. \tag{3.6}$$

Now, we consider the self-map $T : B \to B$ defined by $T(z) = u_z$, over the set $B := \left\{v \in L^2(\Omega) : \|v\|_{L^2(\Omega)} \leq C\right\}$. The embedding $W_0^{1,\beta}(\Omega) \hookrightarrow L^2(\Omega)$ means that $T(B)$ is relatively compact in B. Using the Schauder fixed point theorem, we know that to obtain a fixed point from T, we need continuity from T.
Suppose we are working on a sequence $\{z_n\}$ in $L^2(\Omega)$ which satisfies

$$z_n \to z \text{ in } L^2(\Omega) \quad \text{as} \quad n \to \infty, \tag{3.7}$$

we mean by u_n, for all $n \in \mathbb{N}$, the solution of (3.4) related to $z := z_n$. Therefore the inequality in (3.5) gives

$$\|u_n\|_{1,\beta} \leq C, \quad \text{for some constant (without } n\text{-dependence)}\,.$$

For a given $u \in W_0^{1,\beta}(\Omega)$, passing to a subsequence if necessary (namely again $\{u_n\}$), we get

$$u_n \rightharpoonup u \quad \text{in} \quad W_0^{1,\beta}(\Omega), \quad \text{as} \quad n \to \infty, \tag{3.8}$$

$$u_n \to u \quad \text{in} \quad L^2(\Omega), \quad \text{as} \quad n \to \infty. \tag{3.9}$$

We return to (3.4), so that considering (u_n, z_n) instead of (u, z), we get

$$\begin{aligned}&\int_\Omega \left(|\nabla u_n|^{p(z_n)-2}\nabla u_n + |\nabla u_n|^{q(z_n)-2}\nabla u_n + \varepsilon|\nabla u_n|^{\beta-2}\nabla u_n\right)\cdot\nabla v dx\\&+\int_\Omega \left(|u_n|^{p(z_n)-2}u_n + |u_n|^{q(z_n)-2}u_n + \varepsilon|u_n|^{\beta-2}u_n\right) v dx = \langle f, v\rangle \quad \forall v \in W_0^{1,\beta}(\Omega).\end{aligned} \tag{3.10}$$

As the operator on the left side of (3.10) is monotone, therefore

$$\begin{aligned}&\int_\Omega \left(|\nabla u_n|^{p(z_n)-2}\nabla u_n + |\nabla u_n|^{q(z_n)-2}\nabla u_n + \varepsilon|\nabla u_n|^{\beta-2}\nabla u_n\right)\cdot\nabla (u_n - v) dx\\&\qquad+\int_\Omega \left(|u_n|^{p(z_n)-2}u_n + |u_n|^{q(z_n)-2}u_n + \varepsilon|u_n|^{\beta-2}u_n\right)(u_n - v) dx\\&\qquad-\int_\Omega \left(|\nabla v|^{p(z_n)-2}\nabla v + |\nabla v|^{q(z_n)-2}\nabla v + \varepsilon|\nabla v|^{\beta-2}\nabla v\right)\cdot\nabla (u_n - v) dx\\&\qquad-\int_\Omega \left(|v|^{p(z_n)-2}v + |v|^{q(z_n)-2}v + \varepsilon|v|^{\beta-2}v\right)(u_n - v) dx \geq 0 \quad \forall v \in W_0^{1,\beta}(\Omega).\end{aligned} \tag{3.11}$$

Consider (3.10) with $v = u_n - v$ as a test function. We use (3.11) to get

$$\begin{aligned}&\langle f, u_n - v\rangle - \int_\Omega \left(|\nabla v|^{p(z_n)-2}\nabla v + |\nabla v|^{q(z_n)-2}\nabla v + \varepsilon|\nabla v|^{\beta-2}\nabla v\right)\cdot\nabla (u_n - v) dx\\&\qquad-\int_\Omega \left(|v|^{p(z_n)-2}v + |v|^{q(z_n)-2}v + \varepsilon|v|^{\beta-2}v\right)(u_n - v) dx \geq 0 \quad \forall v \in W_0^{1,\beta}(\Omega).\end{aligned} \tag{3.12}$$

The convergence in (3.7) implies

$$z_n \to z \quad \text{a.e. in} \quad \Omega, \quad \text{as} \quad n \to \infty.$$

We can apply Lebesgue's theorem, since p is a continuous function, so we have

$$\lim_{n\to+\infty}\left[|\nabla v|^{p(z_n)-2} + |\nabla v|^{q(z_n)-2}\right]\nabla v = \left[|\nabla v|^{p(z)-2} + |\nabla v|^{q(z)-2}\right]\nabla v, \tag{3.13}$$

and

$$\lim_{n\to+\infty}\left(|v|^{p(z_n)-2} + |v|^{q(z_n)-2}\right)v = \left(|v|^{p(z)-2} + |v|^{q(z)-2}\right)v, \tag{3.14}$$

for all $v \in W_0^{1,\beta}(\Omega)$. By means of (3.13) and (3.14) we can go to the limit in (3.12) to get

$$\begin{aligned}&\langle f, u - v\rangle - \int_\Omega \left(|\nabla v|^{p(z)-2}\nabla v + |\nabla v|^{q(z)-2}\nabla v + \varepsilon|\nabla v|^{\beta-2}\nabla v\right)\cdot\nabla (u - v) dx\\&-\int_\Omega \left(|v|^{p(z)-2}v + |v|^{q(z)-2}v + \varepsilon|v|^{\beta-2}v\right)(u - v) dx \geq 0 \quad \forall v \in W_0^{1,\beta}(\Omega).\end{aligned} \tag{3.15}$$

Taking $v = u \pm \delta y$, where $y \in W_0^{1,\beta}(\Omega)$ and $\delta > 0$, we obtain

$$\pm \Bigg[\langle f, y\rangle - \int_\Omega \Big(|\nabla(u \pm \delta y)|^{p(z)-2}\nabla(u \pm \delta y) + |\nabla(u \pm \delta y)|^{q(z)-2}\nabla(u \pm \delta y) + \varepsilon|\nabla(u \pm \delta y)|^{\beta-2}\nabla(u \pm \delta y)\Big) \cdot \nabla y dx - \int_\Omega \Big(|u \pm \delta y|^{p(z)-2}(u \pm \delta y) + |u \pm \delta y|^{q(z)-2}(u \pm \delta y) + \varepsilon|v|^{\beta-2}(u \pm \delta y)\Big) y dx\Bigg] \geq 0. \quad (3.16)$$

We go to the limit, since δ goes to zero in (3.16), and conclude that

$$\int_\Omega \left(|\nabla(u)|^{p(z)-2}\nabla u + |\nabla(u)|^{q(z)-2}\nabla u + \varepsilon|\nabla u|^{\beta-2}\nabla u\right) \cdot \nabla y dx + \int_\Omega \left(|u|^{p(z)-2}u + |u|^{q(z)-2}u + \varepsilon|v|^{\beta-2}u\right) y dx = \langle f, y\rangle \quad \forall y \in W_0^{1,\beta}(\Omega).$$

Consequently $u = u_z$. In view of (3.9) and by the strong convergence in (3.9), we conclude that

$$u_{z_n} \to u_z \quad \text{strongly in} \quad L^2(\Omega), \quad \text{as} \quad n \to \infty,$$

It follows that T is continuous, and this establishes the existence of the fixed point which is the exact weak solution to $(\mathcal{P}_\varepsilon)$. □

We introduce the following revised definition before we state the main result.

Definition 3.2. Assume that p and q verifies (1.2) and

$$f \in W^{-1,\alpha'}(\Omega). \quad (3.17)$$

A function $u \in W_0^{1,p(u)}(\Omega)$ is said to be a weak solution to the problem (1.1), if

$$\int_\Omega |\nabla u|^{p(u)-2}\nabla u \cdot \nabla v dx + \int_\Omega |\nabla u|^{q(u)-2}\nabla u \cdot \nabla v dx + \int_\Omega |u|^{p(u)-2}uv dx + \int_\Omega |u|^{q(u)-2}uv dx = \langle f, v\rangle \quad \forall v \in W_0^{1,p(u)}(\Omega),$$

where $\langle \cdot, \cdot \rangle$ is the duality brackets for the pair $\left((W_0^{1,p(u)}(\Omega))', W_0^{1,p(u)}(\Omega)\right)$.

Theorem 3.3. *Assume that* (3.17) *hold together with*

$$N < \alpha \leq q(u) \leq p(u) \leq \beta < +\infty \quad (3.18)$$

and

$$p, q : \mathbb{R} \longrightarrow [1, +\infty) \text{ are Lipschitz-continuous functions.} \quad (3.19)$$

Then there exists at least one weak solution to problem (1.1) *in the sense of Definition* 3.2.

Proof. From Proposition 3.1, for each $\varepsilon > 0$ there exists $u_\varepsilon \in W_0^{1,\beta}(\Omega)$ such that

$$\int_\Omega |\nabla u_\varepsilon|^{p(u_\varepsilon)-2}\nabla u_\varepsilon \nabla v dx + \int_\Omega |\nabla u_\varepsilon|^{q(u_\varepsilon)-2}\nabla u_\varepsilon \nabla v dx + \int_\Omega |u_\varepsilon|^{p(u_\varepsilon)-2} u_\varepsilon v dx$$
$$+ \int_\Omega |u_\varepsilon|^{q(u_\varepsilon)-2} u_\varepsilon v dx + \varepsilon \left(\int_\Omega |\nabla u_\varepsilon|^{\beta-2}\nabla u_\varepsilon \cdot \nabla v dx + \int_\Omega |u_\varepsilon|^{\beta-2} u_\varepsilon v dx \right) = \langle f, v\rangle \tag{3.20}$$

for every $\forall v \in W_0^{1,\beta}(\Omega)$, and

$$N < \alpha \leq p(u_\varepsilon(x)) \leq \beta < \infty \quad \forall \varepsilon > 0, \quad \text{for a.e.} \quad x \in \Omega.$$

Next, we choose $v = u_\varepsilon$ as a test function in (3.20) to obtain

$$\int_\Omega \left(|\nabla u_\varepsilon|^{p(u_\varepsilon)} + |u_\varepsilon|^{p(u_\varepsilon)}\right) dx + \varepsilon \left(\|\nabla u_\varepsilon\|_\beta^\beta + \|u_\varepsilon\|_\beta^\beta \right) = \langle f, u_\varepsilon\rangle. \tag{3.21}$$

From (2.7), we deduce that

$$\|u_\varepsilon\|_{p(u_\varepsilon)} \leq (\rho_{p(u_\varepsilon)}(u_\varepsilon) + 1)^{\frac{1}{p^-(u_\varepsilon)}} = \left(\int_\Omega |u_\varepsilon|^{p(u_\varepsilon)} dx + 1 \right)^{\frac{1}{p^-(u_\varepsilon)}}.$$

By using the Hölder inequality, we get

$$\begin{aligned}\int_\Omega |\nabla u_\varepsilon|^\alpha dx \leq C \| |\nabla u_\varepsilon|^\alpha \|_{\frac{p(u_\varepsilon)}{\alpha}} &\leq C \left(\int_\Omega |\nabla u_\varepsilon|^{p(u_\varepsilon)} dx + 1 \right)^{\frac{1}{\left(\frac{p(u_\varepsilon)}{\alpha}\right)^-}} \\ &\leq C \left(\int_\Omega |\nabla u_\varepsilon|^{p(u_\varepsilon)} dx + 1 \right),\end{aligned} \tag{3.22}$$

where $C = C(\alpha, \beta, \Omega)$. Therefore

$$\langle f, u_\varepsilon\rangle \leq \|f\|_{-1,\alpha'} ||\nabla u_\varepsilon||_\alpha \leq C \|f\|_{-1,\alpha'} \Big(\int_\Omega |\nabla u_\varepsilon|^{p(u_\varepsilon)} dx + 1 \Big)^{\frac{1}{\alpha}}. \tag{3.23}$$

From (3.21), (3.23) and by using Young's inequality, we obtain

$$\int_\Omega \left(|\nabla u_\varepsilon|^{p(u_\varepsilon)} + |u_\varepsilon|^{p(u_\varepsilon)}\right) dx + \int_\Omega \left(|\nabla u_\varepsilon|^{q(u_\varepsilon)} + |u_\varepsilon|^{q(u_\varepsilon)}\right) dx + \varepsilon \left(\|\nabla u_\varepsilon||_\beta^\beta + \|u_\varepsilon||_\beta^\beta \right) \leq C. \tag{3.24}$$

Using (3.22) and (3.23), we can deduce the estimate

$$||u_\varepsilon||_{1,\alpha} \leq C, \tag{3.25}$$

where C is a positive constant without dependence of ε.
Now let us consider a sequence $\{\varepsilon_n\}$ of positive real numbers. For each$n \in \mathbb{N}$, let u_{ε_n} be the solution to the problem $(\mathcal{P}_\varepsilon)$ associated with ε_n. Since $W_0^{1,\alpha}(\Omega) \hookrightarrow L^2(\Omega)$ compactly, so by passing to a subsequence, for some $u \in W_0^{1,\alpha}(\Omega)$ we will

have

$$u_{\varepsilon_n} \rightharpoonup u \quad \text{in} \quad W_0^{1,\alpha}(\Omega), \quad \text{as} \quad n \to \infty \tag{3.26}$$

$$\nabla u_{\varepsilon_n} \rightharpoonup \nabla u \quad \text{in} \quad L^{\alpha}(\Omega)^N, \quad \text{as} \quad n \to \infty \tag{3.27}$$

$$u_{\varepsilon_n} \to u \quad \text{in} \quad L^2(\Omega), \quad \text{as} \quad n \to \infty$$

$$u_{\varepsilon_n} \to u \quad \text{a.e. in} \quad \Omega, \quad \text{as} \quad n \to \infty. \tag{3.28}$$

The condtion on the exponent range in (3.18) mean that u is Hölder-continuous, then from the condition (3.19), the same holds for $p(u)$ and $q(u)$.
From (3.28), we conclude that

$$\lim_{n\to\infty} p(u_{\varepsilon_n}) = p(u), \quad \lim_{n\to\infty} q(u_{\varepsilon_n}) = q(u) \quad \text{a.e. in } \Omega. \tag{3.29}$$

The following chain of inequalities is satisfied

$$N < \alpha \leq q(u_{\varepsilon_n}) \leq p(u_{\varepsilon_n}) \leq \beta < \infty \quad \forall n \in \mathbb{N}, \quad for \quad a.e. \quad x \in \Omega. \tag{3.30}$$

Using (3.24) written for u_{ε_n}, together with (3.27), (3.29) and (3.30), we conclude that (by Lemma 2.3)

$$u \in W_0^{1,p(u)}(\Omega), \tag{3.31}$$

and therefore

$$u \in W_0^{1,q(u)}(\Omega). \tag{3.32}$$

From the theory of monotone operators, we have

$$\begin{aligned}
&\int_\Omega \left(|\nabla u_{\varepsilon_n}|^{p(u_{\varepsilon_n})-2}\nabla u_{\varepsilon_n} + |\nabla u_{\varepsilon_n}|^{q(u_{\varepsilon_n})-2}\nabla u_{\varepsilon_n} + \varepsilon_n |\nabla u_{\varepsilon_n}|^{\beta-2}\nabla u_{\varepsilon_n}\right) \nabla(u_{\varepsilon_n} - v)dx \\
&\quad + \int_\Omega \left(|u_{\varepsilon_n}|^{p(u_{\varepsilon_n})-2}u_{\varepsilon_n} + |u_{\varepsilon_n}|^{q(u_{\varepsilon_n})-2}u_{\varepsilon_n} + \varepsilon_n |u_{\varepsilon_n}|^{\beta-2}u\right)(u_{\varepsilon_n} - v)dx \\
&\quad - \Bigg(\int_\Omega \left(|\nabla v|^{p(u_{\varepsilon_n})-2}\nabla v + |\nabla v|^{q(u_{\varepsilon_n})-2}\nabla v + \varepsilon_n |\nabla v|^{\beta-2}\nabla v\right) \nabla(u_{\varepsilon_n} - v)dx \\
&+ \int_\Omega \left(|v|^{p(u_{\varepsilon_n})-2}v + |v|^{q(u_{\varepsilon_n})-2}v + \varepsilon_n |v|^{\beta-2}v\right)(u_{\varepsilon_n} - v)dx\Bigg) \geq 0 \quad \forall v \in W_0^{1,\beta}(\Omega).
\end{aligned} \tag{3.33}$$

By replacing u_ε with u_{ε_n} and choosing $u_{\varepsilon_n} - v$ as a test function in (3.20), we can reduce (3.33) to the form

$$\begin{aligned}
\langle f, u_{\varepsilon_n} - v\rangle - \Bigg(&\int_\Omega \left(|\nabla v|^{p(u_{\varepsilon_n})-2}\nabla v + |\nabla v|^{q(u_{\varepsilon_n})-2}\nabla v + \varepsilon |\nabla v|^{\beta-2}\nabla v\right) \cdot \nabla(u_{\varepsilon_n} - v)dx \\
&+ \int_\Omega \left(|v|^{p(u_{\varepsilon_n})-2}v + |v|^{q(u_{\varepsilon_n})-2}v + \varepsilon |v|^{\beta-2}v\right)(u_{\varepsilon_n} - v)dx\Bigg) \geq 0,
\end{aligned} \tag{3.34}$$

for all $v \in C_0^\infty(\Omega)$. By using (3.29) and the Lebesgue theorem, we have

$$\lim_{n\to+\infty}\left[|\nabla v|^{p(u_{\varepsilon_n})-2}+|\nabla v|^{q(u_{\varepsilon_n})-2}\right]\nabla v=\left[|\nabla v|^{p(u)-2}+|\nabla v|^{q(u)-2}\right]\nabla v, \quad (3.35)$$

and

$$\lim_{n\to+\infty}\left(|v|^{p(u_{\varepsilon_n})-2}+|v|^{q(u_{\varepsilon_n})-2}\right)v=\left(|v|^{p(u)-2}+|v|^{q(u)-2}\right)v. \quad (3.36)$$

We take the limit as n goes to infinity in (3.34), and use (3.25), (3.26), (3.35) and (3.36), therefore

$$\langle f,u-v\rangle-\left[\int_\Omega\left(|\nabla v|^{p(u)-2}+|\nabla v|^{q(u)-2}\right)\nabla v\cdot\nabla(u-v)dx\right.$$
$$\left.+\int_\Omega\left(|v|^{p(u)-2}+|v|^{q(u)-2}\right)v(u-v)dx\right]\geq 0, \quad (3.37)$$

for all $v \in C_0^\infty(\Omega)$. From the assumptions (3.18) and (3.19), the functions $p(u)$ and $q(u)$ is Hölder-continuous which implies that $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(u)}(\Omega)$. Thus, (3.37) holds true also for all $v \in W_0^{1,p(u)}(\Omega)$.
So we can take $v = u \pm \delta y$, where $y \in W_0^{1,p(u)}(\Omega)$ and $\delta > 0$, as a test function in (3.37) we get

$$\pm\left[\langle f,y\rangle-\left(\int_\Omega\left(|\nabla u|^{p(u)-2}+|\nabla u|^{q(u)-2}\right)\nabla u\cdot\nabla y dx\right.\right.$$
$$\left.\left.+\int_\Omega\left(|u|^{p(u)-2}+|u|^{q(u)-2}\right)uydx\right)\right]\geq 0. \quad (3.38)$$

This implies that,

$$\int_\Omega\left(|\nabla u|^{p(u)-2}+|\nabla u|^{q(u)-2}\right)\nabla u\cdot\nabla ydx+\int_\Omega\left(|u|^{p(u)-2}+|u|^{q(u)-2}\right)uydx=\langle f,y\rangle, \quad (3.39)$$

$\forall y \in W_0^{1,p(u)}(\Omega)$.In the end, we came up with a solution to our local problem (1.1) (See definition 3.2). □

Remark 3.4. The main result above remains true if we relax the assumption $q(u) \leq p(u)$ for all $u \in \mathbb{R}$. We replace (3.18) with the condition $N < \alpha \leq q(u), p(u) \leq \beta < +\infty$ for all $u \in \mathbb{R}$. Then problem (1.1) admits at least one weak solution $u \in W_0^{1,p(u)}(\Omega) \cap W_0^{1,q(u)}(\Omega)$.

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