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A UNIFIED STUDY OF CERTAIN SUBCLASSES OF α-UNIFORMLY CONVEX FUNCTIONS OF ORDER β WITH NEGATIVE COEFFICIENTS

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ABSTRACT. In this paper, we introduce a unified presentation of certain subclasses of starlike and uniformly convex functions with negative coefficients. The unified class of starlike and uniformly convex functions with negative coefficients in the open unit disc is denoted by $\alpha - T(\beta,\mu)$. It is interesting to note that the results of this paper are sharp and generalize the corresponding results of Bharti et al. [1], Dixit and Misra [2] and hence corresponding results of Murugusundaramoorthy [9], Goodman [3, 4], Ma and Minda [7, 8], Rønning [10], Kanas and Wisniowska [5, 6].

1. INTRODUCTION AND PRELIMINARIES

Let T denote the class of functions of the form

$$f(z) = z - \sum_{n=k}^{\infty} a_n z^n, \ a_n \geq 0. \tag{1.1}$$

The classes $\alpha - UCT(\beta)$ and $\alpha - TS_p(\beta)$ were introduced by Bharti et al. [1] as $\alpha - UCT(\beta) = \alpha - UCV(\beta) \cap T$ $(0 \leq \alpha < \infty$ and $0 \leq \beta < 1)$ and $\alpha - TS_p(\beta) = \alpha - S_p(\beta) \cap T$.

Bharti et al. [1] gave the following Lemmas.

Lemma 1.1. *A function $f(z)$ defined by (1.1) is in $\alpha - UCT(\beta)$ if and only if*

$$\sum_{n=2}^{\infty} n[n(1+\alpha) - (\alpha+\beta)]a_n \leq 1-\beta. \tag{1.2}$$

The result (1.2) is sharp for functions

$$f_n(z) = z - \frac{(1-\beta)}{n[n(1+\alpha)-(\alpha+\beta)]} z^n.$$

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Lemma 1.2. *A function $f(z)$ defined by (1.1) is in $\alpha - TS_p(\beta)$ if and only if*

$$\sum_{n=2}^{\infty}[n(1+\alpha)-(\alpha+\beta)]a_n \leq 1-\beta. \tag{1.3}$$

The result (1.3) is sharp for functions

$$f_n(z) = z - \frac{(1-\beta)}{n[(1+\alpha)-(\alpha+\beta)]}z^n.$$

In view of the Lemma 1.1 and Lemma 1.2 it would seem to be natural to introduce an elegant unification of the classes $\alpha - UCT(\beta)$ and $\alpha - TS_p(\beta)$.

2. The class $\alpha - T(\beta, \mu)$

A function $f(z)$ defined by (1.1) belongs to the class $\alpha - T(\beta,\mu)$ if and only if

$$\sum_{n=2}^{\infty}[n(1+\alpha)-(\alpha+\beta)][1-\mu+\mu n]a_n \leq 1-\beta, \tag{2.1}$$

for some $\alpha \geq 0,\ 0 \leq \beta < 1$ and some $\mu(0 \leq \mu \leq 1)$.

Remark 2.1. Since $1-\mu+\mu n \geq 1$ $(0 \leq \mu \leq 1$, and $n = 2,3,...)$. Therefore $\alpha - T(\beta,0) = \alpha - TS_p(\beta)$ and $\alpha - T(\beta,1) = \alpha - UCT(\beta)$.

Following properties are easy consequences of definition (2.1).

Theorem 2.2. *Let $0 \leq \alpha_1 \leq \alpha_2 \leq 1$. Then $\alpha_2 - T(\beta,\mu) \subseteq \alpha_1 - T(\beta,\mu)$.*

Proof. Let the function $f(z)$ defined by (1.1) be in the class $\alpha_2 - T(\beta,\mu)$. Then by (2.1) we have

$$\sum_{n=2}^{\infty}[n(1+\alpha_2)-(\alpha_2+\beta)][1-\mu+\mu n]a_n \leq 1-\beta. \tag{2.2}$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n(1+\alpha_1)-(\alpha_1+\beta)][1-\mu+\mu n]a_n\\
=&\sum_{n=2}^{\infty}[\alpha_1(n-1)+n-\beta][1-\mu+\mu n]a_n\\
\leq&\sum_{n=2}^{\infty}[\alpha_2(n-1)+n-\beta][1-\mu+\mu n]a_n\\
\leq&1-\beta,\ \text{by (2.2)}.
\end{aligned}$$

Hence $f(z)$ is in the class $\alpha_1 - T(\beta,\mu)$. Therefore $\alpha_2 - T(\beta,\mu) \subseteq \alpha_1 - T(\beta,\mu)$. Similarly, we can prove the following theorem. □

Theorem 2.3. *Let $\alpha \geq 0$, $0 \leq \mu \leq 1$ and $0 \leq \beta_1 \leq \beta_2 < 1$. Then $\alpha - T(\beta_1,\mu) \subseteq \alpha - T(\beta_2,\mu)$.*

Theorem 2.4. *Let $\alpha \geq 0$, $0 \leq \beta < 1$ and $0 \leq \mu_1 \leq \mu_2 \leq 1$. Then $\alpha - T(\beta,\mu_1) \subseteq \alpha - T(\beta,\mu_2)$.*

Proof. It follows from (2.1), that

$$\begin{aligned}&\sum_{n=2}^{\infty}[n(\alpha+1)-(\alpha+\beta)][1-\mu_1+\mu_1 n]a_n\\ \leq&\sum_{n=2}^{\infty}[n(1+\alpha)-(\alpha+\beta)][1-\mu_2+\mu_2 n]a_n\leq 1-\beta.\end{aligned}$$

Therefore $\alpha-T(\beta,\mu_1)\subseteq\alpha-T(\beta,\mu_2)$. □

Theorem 2.5. Distortion inequalities *If a function $f(z)$ defined by (1.1) is in the class $\alpha-T(\beta,\mu)$, then*

$$r-\frac{(1-\beta)r^2}{(\alpha-\beta+2)(1+\mu)}\leq|f(z)|\leq r+\frac{(1-\beta)r^2}{(\alpha-\beta+2)(1+\mu)} \tag{2.3}$$

and

$$1-\frac{2r(1-\beta)}{(\alpha-\beta+2)(1+\mu)}\leq|f'(z)|\leq 1+\frac{2r(1-\beta)}{(\alpha-\beta+2)(1+\mu)}. \tag{2.4}$$

The estimates are sharp.

Proof. Since

$$(\alpha-\beta+2)(1+\mu)\sum_{n=2}^{\infty}a_n\leq\sum_{n=2}^{\infty}[n(\alpha+1)-(\alpha+\beta)][1-\mu+n\mu]a_n\leq(1-\beta),$$

we have

$$\begin{aligned}|f(z)|&\leq|z|+|z|^2\sum_{n=2}^{\infty}a_n\\ &\leq|z|+\frac{(1-\beta)}{(\alpha-\beta+2)(1+\mu)}|z|^2\\ &=r+\frac{(1-\beta)r^2}{(\alpha-\beta+2)(1+\mu)}.\end{aligned}$$

Likewise

$$|f(z)|\geq r-\frac{(1-\beta)r^2}{(\alpha-\beta+2)(1+\mu)}.$$

This proves the estimate (2.3) of Theorem 2.4. Further, we note that

$$\begin{aligned}\frac{(\alpha-\beta+2)(1+\mu)}{2}\sum_{n=2}^{\infty}na_n&\leq\sum_{n=2}^{\infty}\left(1+\frac{\alpha-\beta}{n}\right)(1-\mu+\mu n)na_n\\ &=\sum_{n=2}^{\infty}(n+\alpha-\beta)(1-\mu+\mu n)a_n\\ &\leq\sum_{n=2}^{\infty}[n(\alpha+1)-(\alpha+\beta)](1-\mu+\mu n)a_n\\ &\leq(1-\beta).\end{aligned}$$

Now, we have

$$|f'(z)| \leq 1 + |z| \sum_{n=2}^{\infty} n a_n$$
$$\leq 1 + \frac{2(1-\beta)}{(\alpha-\beta+2)(1+\mu)}|z|$$
$$= 1 + \frac{2r(1-\beta)}{(\alpha-\beta+2)(1+\mu)}.$$

Likewise

$$|f'(z)| \geq 1 - \frac{2(1-\beta)}{(\alpha-\beta+2)(1+\mu)}.$$

This proves the estimate (2.4) of the Theorem 2.4. The bounds are sharp, since equalities are attained for the function

$$f(z) = z - \frac{(1-\beta)}{(\alpha-\beta+2)(1+\mu)} z^2.$$

□

Remark 1 Putting $\mu = 0$ and $\mu = 1$ in the above theorem we obtain the distortion inequalities for the classes $\alpha - TS_p(\beta)$ and $\alpha - UCT(\beta)$ given earlier by Bharti et. al. [1].

Remark 2 Putting $\beta = 2,\ \mu = 2$ and $\beta = 0,\ \mu = 1$ in Theorem 2.4, we obtain the corresponding results for the classes $\alpha - TS_p$ and $\alpha - UCT$ given earlier by Murugusundaramoorthy [9].

Radius of close-to-convexity, starlikeness and convexity.

Let the function $f(z)$ defined by (1.1) be in the class $\alpha - T(\beta, \mu)$. Then $f(z)$ is close to convex of order $\rho (0 \leq \rho < 1)$ in $|z| < r$, where

$$r_1 = r_1(\alpha, \beta, \mu, \rho) = \inf \left[\frac{(1-\rho)[n(\alpha+1) - (\alpha+\beta)][1-\mu+\mu n]}{n(1-\beta)} \right]^{1/n-1}. \quad (2.5)$$

The result is sharp, the extremal function $f(z)$ being given by

$$f(z) = z - \frac{(1-\beta)}{[n(\alpha+1) - (\alpha+\beta)][1-\mu+\mu n]} z^n,\ (n \geq 2). \quad (2.6)$$

Proof. We must show that

$$|f'(z) - 1| \leq 1 - \rho \text{ for } |z| < r(\alpha, \beta, \mu, \rho)$$

where r is given by (2.5). Indeed we find from (1.1) that

$$|f'(z) - 1| \leq \sum_{n=2}^{\infty} n a_n |z|^{n-1}.$$

Thus $|f'(z) - 1| \leq 1 - \rho$

$$\text{if } \sum_{n=2}^{\infty} \left(\frac{n}{1-\rho} \right) a_n |z|^{n-1} \leq 1. \quad (2.7)$$

But by (2.1), (2.7) will be true if

$$\left(\frac{n}{1-\rho}\right)|z|^{n-1} \leq \frac{[n(\alpha+1)-(\alpha+\beta)][1-\mu+\mu n]}{1-\beta},$$

that if

$$|z| \leq \left[\frac{(1-\rho)[n(\alpha+1)-(\alpha+\beta)][1-\mu+\mu n]}{n(1-\beta)}\right]^{1/n-1}, \ (n \geq 2). \tag{2.8}$$

Hence, Theorem follows easily from [2.8]. □

Theorem 2.6. *Let the function $f(z)$ defined by (1.1) be in the class $\alpha - T(\beta, \mu)$. Then $f(z)$ is starlike of order $\rho (0 \leq \rho < 1)$ in $|z| < r_2$ where*

$$r_2 = r_2(\alpha, \beta, \mu, \rho) = \inf_n \left[\frac{(1-\rho)[n(\alpha+1)-(\alpha+\beta)][1-\mu+\mu n]}{(n-\rho)(1-\beta)}\right]^{1/n-\rho}. \tag{2.9}$$

The result is sharp with the extremal function $f(z)$ given by (2.6).

Proof. It is sufficient to show

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1-\rho, \text{ for } |z| < r_2(\alpha, \beta, \mu, \rho)$$

where $r_2(\alpha, \beta, \mu, \rho)$ is given by (2.9).

Indeed we find again from (1.1), that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum_{n=2}^{\infty} a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n |z|^{n-1}}.$$

Thus, $\left|\dfrac{zf'(z)}{f(z)} - 1\right| \leq 1-\rho$

$$\text{if } \sum_{n=2}^{\infty} \frac{(n-\rho)}{(1-\rho)} a_n |z|^{n-1} \leq 1. \tag{2.10}$$

But by (2.1), (2.10) will be true if

$$\left(\frac{n-\rho}{1-\rho}\right)|z|^{n-1} \leq [n(\alpha+1)-(\alpha+\beta)][1-\mu+\mu n]/(1-\beta)$$

that if

$$|z| \leq \left[\left(\frac{n-\rho}{1-\rho}\right)\frac{[n(\alpha+1)-(\alpha+\beta)][1-\mu+\mu n]}{(1-\beta)}\right]^{\frac{1}{n-\rho}}. \tag{2.11}$$

Theorem 2.6 follows from (2.11). □

Theorem 2.7. *Let the function $f(z)$ defined by (2.1) be in the class $\alpha - T(\beta, \mu)$. Then $f(z)$ is convex of order $\rho(0 \le \rho < 1)$ in $|z| < r_3$ where*

$$r_3 = r_3(\alpha, \beta, \mu, \rho) = \inf_n \left[\frac{(1-\rho)}{n(n-\rho)} \frac{[n(\alpha+1)-(\alpha+\beta)][1-\mu+\mu n]}{(1-\beta)} \right]^{\frac{1}{n-1}}, \ n \ge 2. \tag{2.12}$$

The result is sharp, with the extremal function given by (2.6).

Proof. We must show that

$$\left| \frac{zf''(z)}{f'(z)} \right| \le 1-\rho \text{ for } |z| < r_3(\alpha, \beta, \mu, \rho)$$

where $r_3(\alpha, \beta, \mu, \rho)$ is given by (2.12). Indeed we find from (1.1) that

$$\left| \frac{zf''(z)}{f'(z)} \right| \le \sum_{n=2}^{\infty} \frac{n(n-1)a_n|z|^{n-1}}{1 - \sum_{n=2}^{\infty} na_n|z|^{n-1}}.$$

Thus, $\left| \frac{zf''(z)}{f'(z)} \right| \le 1-\rho$

$$\text{if } \sum_{n=2}^{\infty} \frac{n(n-\rho)}{(1-\rho)} a_n|z|^{n-1} \le 1.$$

But by (2.1), (2.12) will be true if

$$\frac{n(n-\rho)}{(1-\rho)}|z|^{n-1} \le \frac{[n(\alpha+1)-(\alpha+\beta)][1-\mu+\mu n]}{(1-\beta)}$$

that is, if

$$|z| \le \left[\frac{(1-\rho)}{n(n-\rho)} \frac{[n(\alpha+1)-(\alpha+\beta)][1-\mu+\mu n]}{(1-\beta)} \right]^{\frac{1}{n-\rho}}, \ n \ge 2.$$

Theorem 2.7 follows from the above expression.

Next, we shall show that the class $\alpha - T(\beta, \mu)$ is closed under convex linear combination. ∎

Theorem 2.8. *$\alpha - T(\beta, \mu)$ is a convex set.*

Proof. Let the functions

$$f_v(z) = z - \sum_{n=2}^{\infty} a_{n,v} z^n, \ (a_{n,v} \ge 0, \ v = 1, 2)$$

be in the class $\alpha - T(\beta, \mu)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z) = \lambda f_1(z) + (1-\lambda) f_2(z) \ (0 \le \lambda \le 1)$$

is also in the class $\alpha - T(\beta, \mu)$. Since $0 \le \lambda \le 1$

$$h(z) = z - \sum_{n=2}^{\infty} [\lambda a_{n,1} + (1-\lambda) a_{n,2}] z^n.$$

With that of (2.1), we have

$$\sum_{n=2}^{\infty}[n(\alpha+1)-(\alpha+\beta)][1-\mu+\mu n][\lambda a_{n,1}+(1-\lambda)a_{n,2}] \leq 1-\beta,$$

which implies that $h(z) \in \alpha - T(\beta,\mu)$.

Hence $\alpha - T(\beta,\mu)$ is convex set. □

Theorem 2.9. *Let* $f_1(z) = z$ *and*

$$f_n(z) = z - \frac{(1-\beta)}{[n(\alpha+1)-(\alpha+\beta)][1-\mu+\mu n]}z^n, \ \ n \geq 2$$

for $\alpha \geq 0, \ 0 \leq \beta < 1$ *and* $0 \leq \mu \leq 1$.

Then $f(z)$ *is in the class* $\alpha - T(\beta,\mu)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=1}^{\infty}\lambda_n f_n(z), \tag{2.13}$$

where $\lambda_n \geq 0 \ (n \geq 1)$ *and* $\sum_{n=1}^{\infty}\lambda_n = 1$.

Proof. Assume that

$$f(z) = \sum_{n=1}^{\infty}\lambda_n f(z) = z - \sum_{n=1}^{\infty}\frac{(1-\beta)\lambda_n z^n}{[n(\alpha+1)-(\alpha+\beta)][1-\mu+\mu n]}.$$

Then it follows that

$$\sum_{n=2}^{\infty}[n(\alpha+1)-(\alpha+\beta)][1-\mu+\mu n]\frac{(1-\beta)\lambda_n}{[n(\alpha+1)-(\alpha+\beta)][1-\mu+\mu n]}$$

$$=\sum_{n=2}^{\infty}\lambda_n = 1-\lambda_1 \leq 1.$$

Hence $f \in \alpha - T(\beta,\mu)$ (by using (2.1).

Conversely, assume that the function $f(z)$ defined by (1.1) belongs to the $\alpha - T(\beta,\mu)$, then

$$a_n \leq \frac{(1-\beta)}{[n(\alpha+1)-(\alpha+\beta)][1-\mu+\mu n]}, \ \ n \geq 2$$

and $\lambda_1 = 1 - \sum_{n=2}^{\infty}\lambda_n$.

We can see that $f(z)$ can be expressed in the form (2.13). This completes the proof of Theorem 2.8.

Next, we present our studies for the class $\alpha - T(\beta,\mu)$ by using modified Hadmard product (or convolution product). For this we need the following definition.

Let the function $f_j(z)(j = 1,2) \in T$ is of the form

$$f_j(z) = z - \sum_{n=2}^{\infty}a_{j,n}z^n, \ \ (a_{j,n} \geq 0, \ j = 1,2).$$

The modified Hadamard product or (convolution) of $f_j(z),(j=1,2)$ is denoted by $f_1 * f_2(z)$ and is defined by

$$f_1 * f_2(z) = z - \sum_{n=2}^{\infty} a_{1,n} \cdot a_{2,n} z^n.$$

□

Theorem 2.10. *Let $f_j(z)(j=1,2)$ be in the class $\alpha - T(\beta,\mu)$. Then $f_1 * f_2(z)$ belongs to the class $\gamma - T(\beta,\mu)$, where*

$$\gamma = \frac{(\alpha+2-\beta)^2(1-\mu)-(2-\beta)(1-\beta)}{(1-\beta)}.$$

Proof. Applying the technique used earlier by Schild and Silverman [11], we need to find largest r such that

$$\sum_{n=2}^{\infty}[n(\gamma+1)-(\gamma+\beta)][1-\mu+\mu n]a_{1,n}a_{2,n} \leq 1-\beta.$$

Since $f_j(z)(j=1,2) \in \alpha - T(\beta,\mu)$, therefore we have

$$\sum_{n=2}^{\infty}[n(\alpha+1)-(\alpha+\beta)][1-\mu+\mu n]a_{j,n} \leq 1-\beta,\ (j=1,2).$$

Therefore by Cauchy-Schwartz inequality, we have

$$\sum_{n=2}^{\infty}[n(\alpha+1)-(\alpha+\beta)][1-\mu+\mu n]\sqrt{a_{1,n}a_{2,n}} \leq 1-\beta. \tag{2.14}$$

Thus, it is sufficient to show that

$$\begin{aligned}&[n(\gamma+1)-(\gamma+\beta)][1-\mu+\mu n]a_{1,n}a_{2,n}\\ &\qquad \leq [n(\alpha+1)-(\alpha+\beta)][1-\mu+\mu n]\sqrt{a_{1,n}a_{2,n}}\end{aligned}$$

that is,

$$\sqrt{a_{1,n}a_{2,n}} \leq \frac{[n(\alpha+1)-(\alpha+\beta)]}{[n(\gamma+1)-(\gamma+\beta)]}$$

$$\frac{(1-\beta)}{[n(\alpha+1)-(\alpha+\beta)][1-\mu+\mu n]} \leq \frac{[n(\alpha+1)-(\alpha+\beta)]}{[n(\gamma+1)-(\gamma+\beta)]}, \quad \text{from 2.14}$$

that is, if

$$(1-\beta)[n(\gamma+1)-(\gamma+\beta)] \leq [n(\alpha+1)-(\alpha+\beta)]^2[1-\mu+\mu n]$$

or, $\gamma(n-1)+n-\beta \leq \dfrac{[n(\alpha+1)-(\alpha+\beta)]^2[1-\mu+\mu n]}{1-\beta}$

or, $\gamma \leq \dfrac{[n(\alpha+1)-(\alpha+\beta)]^2[1-\mu+\mu n]}{(1-\beta)(n-1)} - \dfrac{n-\beta}{n-1}$.

If we set

$$\pi(n) = \frac{[n(\alpha+1)-(\alpha+\beta)]^2[1-\mu+\mu n]}{(1-\beta)(n-1)} - \frac{n-\beta}{n-1}.$$

We easily see that $\pi(n)$ is an increasing function of n. On putting $n = 2$, we get

$$\begin{aligned}\gamma \leq \pi(2) &= \frac{[\alpha+2-\beta]^2[1+\mu]}{1-\beta} - (2-\beta)\\ &= \frac{[\alpha+2-\beta]^2[1+\mu]-(2-\beta)(1-\beta)}{1-\beta}.\end{aligned}$$

This completes the proof.

Finally, by taking the function given by

$$f_j(z) = z - \frac{1-\beta}{(\alpha+2-\beta)(1+\mu)}z^2, \ (j = 1, 2).$$

The result is sharp. □

Remark 1 Putting $\mu = 0$ and $\mu = 1$ with $\beta = 0$ in the above theorem, we obtain the corresponding results for class $TS_p(\alpha)$ and $UCT(\alpha)$ respectively given earlier by Murugusundaramoorthy [9].

Remark 2 Putting $\beta = 0$ we obtain the corresponding results for class $\alpha - T(\mu) = T(\alpha, \mu)$ given by Dixit and Misra [2].

Theorem 2.11. *Let the function $f_j(z)(j = 1, 2)$ be in class $\alpha - T(\beta, \mu)$, then the function $h(z)$ defined by $h(z) = z - \sum_{n=2}^{\infty}(a_{1,n}^2 + a_{2,n}^2)z^n$ belongs to the class $T(\gamma, \beta)$, where*

$$\gamma = \frac{[\alpha+2-\beta]^2(1+\mu)-2(2-\beta)(1-\beta)}{2(1-\beta)}.$$

Proof. Since $\displaystyle\sum_{n=2}^{\infty}\frac{[n(\alpha+1)-(\alpha+\beta)]^2[1-\mu+\mu n]^2}{(1-\beta)^2}a_{j,n}^2$

$$\begin{aligned}&\leq \left[\sum_{n=2}^{\infty}\frac{[n(\alpha+1)-(\alpha+\beta)][1-\mu+\mu n]}{(1-\beta)}a_{j,n}\right]^2\\ &\leq \frac{(1-\beta)^2}{(1-\beta)^2} = 1 \ (j = 1, 2).\end{aligned}$$

Thus we have $\displaystyle\sum_{n=2}^{\infty}[n(\alpha+1)-(\alpha+\beta)]^2[1-\mu+\mu n]^2[a_{1,n}^2 + a_{1,n}^2] \leq 2(1-\beta)^2$.

Thus, it is sufficient to find a largest r such that

$$\frac{[n(\gamma+1)-(\gamma+\beta)][1-\mu+\mu n]}{(1-\beta)} \leq \frac{[n(\alpha+1)-(\alpha+\beta)]^2[1-\mu+\mu n]^2}{2(1-\beta)^2}$$

$$\gamma(n-1)+(n-\beta) \leq \frac{[n(\alpha+1)-(\alpha+\beta)]^2[1-\mu+\mu n]}{2(1-\beta)}$$

that is

$$\gamma(n-1) \leq \frac{[n(\alpha+1)-(\alpha+\beta)]^2[1-\mu+\mu n]}{2(1-\beta)} - (n-\beta)$$

or

$$\gamma \leq \frac{[n(\alpha+1)-(\alpha+\beta)]^2[1-\mu+\mu n]}{2(1-\beta)(n-1)} - \frac{(n-\beta)}{(n-1)}$$

or

$$\gamma \leq \frac{[n(\alpha+1)-(\alpha+\beta)]^2[1-\mu+\mu n]-2(n-\beta)(1-\beta)}{2(1-\beta)(n-1)}.$$

If we set

$$\psi(n) = \frac{[n(\alpha+1)-(\alpha+\beta)]^2[1-\mu+\mu n]-2(n-\beta)(1-\beta)}{2(1-\beta)(n-1)}$$

we easily see that $\psi(n)$ is an increasing function of n. On setting $n = 2$, we get

$$\gamma \leq \psi(2) = \frac{[\alpha+2-\beta]^2(1+\mu)-2(2-\beta)(1-\beta)}{2(1-\beta)}.$$

This completes the proof. □

Remark 1 Putting $\beta = 0$ we get the corresponding result for the class $\alpha - T(\mu) = T(\alpha, \mu)$ given earlier by Dixit and Misra [2].
Remark 2 Taking $\mu = 0$ and $\mu = 1$ with $\beta = 0$, we get corresponding results for the classes $TS_p(\alpha)$ and $UCT(\alpha)$ respectively given by Murugusundaramoorthy [9].

Theorem 2.12. (A family of Integral Operators) *Let the function $f(z)$ defined by (1.1) be in the class $\alpha - T(\beta, \mu)$, and let c be a real number such that $c > -1$. Then the function $F(z)$ defined by*

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt, \ (c > -1) \tag{2.15}$$

also belongs to the class $\alpha - T(\beta, \mu)$.

Proof. From the representation of (2.15) of $F(z)$, it follows that

$$F(z) = z - \sum_{n=2}^{\infty} b_n z^n, \text{ where } b_n = \left(\frac{c+1}{c+n}\right) a_n.$$

Therefore, we have

$$\begin{aligned} &\sum_{n=2}^{\infty}[n(\alpha+1)-(\alpha+\beta)][1-\mu+\mu n]b_n \\ =&\sum_{n=2}^{\infty}[n(\alpha+1)-(\alpha+\beta)][1-\mu+\mu n]\left(\frac{c+1}{c+n}\right)a_n \\ \leq&\sum_{n=2}^{\infty}[n(\alpha+1)-(\alpha+\beta)][1-\mu+\mu n]a_n \leq 1-\beta, \end{aligned}$$

since $f(z) \in \alpha - T(\beta, \mu)$.
Hence by (2.1), $F(z) \in \alpha - T(\beta, \mu)$. □

Theorem 2.13. *Let the function* $F(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $(a_n \geq 0)$ *be in the class* $\alpha - T(\beta, \mu)$ *and let* c *be a real number such that* $c > -1$. *Then the function* $f(z)$ *given by* (2.15) *is univalent in* $|z| < R^*$, *where*

$$R^* = \inf_n \left[\frac{\{n(\alpha+1) - (2-\beta)\}(1-\mu+\mu n)(c+1)}{(1-\beta)n(c+n)} \right], \ (n \geq 2). \tag{2.16}$$

The result is sharp.

Proof. From (2.15), we have

$$\begin{aligned} f(z) &= \frac{z^{1-c}(z^c F(z))'}{c+1} \\ &= z - \sum_{n=2}^{\infty} \left(\frac{c+n}{c+1} \right) a_n z^n. \end{aligned}$$

In order to obtain the required result, it suffices to show that $|f'(z) - 1| < 1$, for $|z| < R^*$, where R^* is given by (2.16).

Now $|f'(z) - 1| \leq \sum_{n=2}^{\infty} \frac{n(c+n)}{(c+1)} a_n |z|^{n-1}$.

Thus $|f'(z) - 1| < 1$ if

$$\sum_{n=2}^{\infty} \frac{n(c+n)}{(c+1)} a_n |z|^{n-1} < 1. \tag{2.17}$$

But (2.1) confirms that

$$\sum_{n=2}^{\infty} [n(\alpha+1) - (\alpha+\beta)][1-\mu+\mu n] a_n \leq (1-\beta).$$

Hence, (2.17) will be satisfied if

$$\frac{n(c+n)}{(c+1)} |z|^{n-1} < \frac{[n(\alpha+1) - (\alpha+\beta)][1-\mu+\mu n]}{(1-\beta)}$$

that is, if

$$|z| < \left[\frac{[n(\alpha+1) - (\alpha+\beta)][1-\mu+\mu n](c+1)}{(1-\beta)n(c+n)} \right]^{1/n-1}, \ (n \geq 2).$$

Therefore, the function $f(z)$ given by (2.15) is univalent in $|z| < R^*$. Sharpness of the result follows, if we take

$$f(z) = z - \frac{(c+n)z^n}{(c+1)[n(\alpha+1) - (\alpha+\beta)][1-\mu+\mu n]}, \ n \geq 2.$$

$\square$

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