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ON THE SOLUTIONS OF THE SYSTEM OF RATIONAL DIFFERENCE EQUATIONS $x_{n+1} = \frac{x_{n-1}}{x_{n-1}-1}$, $y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1}-1}$, AND $z_{n+1} = \frac{z_{n-1}}{x_n y_n z_{n-1}-1}$

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ABSTRACT. In this paper, we introduce systems of difference equations, namely $x_{n+1} = \frac{x_{n-1}}{x_{n-1}-1}$, $y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1}-1}$, and $z_{n+1} = \frac{z_{n-1}}{x_n y_n z_{n-1}-1}$. In addition, we obtain the general form of the solutions for these systems.

1. INTRODUCTION AND PRELIMINARIES

Recently many researchers are interested in the study of the behavior of solutions of difference equations. In [1], the difference equation

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A_n}{x_{n-1}}\right\}$$

was considered and the general form of the solutions was explicitly computed in some cases. In [2], the general form of the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{\pm 1 \pm x_{n-1}x_{n-3}x_{n-5}}$$

was computed. In [3], Ginar studied the solutions of the system of rational difference equations

$$\begin{aligned} x_{n+1} &= \frac{1}{y_n}, \\ y_{n+1} &= \frac{y_n}{x_{n-1}y_{n-1}} \end{aligned}$$

In [7], Kurbanli, Ginar and Yalcinkaya studied the behavior of positive solutions of the system of rational difference equations

$$\begin{aligned} x_{n+1} &= \frac{x_{n-1}}{y_n x_{n-1}+1}, \\ y_{n+1} &= \frac{y_{n-1}}{x_n y_{n-1}+1} \end{aligned}$$

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In [5], Kurbanli studied the system

$$\begin{aligned} x_{n+1} &= \frac{x_{n-1}}{y_n x_{n-1} - 1} \\ y_{n+1} &= \frac{y_{n-1}}{x_n y_{n-1} - 1} \\ z_{n+1} &= \frac{z_{n-1}}{y_n z_{n-1} - 1} \end{aligned}$$

In this system, there is a similarity between x and y. The general formulae for x_{n+1}, y_{n+1}, and z_{n+1} were computed. Using these formulae some limit properties were proved. In [6], Kurbanli also studied the system

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}, \quad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1},$$

$$z_{n+1} = \frac{1}{y_n z_n}.$$

Motivated by these works, we consider the two systems

$$x_{n+1} = \frac{x_{n-1}}{x_{n-1} \pm 1},\ y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} \pm 1}$$

In the case of negative sign, we obtain simple results, which can be used in the study of the extension of the system to the system

$$x_{n+1} = \frac{x_{n-1}}{x_{n-1} - 1},\ y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1},\ \text{and } z_{n+1} = \frac{z_{n-1}}{x_n y_n z_{n-1} - 1}$$

1.1. **helpful result.** We use basically the following two relations, which are easily proven by mathematical induction on n.

Lemma 1.1. *Let $p, q \in \mathbb{R}$ such that $p > 0$. For every positive integers m and n the following holds*

$\prod_{l=0}^{n}(p+ql) = q^{n+1}\frac{\Gamma(n+\frac{p+q}{q})}{\Gamma(\frac{p}{q})}$, *and*

$\prod_{l=m+1}^{n}(p+ql) = \frac{q^{n+1}}{q^{m+1}\Gamma(m+\frac{p+q}{q})}\Gamma(n+\frac{p+q}{q})$.

2. Systems of two equations

We denote r by a positive fixed real number. We consider now the following system of equations

$$\begin{aligned} x_{n+1} &= \frac{x_{n-1}}{x_{n-1} + r} \\ y_{n+1} &= \frac{y_{n-1}}{x_n y_{n-1} + r} \end{aligned} \tag{2.1}$$

We take the initial values

$$\begin{aligned} x_0 &= b,\ x_{-1} = c, \\ y_0 &= a, \text{ and } y_{-1} = d \end{aligned}$$

In this case, we obtain

$$x_1 = \frac{x_{-1}}{x_{-1}+r} = \frac{c}{c+r}, \quad y_1 = \frac{y_{-1}}{x_0 y_{-1}+r} = \frac{d}{bd+r}$$

$$x_2 = \frac{x_0}{x_0+1} = \frac{b}{b+r}, y_2 = \frac{y_0}{x_1 y_0 + r} = \frac{a(c+r)}{ac+cr+r^2} = \frac{a(c+r)}{ac+r(c+r)}$$

$$x_3 = \frac{c}{(r+1)c+r^2}, \quad y_3 = \frac{d(b+r)}{bd+r(bd+r)(b+r)}$$

$$x_4 = \frac{b}{b(1+r)+r^2},$$

$$y_4 = \frac{a(c+r)(r^2+c(r+1))}{ac(c+r)+acr((r+1)c+r^2)+r^2(c+r)((r+1)c+r^2)}$$

$$x_5 = \frac{c}{r^3+c(r^2+r+1)},$$

$$y_5 = \frac{d(b+r)(b(1+r)+r^2)}{bd(b+r)+bdr(b(1+r)+r^2)+r^2(bd+r)(b+r)(b(1+r)+r^2)},$$

We use the notation

$$G(b,j) = (r^j + (1+r+...+r^{j-1})b)$$

Hence, using our notation we can rewrite y_4 and y_5 as follows

$$y_4 = \frac{aG(c,1)G(c,2)}{acG(c,1)+G(c,2)(acr+cr^2+r^3)}$$

$$y_5 = \frac{dG(b,1)G(b,2)}{bdG(b,1)+bdrG(b,2)+r^2(bd+1)G(b,1)G(b,2)}$$

From these calculations, we conclude that the general form of the solution of x_n is

$$\begin{aligned} x_{2k} &= \frac{b}{G(b,k)}, \text{ for } k = 1,2,... \\ x_{2k+1} &= \frac{c}{G(c,k+1)}, \text{ for } k = 1,2,... \end{aligned}$$

Further, we conclude that

$$y_{2k} = a\prod_{j=1}^{k} G(c,j)(\sum_{j=0}^{k-1} acr^j \frac{\prod_{j=1}^{k} G(c,j)}{G(c,k-j)} + r^k \prod_{j=1}^{k} G(c,j))^{-1}$$

$$y_{2k+1} = d\prod_{j=1}^{k} G(b,j)(\sum_{j=0}^{k-1} bdr^j \frac{\prod_{j=1}^{k} G(b,j)}{G(b,k-j)} + r^k(bd+1) \prod_{j=1}^{k} G(b,j))^{-1}$$

We study now the special case that $r = 1$. It is well-known that

$$\prod_{j=1}^{k} G(b,j) = \prod_{j=1}^{k}(1+jb) = b^k \frac{\Gamma(k+1+\frac{1}{b})}{\Gamma(1+\frac{1}{b})}$$

On the other hand, we obtain for all $0 \leq j < k$ that

$$\frac{\prod_{j=1}^{k} G(b,j)}{G(b,k-j)} = \prod_{l=0}^{k-j-1}(1+lb) \prod_{l=k-j+1}^{k}(1+lb) =$$

$$b^{k-j}\frac{\Gamma(k-j-1+\frac{1+b}{b})}{\Gamma(\frac{1}{b})} \frac{b^{k+1}}{b^{k-j+1}\Gamma(k-j+\frac{1+b}{b})} \Gamma(k+\frac{1+b}{b}) =$$

$$(\Gamma(\frac{1}{b}))^{-1} b^k \Gamma(k+\frac{1+b}{b})(k-j-1+\frac{1+b}{b})^{-1}$$

Hence,

$$\sum_{j=0}^{k-1} bdr^j \frac{\prod_{j=1}^{k} G(b,j)}{G(b,k-j)} = \frac{b^{k+1}d}{\Gamma(\frac{1}{b})}\Gamma(k+\frac{1+b}{b}) \sum_{j=0}^{k-1} \frac{1}{k-j-1+\frac{1+b}{b}} =$$

$$\frac{b^{k+1}d}{\Gamma(\frac{1}{b})}\Gamma(k+\frac{1+b}{b})(\Psi(k+\frac{1+b}{b}) - \Psi(\frac{1+b}{b}))$$

where the symbol Ψ denotes the Psi (Digamma) function. We obtain now the formula

$$y_{2k+1} = (\Psi(k+\frac{1+b}{b}) - \Psi(\frac{1+b}{b}) + b + \frac{1}{d})^{-1}$$

Similarly, we obtain

$$y_{2k} = (\Psi(k+\frac{1+c}{c}) - \Psi(\frac{1+c}{c}) + \frac{1}{a})^{-1}$$

We consider also the special case that $r = 1$. We conclude first that the general form of the solution for x_n is

$$x_{2k} = \begin{cases} \frac{b}{b-1}, & k = 2l+1 \\ b\ , & k = 2l \end{cases}, \text{ and}$$
$$x_{2k+1} = \begin{cases} c, & k = 2l+1 \\ \frac{c}{c-1}, & k = 2l \end{cases}$$

where $l = 0, 1, 2, ...$

Secondly, the general form of y_{2k+1} (for $k = 1, 2, 3, ...$) is $\frac{d(b-1)}{E(k)}$, where

$$E(k) = \begin{cases} b - (l+1)db^2 + 2(l+1)bd - 1\ , & k = 2l+1 \\ (l+2)db^2 - b - (2l+3)bd + 1, & k = 2l+2 \end{cases} \quad \text{for } l = 0, 1, 2, ...$$

The general form of y_{2k} (for $k = 2, 3, 4, 5, ...$) is $\frac{a(c-1)}{D(k)}$, where

$$D(k) = \begin{cases} c + (l+1)ac^2 - 2(l+1)ac - 1\ , & k = 2l+2 \\ (2l+3)ac - c - (l+1)ac^2 + 1, & k = 2l+3 \end{cases} \quad \text{for } k = 1, 2, ...$$

3. Systems of three equations

In this section, we extend the results for the system (2.1) in the special case that $r = -1$. To perform this extension, we use the existing formulae of system (2.1). We consider now the following system

$$\begin{aligned} x_{n+1} &= \frac{x_{n-1}}{x_{n-1} - 1}, \\ y_{n+1} &= \frac{y_{n-1}}{x_n y_{n-1} - 1}, \\ z_{n+1} &= \frac{z_{n-1}}{x_n y_n z_{n-1} - 1}. \end{aligned} \tag{3.1}$$

In the general case, we take the initial values as

$$\begin{aligned} x_0 &= b,\ x_{-1} = c \\ y_0 &= a,\ y_{-1} = d \\ z_0 &= f,\ z_{-1} = g \end{aligned} \tag{3.2}$$

Let

$$p = b - b^2d + 2bd - 1, q = 2bd - b^2d \text{ and } r = 2b^2d - b - 3bd + 1.$$

Then, we express the $E(k)$'s as follows

$$E(2l+1) = b-(l+1)db^2+2(l+1)bd-1 = (2bd-b^2d)l+(b-b^2d+2bd-1) = p+ql,$$
$$E(2l+2) = (l+2)db^2-b-(2l+3)bd+1 = (b^2d-2bd)l+(2b^2d-b-3bd+1) = r-ql.$$

According to system (3.1) and (3.2) we obtain

$$\begin{aligned} x_1 &= \frac{x_{-1}}{x_{-1}-1} = \frac{c}{c-1} \\ y_1 &= \frac{y_{-1}}{x_0 y_{-1}-1} = \frac{d}{bd-1} \\ z_1 &= \frac{z_{-1}}{x_0 y_0 z_{-1}-1} = \frac{g}{bag-1} \end{aligned}$$

$$x_2 = \frac{x_0}{x_0-1} = \frac{b}{b-1},$$

$$y_2 = \frac{y_0}{x_1 y_0 - 1} = \frac{a}{\frac{c}{c-1}a-1} = \frac{a(c-1)}{ac-c+1} = \frac{a(c-1)}{D(1)}$$

$$z_2 = \frac{z_0}{x_1 y_1 z_0 - 1} = \frac{f}{\frac{c}{c-1}\frac{d}{bd-1}f-1} = \frac{f(c-1)(bd-1)}{cdf-(c-1)(bd-1)}$$

$$x_3 = \frac{x_1}{x_1-1} = c,$$

$$y_3 = \frac{y_1}{x_2 y_1 - 1} = \frac{\frac{d}{bd-1}}{\frac{b}{b-1}\frac{d}{bd-1}-1} = \frac{d(b-1)}{bd-(b-1)(bd-1)} = \frac{d(b-1)}{E(1)},$$

$$z_3 = \left(\frac{b}{b-1}\frac{a(c-1)}{D(1)}\frac{g}{bag-1}-1\right)^{-1}\frac{g}{bag-1} =$$

$$\frac{g(b-1)D(1)}{bag(c-1)-(b-1)(bag-1)D(1)}$$

$$x_4 = \frac{x_2}{x_2-1} = b$$

$$y_4 = \frac{y_2}{x_3 y_2 - 1} = \frac{\frac{a(c-1)}{ac-c+1}}{c\frac{a(c-1)}{ac-c+1}-1} = \frac{a(c-1)}{ac(c-1)-(ac-c+1)} = \frac{a(c-1)}{D(2)}$$

$$\begin{aligned} z_4 &= \frac{z_2}{x_3 y_3 z_2 - 1} = \frac{\frac{f(c-1)(bd-1)}{cd-(c-1)(bd-1)}}{c\frac{d(b-1)}{E(1)}\frac{f(c-1)(bd-1)}{cdf-(c-1)(bd-1)}-1} = \\ &\frac{f(c-1)(bd-1)E(1)}{cdf(b-1)(c-1)(bd-1)-E(1)(cdf-(c-1)(bd-1))} = \\ &\frac{f(c-1)(bd-1)E(1)}{H} \end{aligned}$$

We adopt the following notations for the next work

$$H = cdf(b-1)(c-1)(bd-1) - E(1)(cdf - (c-1)(bd-1)),$$

$$T = bag(c-1) - (b-1)(bag-1)D(1).$$

Thus

$$z_4 = \frac{f(c-1)(bd-1)E(1)}{H}$$

$$x_5 = \frac{x_3}{x_3-1} = \frac{c}{c-1}$$

$$y_5 = \frac{\frac{d(b-1)}{bd-(b-1)(bd-1)}}{b\frac{d(b-1)}{bd-(b-1)(bd-1)}-1} = \frac{d(b-1)}{bd(b-1)-(bd-(b-1)(bd-1))} = \frac{d(b-1)}{E(2)}$$

$$z_5 = \frac{z_3}{x_4y_4z_3-1} = \frac{\frac{g(b-1)D(1)}{bag(c-1)-(b-1)(bag-1)D(1)}}{b\frac{a(c-1)}{D(2)}\frac{g(b-1)D(1)}{bag(c-1)-(b-1)(bag-1)D(1)}-1} =$$

$$\frac{g(b-1)D(1)D(2)}{bag(b-1)(c-1)D(1)-(bag(c-1)-(b-1)(bag-1)D(1))D(2)} =$$

$$\frac{g(b-1)D(1)D(2)}{bag(b-1)(c-1)D(1)-TD(2)}$$

$$y_6 = \frac{a(c-1)}{D(3)}$$

$$z_6 = \frac{z_4}{x_5y_5z_4-1} = \frac{\frac{f(c-1)(bd-1)E(1)}{H}}{\frac{c}{c-1}\frac{d(b-1)}{E(2)}\frac{f(c-1)(bd-1)E(1)}{H}-1}$$

$$= \frac{f(c-1)(bd-1)E(1)E(2)}{cdf(b-1)(bd-1)E(1)-HE(2)}$$

$$y_7 = \frac{d(b-1)}{E(3)}$$

$$z_7 = \frac{z_5}{x_6y_6z_5-1} = \frac{\frac{g(b-1)D(1)D(2)}{bag(b-1)(c-1)D(1)-TD(2)}}{\frac{b}{b-1}\frac{a(c-1)}{D(3)}\frac{g(b-1)D(1)D(2)}{bag(b-1)(c-1)D(1)-TD(2)}-1} =$$

$$\frac{g(b-1)D(1)D(2)D(3)}{bag(c-1)D(1)D(2)-bag(b-1)(c-1)D(1)D(3)+TD(2)D(3)}$$

$$z_8 = \frac{z_6}{x_7y_7z_6-1} = \frac{\frac{f(c-1)(bd-1)E(1)E(2)}{cdf(b-1)(bd-1)E(1)-HE(2)}}{c\frac{d(b-1)}{E(3)}\frac{f(c-1)(bd-1)E(1)E(2)}{cdf(b-1)(bd-1)E(1)-HE(2)}-1} =$$

$$\frac{f(c-1)(bd-1)E(1)E(2)E(3)}{cdf(b-1)(c-1)(bd-1)E(1)E(2)-cdf(b-1)(bd-1)E(1)E(3)+HE(2)E(3)}$$

We denote the denominator of z_{2k} by $F(k)$. We set

$$K = cdf(b-1)(bd-1).$$

We set

$$\text{Product(1,...,M)}=E(1)E(2)...E(M)$$

For all numbers $k = 2n+3$ we conclude in general

$$F(k) = K\{S_1 - (c-1)S_2\} - H\frac{\text{Product(1,...,2n+2)}}{E(1)}$$

and for all numbers $k = 2n+2$:

$$F(k) = K\{S_3 + (c-1)S_4\} + H\frac{\text{Product(1,...,2n+1)}}{E(1)}.$$

$$S_1 = \sum_{m=0}^{n} \frac{\text{Product(1,...,2n+2)}}{E(2m+2)},$$

$$S_2 = \sum_{m=1}^{n} \frac{\text{Product(1,...,2n+2)}}{E(2m+1)},$$

$$S_3 = -\sum_{m=0}^{n-1} \frac{\text{Product(1,...,2n+1)}}{E(2m+2)},$$

$$S_4 = \sum_{m=1}^{n} \frac{\text{Product(1,...,2n+1)}}{E(2m+1)}.$$

Our helpful lemma (1.1) implies

$$1)\ \frac{\text{Product(1,...,2n+1)}}{E(2m+2)} = (\prod_{l=0}^{m-1}(r-ql))(\prod_{l=m+1}^{n-1}(r-ql))(\prod_{l=0}^{n}(p+ql)) =$$

$$\prod_{l=0}^{m-1}(r-ql) = (-q)^m\frac{\Gamma(m-1-\frac{r-q}{q})}{\Gamma(-\frac{r}{q})}$$

$$\prod_{l=m+1}^{n-1}(r-ql) = \frac{(-q)^n}{(-q)^{m+1}\Gamma(m-\frac{r-q}{q})}\Gamma(n-1-\frac{r-q}{q})$$

$$\prod_{l=0}^{n}(p+ql) = q^{n+1}\frac{\Gamma(n+\frac{p+q}{q})}{\Gamma(\frac{p}{q})}$$

Hence, we obtain

$$\frac{\text{Product(1,...,2n+1)}}{E(2m+2)} =$$

$$-(-q^2)^n(\Gamma(-\tfrac{r}{q})\Gamma(\tfrac{p}{q}))^{-1}\Gamma(n-1-\tfrac{r-q}{q})\Gamma(n+\tfrac{p+q}{q})\frac{\Gamma(m-1-\frac{r-q}{q})}{\Gamma(m-\frac{r-q}{q})} =$$

$$-L(n)\Gamma(n-1-\tfrac{r-q}{q})(m-1-\tfrac{r-q}{q})^{-1}$$

$$2)\ \frac{\text{Product(1,...,2n+1)}}{E(2m+1)} = (\prod_{l=0}^{n-1}(r-ql))(\prod_{l=0}^{m-1}(p+ql))(\prod_{l=m+1}^{n}(p+ql)) =$$

$$L(n)\Gamma(n-1-\tfrac{r-q}{q})(m-1+\tfrac{p+q}{q})^{-1}$$

$$3)\ \frac{\text{Product(1,...,2n+2)}}{E(2m+2)} = (\prod_{l=0}^{n}(p+ql))(\prod_{l=0}^{m-1}(r-ql))(\prod_{l=m+1}^{n}(r-ql)) =$$

$$qL(n)\Gamma(n-\tfrac{r-q}{q})(m-1-\tfrac{r-q}{q})^{-1}$$

$$4)\ \frac{\text{Product(1,...,2n+2)}}{E(2m+1)} = (\prod_{l=0}^{n}(r-ql))(\prod_{l=0}^{m-1}(p+ql))(\prod_{l=m+1}^{n}(p+ql)) =$$

$$-qL(n)\Gamma(n-\tfrac{r-q}{q})(m-1+\tfrac{p+q}{q})^{-1}$$

where

$$L(n) = (-q^2)^n(\Gamma(-\tfrac{r}{q})\Gamma(\tfrac{p}{q}))^{-1}\Gamma(n+\tfrac{p+q}{q}).$$

In general we obtain for even numbers $k = 2n+2$:

$$F(k) = L(n)\Gamma(n-1-\tfrac{r-q}{q})(K(S_5+(c-1)S_6)+\tfrac{q}{p}H)$$

where

$$S_5 = \sum_{m=0}^{n-1}(m-1-\tfrac{r-q}{q})^{-1},$$
$$S_6 = \sum_{m=1}^{n}(m-1+\tfrac{p+q}{q})^{-1}.$$

Hence

$$F(k) = L(n)\Gamma(n-1-\tfrac{r-q}{q})Y(k)$$

where

$$Y(n) = K\{\Psi(-\tfrac{r-nq}{q}) - \Psi(-\tfrac{r}{q}) + (c-1)(\Psi(\tfrac{p+q+nq}{q}) - \Psi(\tfrac{p+q}{q}))\} + \tfrac{q}{p}H.$$

On the other hand we have

$$\sum_{m=0}^{n}\tfrac{\text{Product}(1,...,2n+2)}{E(2m+2)} = qL(n)\Gamma(n-\tfrac{r-q}{q})(\Psi(\tfrac{q-r+nq}{q}) - \Psi(-\tfrac{r}{q}))$$
$$\sum_{m=1}^{n}\tfrac{\text{Product}(1,...,2n+2)}{E(2m+1)} = -qL(n)\Gamma(n-\tfrac{r-q}{q})(\Psi(\tfrac{p+q+nq}{q}) - \Psi(\tfrac{p+q}{q}))$$

In general we obtain the following formula for $F(k)$ in case of $k = 2n+3$:

$$K\{\sum_{m=0}^{n}\tfrac{\text{Product}(1,...,2n+2)}{E(2m+2)} - (c-1)\sum_{m=1}^{n}\tfrac{\text{Product}(1,...,2n+2)}{E(2m+1)}\} - H\tfrac{\text{Product}(1,...,2n+2)}{E(1)} = -q^3L(n)\Gamma(n-\tfrac{r-q}{q})Z(k)$$

where

$$Z(n) = K\{\Psi(\tfrac{q-r+nq}{q}) - \Psi(-\tfrac{r}{q}) + (c-1)(\Psi(\tfrac{p+q+nq}{q}) - \Psi(\tfrac{p+q}{q}))\} + \tfrac{q}{p}H.$$

The numerator of z_{2k} is equal to

$$f(c-1)(bd-1)E(1)E(2)...E(k-1).$$

Regarding the numerator of z_{2k}, we have two cases:

(1) Assume $k = 2n+3$: In this case

$$E(1)E(2)...E(k-1) = E(1)E(2)...E(2n+2),$$
$$E(1)E(3)...E(2n+1) = \prod_{l=0}^{n}(p+ql\ ) = q^{n+1}\frac{\Gamma(n+\frac{p+q}{q})}{\Gamma(\frac{p}{q})},$$
$$E(2)E(4)...E(2n+2) = \prod_{l=0}^{n}(r-ql) = (-q)^{n+1}\frac{\Gamma(n-\frac{r-q}{q})}{\Gamma(-\frac{r}{q})}.$$

Hence, we obtain the relation

$$E(1)...E(2n+2) = -q^2L(n)\Gamma(n-\tfrac{r-q}{q}).$$

(2) Assume $k = 2n+2$: In this case

$$E(1)E(2)...E(k-1) = E(1)...E(2n+1) = \prod_{l=0}^{n-1}(r-ql)\prod_{l=0}^{n}(p+ql) = (-q)^n\frac{\Gamma(n-1-\frac{r-q}{q})}{\Gamma(-\frac{r}{q})}q^{n+1}\frac{\Gamma(n+\frac{p+q}{q})}{\Gamma(\frac{p}{q})} = -q^3L(n)\Gamma(n-1-\tfrac{r-q}{q}).$$

So, we reach the following formulae

$$z_{2k} = qf(c-1)(bd-1)\tfrac{1}{Y(n)} \text{ if } k = 2n+2,$$
$$z_{2k} = -qf(c-1)(bd-1)\tfrac{1}{Z(n)}, \text{ if } k = 2n+3.$$

We set

$$W(n) = K((c-1)(\Psi(\tfrac{p+q+nq}{q}) - \Psi(\tfrac{p+q}{q})) - \Psi(-\tfrac{r}{q})) + \tfrac{q}{p}H.$$

We rewrite the general solution as

$$z_{2k} = -qf(c-1)(bd-1)(K\Psi(\tfrac{1}{q}(q-r+nq)) + W(n))^{-1},$$

if $k = 2l+3$ (for $l = 1, 2, ...$),

$$z_{2k} = qf(c-1)(bd-1)(K\Psi(-\tfrac{1}{q}(r-nq)) + W(n))^{-1},$$

if $k = 2n+2$ (for $n = 2, 3, ...$) .

We rewrite also

$$D(2l+1) = (2ac - ac^2)l + (ac - c + 1) = v + ul$$
$$D(2l+2) = (ac^2 - 2ac)l + (c + ac^2 - 2ac - 1) = s - ul$$

We use the notation

$$X(l) = bag(c-1)\{\Psi(\tfrac{u+v+lu}{u}) - \Psi(\tfrac{u+v}{u}) - (b-1)\Psi(-\tfrac{s}{u}))\} + \tfrac{u}{v}T$$

We can anologous to our previous computations conclude that: The general solution can be written (for $n = 1, 2, 3, ...$) as

$$z_{2k+1} = gu(b-1)(X(n) + bag(b-1)(c-1)\Psi(-\tfrac{1}{u}(s-nu))^{-1}$$

if $k = 2n+1$,

$$z_{2k+1} = -gu(b-1)(X(n) + bag(b-1)(c-1)\Psi(\tfrac{1}{u}(u-s+nu)))^{-1}$$

if $k = 2n+2$.

We summarize our calculations in the next theorem.

Theorem 3.1. *Let $a, b, c, d \in \mathbb{R}$. If*

$$\begin{aligned} x_0 &= b,\ x_{-1} = c \\ y_0 &= a,\ y_{-1} = d \\ z_0 &= f,\ z_{-1} = g \end{aligned}$$

then the solution of the system (3.1) and (3.2) is given by

$$x_{2k+1} = \begin{cases} c\ ,\ k = 2l+1 \\ \frac{c}{c-1},\quad k = 2l \end{cases}$$
$$x_{2k} = \begin{cases} \frac{b}{b-1}\ ,\ k = 2l+1 \\ b,\quad k = 2l \end{cases}$$

for $l = 0, 1, 2, 3, ...$

$$y_{2k+1} = \frac{d(b-1)}{E(k)}$$
$$y_{2k} = \frac{a(c-1)}{D(k)}$$

for $k = 2, 3, 4, ...$

$$z_{2k} = \frac{-qf(c-1)(bd-1)}{K\Psi(\frac{1}{q}(q-r+nq)) + W(l)}, k = 2l+3$$

$$z_{2k+1} = z_{2k+1} = \frac{gu(b-1)}{X(l) + bag(b-1)(c-1)\Psi(-\frac{1}{u}(s-lu))}, k = 2l+1$$

$$z_{2k+1} = \frac{-gu(b-1)}{X(l) + bag(b-1)(c-1)\Psi(\frac{1}{u}(u-s+lu))}, k = 2l+2$$

for $l = 1, 2, 3, ...$

$$z_{2k} = \frac{qf(c-1)(bd-1)}{K\Psi(-\frac{1}{q}(r-lq)) + W(l)}, k = 2l+2$$

for $l = 2, 3, 4, ...$

Next we list some special cases of theorem 3.1.

Corollary 3.2. *Let* $a, b, c, d \in \mathbb{R}$. *If*

$$x_0 = a, \ \ x_{-1} = b, y_0 = y_{-1} = 0, z_0 = c, \ \ z_{-1} = d$$

then the solution of the system (3.1) *and* (3.2) *is*

$$\begin{aligned} x_{2k+1} &= \begin{cases} b \ , & k = 2l+1 \\ \frac{b}{b-1}, & k = 2l \end{cases} \\ x_{2k} &= \begin{cases} \frac{a}{a-1} \ , & k = 2l+1 \\ a, & k = 2l+2 \end{cases} \end{aligned}$$

for $l = 0, 1, 2, 3, ...$

$$y_n = 0$$

for $n = 1, 2, 3, ...$

$$z_{2k+1} = (-1)^{k+1} d, \quad z_{2k+2} = (-)^{k+1} c$$

for $k = 0, 1, 2, 3, ...$

Corollary 3.3. *Let* $a, b, c, d \in \mathbb{R}$. *If*

$$x_0 = a, \ x_{-1} = d, y_0 = b, \ y_{-1} = 0, z_0 = c,, \ \text{and} \ z_{-1} = 0,$$

then the solution of the system (3.1) *and* (3.2) *is*

$$\begin{aligned} x_{2k+1} &= \begin{cases} d \ , & k = 2l+1 \\ \frac{d}{d-1}, & k = 2l \end{cases} \\ x_{2k} &= \begin{cases} \frac{a}{a-1} \ , & k = 2l+1 \\ a, & k = 2l \end{cases} \\ y_{2k} &= \begin{cases} -\frac{b(d-1)}{d+lbd^2-kbd-1}, & k = 2l+1 \\ \frac{b(d-1)}{d+lbd^2-kbd-1}, & k = 2l+2 \end{cases} \end{aligned}$$

for $l = 0, 1, 2, 3, ...$

$$y_{2k+1} = 0, \ \ z_{2k+1} = 0, \ \ z_{2k+2} = (-)^k c$$

for $k = 0, 1, 2, 3, ...$

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