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COUNTABLE DENSE HOMOGENEITY PROPERTY OF CONNECTED COMPACT MANIFOLDS

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Abstract. Let Υ be a connected compact manifold and $H(\Upsilon)$ denote the set of all homeomorphisms from Υ onto itself. In this paper we provide another proof for the known result that every connected compact manifolds are $n-$homogeneous and thereby countable dense homogeneous (CDH). Mainly, we discuss about the denseness of $H_A(\Upsilon) = \{h \in H(\Upsilon) \mid h : A \to A$ is a homeomorphism$\}$ in $H(\Upsilon)$ whenever A is a subset of Υ. In the end we show that if f is a contraction function on Υ, then the set of all homeomorphisms that commutes with f is a nowhere dense subgroup of $H(\Upsilon)$.

1. Introduction and preliminaries

Let Υ be a compact metric space and $H(\Upsilon)$ denote the set of all homeomorphisms from Υ onto itself. The topology on $H(\Upsilon)$ is the compact open topology which has its subbasis as $\{(K;U) \mid K \subseteq \Upsilon$ is compact and $U \subseteq \Upsilon$ is open$\}$ where $(K;U) = \{h \in H(\Upsilon) \mid h(K) \subseteq U\}$. If d is a metric on Υ compatible with its topology, define ρ, on $H(\Upsilon)$ as $\rho(f,t) = sup_{x\in\Upsilon} d(f(x), t(x))$ for $f, t \in H(\Upsilon)$. This metric is compatible with the compact-open topology on $H(\Upsilon)$. From [6], it is known that for a compact manifold Υ its $H(\Upsilon)$ is a separable metric space and a topological group under composition of functions. Hence it is homogeneous and also topologically complete. Therefore $H(\Upsilon)$ is a Baire space (hence non-meager) by Baire's category theorem, which asserts that every complete metric space is Baire.

Our space under consideration will be connected compact manifolds as the space itself and its space of homeomorphisms have many interesting properties. They are $n-$homogeneous as well as CDH. From [5], it is known that in compact connected CDH spaces, dense open subsets are also CDH. Also from [4] we have, if Υ is a separable and homogeneous space having a non-trivial convergent sequence, then it contains a countable dense subset that is sequentially crowded. Thus, if f is a contraction function on a connected compact manifold, then every orbit of f converges to the unique fixed point of f there by claiming the existence of non-trivial convergent sequences. Hence all countable dense subsets of connected compact manifolds are sequentially crowded, being CDH. Also since compact

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Hausdorff spaces are Baire, we have our space is Baire. We shall see below that connected compact manifolds are also productively Baire.

Firstly, we give a proof for the classical result stating that every connected compact manifold Υ of dimension ≥ 2 is $n-$homogeneous [10]. Then, it would follow from [8] that, connected compact manifolds are countable dense homogeneous (CDH). These facts guarantees the existence of the set $\{f \in H(\Upsilon) \mid f(A) = B\}$ whenever A and B are either two countable dense subsets or two finite subsets of the same size. We also show that every compact manifold can be expressed as a disjoint union of countable dense subsets. This paper mainly discusses about the denseness of $H_A(\Upsilon) = \{h \in H(\Upsilon) \mid h : A \to A \text{ is a homeomorphism}\}$ in $H(\Upsilon)$ for $A \subseteq \Upsilon$ and Υ being a connected compact manifold. It can be seen from [3] that, when Υ is a cantor set or a one- dimensional topological manifold, then $H_A(\Upsilon)$ (A being a countable dense subset of Υ) is homeomorphic to $\mathbb{Q}^\infty$, the countable power of the space of rationals. Also when Υ is a topological manifold of dimension atleast 2 or a manifold modelled on a universal Menger continuum or a Hilbert cube manifold, then $H_A(\Upsilon)$ is homeomorphic to the Erdős space having elements as vectors in l^2 (the space of all square summable real sequences) whose coordinates are all rational. Finally, we prove that if k is a contraction function on a connected compact manifold Υ, then the set of all homeomorphisms that commutes with k is a nowhere dense subgroup of $H(\Upsilon)$.

Notations. $A^0, A^c, \overline{A}$ denotes the interior, complement and closure of A respectively.

Definition 1.1. [11] A Hausdorff space Υ is an m-manifold if it has a countable basis, and each x of Υ has a neighborhood that is homeomorphic with an open subset of $\mathbb{R}^m$.

Definition 1.2. [2] Let (Υ, d) be a metric space. A map $K : \Upsilon \to \Upsilon$ is said to be of Kannan type if $\exists$ a $c \in (0, \frac{1}{2}]$ such that $d(Kx, Ky) \leq c\,[d(x, Kx) + d(y, ky)]$ for all $x, y \in \Upsilon$.

Definition 1.3. [12] A topological space Υ is said to be $n-$homogeneous if whenever $x_1, x_2, ..., x_n \in \Upsilon$ and $y_1, y_2, ..., y_n \in \Upsilon$, then $\exists\, h \in H(\Upsilon)$ such that $h(x_i) = y_i$. In particular when $n = 1$, we simply say the space is *homogeneous*.

Definition 1.4. [12] A separable space Υ is said to be countable dense homogeneous (CDH) if given any two countable dense subsets D and E of Υ, $\exists\, h \in H(\Upsilon)$ such that $h(D) = E$.

Definition 1.5. [11] Let Υ be a topological space. Then any set that can be expressed as a countable intersection of open sets of Υ is said to be a G_δ set in Υ.

Definition 1.6. [5] Let Υ be a topological space. Then Υ is said to be a λ-set if every countable set is a G_δ set.

Definition 1.7. [1] A Baire space Υ is productively Baire if $\Upsilon \times \Upsilon'$ is Baire for any Baire space Υ'.

Product of Baire spaces need not be always Baire (See [1]). But, our space is productively Baire. For this, let's summarize about Banach-Mazur game $BM(\Upsilon)$, a game played on a topological space Υ between 2 players namely A and B, who alternately picks only open subsets of Υ that are non-empty. A plays first and selects an open subset P_0 of Υ. B goes next by taking an open subset $Q_0 \subseteq P_0$. In general, A chooses any open subset P_n of the previous move Q_{n-1} of B and then B answers by choosing an open subset Q_n of the set P_n, just chosen by A. In this way, A and B produce a sequence of non-empty open sets $P_0 \supseteq Q_0 \supseteq P_1 \supseteq Q_1 \supseteq ... \supseteq P_n \supseteq Q_n \supseteq ....$ We say B wins a play of $BM(\Upsilon)$ if $\bigcap_{n\in\omega} Q_n = \bigcap_{n\in\omega} P_n \neq \emptyset$. Otherwise, A is said to be the winner of this play.

Theorem 1.8. *Connected compact manifolds are productively Baire.*

Proof. Let Υ be a connected compact manifold. We know from [1] that if B has a winning strategy on $BM(\Upsilon)$, then Υ is productively Baire. The fact that our space is metrizable and every open set in it is uncountable proves that B has a winning strategy. $\square$

Theorem 1.9. *Every connected compact manifold Υ of dimension ≥ 2 is $n-$homogeneous.*

Proof. Consider $n = 3$. Let $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ be any two $3-$element sets. Since a manifold is path connected whenever it is connected, let $\alpha_i : [0, 1] \to \Upsilon$ be a path joining x_i and y_i. That is, α_i is a continuous function such that $\alpha_i(0) = x_i$ and $\alpha_i(1) = y_i$. Let $\delta = inf\{d(\alpha_i[0, 1], \alpha_j[0, 1])\}$. Choose a partition P of $[0, 1]$; $P = \{t_0, t_1, ..., t_r\}$ such that $diam_j(\alpha_i[t_j, t_j + 1]) < \frac{\delta}{4}$ $\forall$ $i = 1, 2, 3$. Let $B^i_j = B(\alpha_i(t_j), \frac{\delta}{4})$ for $j = 0, 1, ..., r-1$. Then clearly $\alpha_i(t_{j+1}) \in (B^i_j)^\circ$. Since Υ is a manifold, $\exists$ a homeomorphism $\Phi^i_j : B^i_j \to B^i_j$ such that $\Phi^i_j(\alpha_i(t_j)) = \alpha_i(t_{j+1})$ and identity outside B^i_j. Define $\Phi_i = \Phi^i_{r-1} \circ \Phi^i_{r-2} \circ ... \circ \Phi^i_0$. Then $\forall i$, $\Phi_i(x_i) = (\Phi^i_{r-1} \circ \Phi^i_{r-2} \circ ... \circ \Phi^i_0)(x_i) = y_i$. So our required homeomorphism will be $\Phi = \Phi_3 \circ \Phi_2 \circ \Phi_1$. $\square$

Theorem 1.10. [7] *Let Υ be a homogeneous space and $x, y \in \Upsilon$. If a is the cardinality of $\{f \in H(\Upsilon) \mid f(x) = y\}$, then for any $z, w \in \Upsilon$ the set $\{f \in H(\Upsilon) \mid f(z) = w\}$ also has cardinality a.*

Theorem 1.11. [7] *If Υ is uncountable, separable, homogeneous and has a countable collection of equivalence classes of countable dense subsets, then for each $x, y \in \Upsilon$ the set $\{f \in H(\Upsilon) \mid f(x) = y\}$ is uncountable.*

However, $A = \{f \in H(\Upsilon) \mid f(x) = y\}$ for x, y in a connected compact manifold Υ are never dense in $H(\Upsilon)$. For, $(K, U) \cap A = \emptyset$ if K, U are compact and open subsets of Υ such that $x \in K$ and $y \notin U$. Same case prevails if we consider $n-$element sets instead of x and y. Surprisingly, this is not the case for $H_A(\Upsilon) = \{h \in H(\Upsilon) \mid h(A) = A\}$ whenever A is a countable dense subset of Υ, which is the main part of our paper.

2. Denseness property of $H_A(\Upsilon)$

Here we shall discuss about the denseness of $H_A(\Upsilon) = \{h \in H(\Upsilon) \mid h : A \to A$ is a homeomorphism$\}$ in $H(\Upsilon)$ for $A \subseteq \Upsilon$ and Υ being a connected compact manifold.

Lemma 2.1. *Let $(\Upsilon, \mathcal{T})$ be a topological space. Then $\forall A \subseteq \Upsilon$, $H_A(\Upsilon) = H_{A^c}(\Upsilon)$.*

Proof. For, $h \in H_A(\Upsilon) \Longrightarrow h : A \to A$ is bijective and thus $h : A^c \to A^c$ is also bijective. Let U be open in A^c, then $\exists\ U' \in \mathcal{T}$ such that $U = U' \cap A^c$. Thus $h^{-1}(U) = h^{-1}(U' \cap A^c) = h^{-1}(U') \cap A^c$. Therefore $h : A^c \to A^c$ is continuous. Similarly, we can show that $h^{-1} : A^c \to A^c$ is continuous and thus $H_A(\Upsilon) \subseteq H_{A^c}(\Upsilon)$. Converse is trivial as $(A^c)^c = A$. $\square$

Theorem 2.2. *Let Υ be a compact manifold and $A \subset \Upsilon$ such that $\overline{H_A(\Upsilon)} = H(\Upsilon)$, then $\overline{A} = \Upsilon$ with $A^0 = \emptyset$.*

Proof. A is dense in Υ. For if not, $\exists$ an open set U such that $U \cap A = \emptyset$. Let $K \subseteq A$ be compact, then $(K; U) \cap H_A(\Upsilon) = \emptyset$, a contradiction. Since $H_A(\Upsilon) = H_{A^c}(\Upsilon) \Longrightarrow \overline{H_{A^c}(\Upsilon)} = H(\Upsilon) \Longrightarrow \overline{A^c} = \Upsilon$. Thus A and A^c are dense in Υ making interior of A empty. $\square$

Remark 2.3. Thus for any set A for which $\overline{H_A(\Upsilon)} = H(\Upsilon)$, every open subset of Υ intersects both A and A^c.

Example 2.4. $A^0 = \emptyset$ is a necessary condition for $H_A(\Upsilon)$ to be dense in $H(\Upsilon)$. For, let $x \in \Upsilon$ and $A = \Upsilon \setminus \{x\}$. Then $\overline{A} = \Upsilon$ and $H_A(\Upsilon) = H_{A^c}(\Upsilon) = \{h \in H(\Upsilon) \mid h(x) = x\}$. Take $K = \{x\}$ and U as any open set not containing x. Then, $(K; U) \cap H_A(\Upsilon) = \emptyset$.

Theorem 2.5. *Every compact manifold Υ can be expressed as a disjoint union of countable dense subsets. That is, $\Upsilon = \cup A_\lambda$ where $A_\lambda \subseteq \Upsilon$ is countable, dense, $A_\lambda^0 = \emptyset$ and $A_\lambda \cap A_\alpha = \emptyset$ for $\lambda \neq \alpha$.*

Proof. For a compact manifold Υ, its $H(\Upsilon)$ is a separable metric space [6] . Thus $\exists\ G = \{h_1, h_2, ...\}$ such that $\overline{G} = H(\Upsilon)$. Let $G^{-1} = \{h_1^{-1}, h_2^{-1}, ...\}$ and $x \in \Upsilon$ be fixed. Define $A(x) = \{g(x) \mid$ g is a finite composition of elements of $G \cup G^{-1}\}$. Then $x \in A(x)$ and for $(h_{i_1}^{\pm 1} \circ h_{i_2}^{\pm 1} \circ ... \circ h_{i_n}^{\pm 1})(x) \in A(x)$ and $h_i \in G$ we have $h_i(h_{i_1}^{\pm 1} \circ h_{i_2}^{\pm 1} \circ ... \circ h_{i_n}^{\pm 1})(x) \in A(x)$. That is, all functions in G maps $A(x)$ to $A(x)$ itself. We shall easily prove (1), (2) and (3).

1. (1) $\forall h \in G, h : A(x) \to A(x)$ is a homeomorphism. For, one-oneness and bicontinuity easily follows as $h \in H(\Upsilon)$ and for $y' = (h_{i_1}^{\pm 1} \circ h_{i_2}^{\pm 1} \circ ... \circ h_{i_n}^{\pm 1})(x) \in A(x)$, $\exists\ y = (h^{-1} \circ h_{i_1}^{\pm 1} \circ h_{i_2}^{\pm 1} \circ ... \circ h_{i_n}^{\pm 1})(x) \in A(x)$ such that $h(y) = y'$.
2. (2) $A(x)$ is countable. For, $G \cup G^{-1}$ is countable and from the construction of the set $A(x)$, it is clear that $A(x)$ is countable. Thus $A(x) \neq \Upsilon$.
3. (3) $A(x)$ is dense. For, $G \subseteq H_{A(x)}(\Upsilon)$ and $\overline{G} = H(\Upsilon) \Longrightarrow \overline{H_{A(x)}(\Upsilon)} = H(\Upsilon)$ with $A(x) \subset \Upsilon$. Therefore by previous theorem $\overline{A(x)} = \Upsilon$ and $A(x)^0 = \emptyset$.

It can be seen that $\forall y \in A(x), A(y) = A(x)$. Thus $A(x)$ and $A(y)$ are either equal or disjoint. For, let $A(x) \neq A(y)$ and suppose that $A(x) \cap A(y) \neq \emptyset$. Then, $z \in A(x)$ and $z \in A(y)$ implies $A(z) = A(x)$ and $A(z) = A(y)$, a contradiction. $\square$

Remark 2.6. Clearly, $A(x)$ is the smallest invariant set containing x on which G is a set of homeomorphisms.

Example 2.7. Let $X = [0, 1]$ and $A = (p_1, q_1), B = (p_2, q_2) \in \mathbf{Q}^2 \cap X^2$ with $p_1 < p_2$ and $q_1 < q_2$. Let l_1, l_2, l_3 be the straight lines joining points $(0, 0)$ and A), (A and B) and (B and $(1, 1)$) resp. Then $h_{A,B} = l_1 \cup l_2 \cup l_3$ is a strictly increasing polygonal path. Also $h_{A,B} : [0, 1] \to [0, 1]$ is a homeomorphism since it is a continuous and bijective function on $[0, 1]$. Let $G = \{h_{A,B} \mid A, B \in \mathbf{Q}^2 \cap X^2\}$. Clearly G is countable and also dense. For, consider an open set $(K; U)$ in $H(X)$. Let K be a closed interval say $[a, b]$ (other cases easily follows) and let $I \subseteq U$ be an open interval. Choose four rational numbers $p', q', r', s' \in [0, 1]$ such that $p' \leq a, q' \geq b, r' < s'$ and $r', s' \in I$. Let $P = (p', r'), Q = (q', s')$. Then clearly $h_{P,Q} \in (K; U)$. For any $x \in \mathbf{Q} \cap X$, $A(x)$ consists of $\mathbf{Q} \cap X$ and countably many elements from $\mathbf{Q^c} \cap X$.

Now let us proceed to the core part of our paper. Since for connected compact metric spaces other than the circle, $n-$homogeneity and countable dense homogeneity are equivalent (refer [12]) it then follows from our second theorem that connected compact manifolds are CDH. Also, in [13] it is shown that the set of all isolated points E in CDH spaces are clopen and open sets intersecting E^c are uncountable. Therefore connected compact manifolds have no isolated points and every open set is uncountable.

Theorem 2.8. *CDH spaces are T_1.*

Proof. Let Υ be a CDH space and $x \neq y \in \Upsilon$. Suppose that all open sets containing x contains y and viceversa. Let $U = \{V \mid V \text{ is open and } x \notin V\}$. Since Υ is separable $\exists$ a dense set $A = \{x, x_1, x_2, ...\}$ with $x_i \in V$ for some $V \in U$ such that both of $\overline{A \setminus \{x\}}$ and $\overline{A \setminus \{x_i\}}$ are not Υ. Thus there exists open sets $V_i \in U$ such that x_i's are separated by V_i's. Take $B = \{x, y, x_1, x_2, ...\}$. Since Υ is CDH, $\exists\, h \in H(\Upsilon)$ such that $h(A) = B$. Now if $h(x_i) = x$ then $h(V_i)$ is an open set containing x and hence y has its preimage in V_i, which is a contradiction. Same is the case when $h(x_i) = y$. This forces either x or y not to have its preimage in A and thereby again leading to a contradiction.

Note that if Υ is the only open set containing x and y, then $A' = \{x_1, x_2, ...\}$ is such that $\overline{A'} = \Upsilon$ and $\overline{A' \setminus \{x_i\}} \neq \Upsilon$. Taking $B' = \{x, x_1, x_2, ...\}$, $\exists\, h \in H(\Upsilon)$ such that $h(A') = B'$. If $h(x_i) = x$, then $h(V_i)$ is an open set containing x which is Υ itself, making a contradiction. Proceeding as above we observe that this case is also not possible. $\square$

Theorem 2.9. *Let Υ be a connected compact manifold. Then $H_A(\Upsilon)$ is dense if A is countable dense.*

Proof. Let $A = \{x_1, x_2, ...\}$. Since Υ is CDH, let us rename elements in $A(x)$ as $A(x) = \{y_1, y_2, ...\}$ such that the homeomorphism (say) $g \in H(\Upsilon)$ for which

$g(A) = A(x)$ maps x_i to y_i. Let $G' = \{g^{-1} \circ h_i \circ g \mid h_i \in G\}$. Then, G' is countable and $\forall i$, $g^{-1} \circ h_i \circ g : A \to A$ is a homeomorphism. For, one-oneness and bicontinuity easily follows as $h \in H(\Upsilon)$. Now to prove it is onto. Let $x_k \in A$, then since $h_i : A(x) \to A(x)$ is a homeomorphism $\exists$ $'y_j'$ such that $h_i(y_j) = y_k$. Thus $x_k = g^{-1}(y_k) = g^{-1}(h_i(y_j)) = g^{-1}(h_i(g(x_j))$. Also G' is dense. For, let $(K; U)$ be an open set in $H(\Upsilon)$, then considering the open set $(g(K), g(U))$, $\exists$ $h_i \in G$ such that $(h_i \circ g)(K) \subseteq g(U)$ and hence $(g^{-1} \circ h_i \circ g)(K) \subseteq U$. □

Corollary 2.10. *Let Υ be a connected compact manifold. If A is an uncountable dense subset of Υ such that $A^0 = \emptyset$ and A^c is countable. Then, $H_A(\Upsilon)$ is dense.*

Proof. Since $H_A(\Upsilon) = H_{A^c}(\Upsilon)$ and $\overline{H_{A^c}(\Upsilon)} = H(\Upsilon)$. □

Observation 2.11. Let Υ be a connected compact manifold. If A, A^c are uncountable dense subsets of Υ. Then, $H_A(\Upsilon)$ need not be dense in $H(\Upsilon)$.

Proof. Let Υ be an $m-$dimensional manifold and $B \subseteq \Upsilon$ be countable dense. Let $B' \subseteq \Upsilon$ be such that B' is homeomorphic to an uncountable subset of $(\mathbf{Q^c})^m$ (where $\mathbf{Q}$ denotes rationals) satisfying $(B \cup B')^0 = \emptyset$. Then $A = B \cup B'$ is uncountable, dense with interior empty. Let U be an open set such that $U \cap B' = \emptyset$. This implies A^c is uncountable. Also A^c is dense with its interior empty.

Consider the open set $(U^c; U)$. Now $B' \subseteq U^c$ implies $A \cap U^c$ is uncountable and $A \cap U$ is atmost countable. Then, there doesnot exist any homeomorphism h such that $h(A \cap U^c) \subseteq A \cap U$. That is, $H_A(\Upsilon) \cap (U^c; U) = \emptyset$. □

Corollary 2.12. *Let M and N be countable dense subsets of a connected compact manifold Υ. Then $H(M, N) = \{g \in H(\Upsilon) \mid g(M) = N\}$ is dense in $H(\Upsilon)$.*

Proof. Let $g \in H(M, N)$ and $h \in H_N(\Upsilon)$. Then $h \circ g : M \to N$ is a homeomorphism. Thus the set $\{h \circ g \mid h \in H_N(\Upsilon)\}$ is dense since $H_N(\Upsilon)$ is dense. For, let $(K; U)$ be an open set in $H(\Upsilon)$, then considering the open set $(g(K); U)$, $\exists$ $h \in H_N(\Upsilon)$ such that $(h \circ g)(K) \subset U$. □

Corollary 2.13. *Let Υ be a connected compact manifold. Then for every $h \in H(\Upsilon)$ $\exists$ a partition of Υ in to countable dense subsets on which it is invariant.*

Proof. Let $h \in H(\Upsilon)$. Consider $G_h = G \cup \{h\}$. Let $A^h(x) = \{g(x) \mid$ g is a finite composition of elements of $G_h \cup G_h^{-1}\}$. Then $\forall x \in \Upsilon$, $G_h \subseteq H_{A^h(x)}(\Upsilon)$ with $\Upsilon = \bigcup_{x \in \Upsilon} A^h(x)$. Thus $H(\Upsilon) = \bigcup_A H_A(\Upsilon)$; A is a countable dense subset of Υ. □

Remark 2.14. If C and D are two countable dense subsets of Υ. Then for homeomorphisms $h \in H(\Upsilon)$ for which $h(C) = D$ implies C and D belong to the same $A^h(x)$ for some $x \in \Upsilon$. This is true for every countable G such that $\overline{G} = H(\Upsilon)$.

Since connected compact manifolds are $n-$homogeneous and CDH there naturally arises a question : If given two countable dense subsets M, N and two finite $n-$element sets F, F' of Υ, does $\exists$ a homeomorphism $g \in H(\Upsilon)$ such that $g(M) = N$ and $g(F) = F'$?. We have a partial answer to this question using the following result.

Theorem 2.15. *Let Υ be a topological space and D be a subset of $H(\Upsilon)$ that makes Υ CDH. Then for any finite $C \subseteq \Upsilon$ and countable dense subsets M, N of $\Upsilon \setminus C$, $\exists\ p, q \in D$ such that $p \mid C = q \mid C$ and $(p^{-1} \circ q)(M) \subseteq N$.* [13]

Here choose $C = F \cup F'$.

Theorem 2.16. *In a connected compact manifold Υ, no countable dense set is a G_δ set.*

Proof. Since Υ is CDH if a countable dense set is a G_δ set, then all countable dense sets are G_δ sets. Let C and D be two countable dense subsets with $C \cap D = \emptyset$. Thus if $C = \bigcap\limits_n O_n$ and $D = \bigcap\limits_n O'_n$ where O_n, O'_n are dense open sets, then $(\bigcap\limits_n O_n) \bigcap (\bigcap\limits_n O'_n)$ is dense as Υ is baire. But clearly this intersection is empty. $\square$

Corollary 2.17. *Connected compact manifolds are not $\lambda-$ sets.*

Corollary 2.18. *In Connected compact manifolds countable intersection of dense open sets are uncountably dense.*

Theorem 2.19. *Let Υ be a connected compact manifold and k be a contraction function on Υ. Then, the set of all homeomorphisms that commutes with k is a nowhere dense subgroup of $H(\Upsilon)$.*

Proof. Since k is a contraction function on the complete metric space Υ, it has a unique fixed point x and no other periodic points. Define $M(k) = \{h \in H(\Upsilon) \mid k \circ h = h \circ k\}$. All functions in $M(k)$ also leaves x fixed. For, $k(x) = x \Rightarrow (h \circ k)(x) = h(x) \Rightarrow k(h(x)) = h(x) \Rightarrow h(x) = x$. Thus all homeomorphisms that commutes with any contraction function has a fixed point. Clearly $M(k)$ is a closed subset of $H(\Upsilon)$ having its interior empty. For, if h is an interior point of $M(k)$ then $\exists$ an open ball $B_\rho(r; h)$ with centre at h and radius r such that $B_\rho(r; h) \subseteq M(k)$. Thus if M and N are two disjoint countable dense subsets of Υ such that $x \in M$ then, $H(M, N) \cap B_\rho(r; h) = \emptyset$ where $H(M, N) = \{g \in H(\Upsilon) \mid g(M) = N\}$ is proven to be a dense subset of $H(\Upsilon)$. Hence $M(k)$ is a nowhere dense subset of $H(\Upsilon)$.

$M(k)$ is a subgroup of $H(\Upsilon)$: $h^{-1} \in M(k)$, $\forall h \in M(k)$ using $I = h \circ h^{-1}$ in the relation $I \circ k = k \circ I$. $\square$

Corollary 2.20. *Let k be any continuous mapping on Υ having a unique fixed point.Then, the set of all homeomorphisms that commutes with k is a nowhere dense subgroup of $H(\Upsilon)$.*

Proof. Obvious from the previous proof, as we only used the fact that contraction functions have unique fixed point. $\square$

Example 2.21. Since every Kannan mapping k on a complete metric space Υ has a unique fixed point [9], the above result holds for k if it is continuous as well.

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