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UNVEILING CHANGE-POINT DETECTION WITH ARMA-GARCH MODELS

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Abstract. This paper delves into the asymptotic theory of a Cumulative Sum (CUSUM)-type test designed for detecting change points in the variance of AutoRegressive Moving-Average models, incorporating Generalized Autoregressive Conditional Heteroscedasticity innovations. We demonstrate that, under the null hypothesis (no change), the CUSUM test statistic converges to the supremum of a standard Brownian bridge. Using a Monte Carlo simulation, we highlight the strong performance of our proposed test when compared to the methods by Song and Kang. Furthermore, we provide a real-world data analysis to showcase the practical application of our test.

1. Introduction

Since [21], the identification and analysis of change-points have gained significance in the field of time series modeling. For historical background and general review, references such as [2], [8], [11] and [23] provide valuable insights. For a more recent overview of change-point detection applications and additional details on the topic, see [1, 3, 5, 13, 14, 15, 16, 25] and the references therein.

While differencing is commonly employed to achieve stationarity in time series (see, e.g., [12]), this work focuses on detecting structural changes, such as shifts in variance, using ARMA-GARCH models.

The ARMA-GARCH processes play an important role in the modeling of financial series and therefore, in financial applications, it is more convenient to fit return series by ARMA with GARCH innovations. Estimating volatility, a key element in asset pricing models, relies on the squared returns. However, sudden shifts in the probability distribution of the underlying stochastic process can occur at specific points in the time series, often leading to unfavorable consequences. Specifically, the mean or covariance structure of this process may undergo sudden changes at unspecified dates. Several researchers have emphasized the dangers that emerge when such points of transition are not analyzed or detected. Hence, it is intriguing to devise a test aimed at identifying variations in the variance within ARMA-GARCH model.

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The literature on the problem of change-point detection in the parameters of ARMA-GARCH models is not extensive enough. The evaluation of a parameter shift in these models has been addressed using the CUSUM test based on the quasi-maximum likelihood estimator (see [19]). The challenge of detecting a parameter change in AR(1)-GARCH(1,1) models has been explored through the application of the residual CUSUM test (see [18]). A change-point estimator for both linear and nonlinear time series models, with an application to ARMA-GARCH models, has also been proposed by [20]. Moreover, a score test and a residual-based CUSUM test have been developed for testing parameter changes in ARMA-GARCH models by [24]. A sequential procedure for detecting parameter changes in these models, building on the testing procedure of [5], has also been investigated by [25].

The authors in [24] addressed the fact that the residual-based test for ARMA-GARCH model shows good performance in testing for GARCH parameter change but can not detect any change in parameters belonging to ARMA part. Indeed the empirical powers of their proposed residual-based test for testing the change in ARMA parameter are close to the significance level 0.05. This means that their proposed test can not detect any change in parameter belonging to ARMA. In this study, we propose square-based CUSUM test as an alternative.

The rest of the paper is structured as follows. In Section 2, we review the construction of the CUSUM test statistic. Section 3 presents the limiting distribution of the test statistic under the null hypothesis. In Section 4, a numerical experiment is performed, and our testing approach is applied to the Korean won-Japanese yen (KRW/JPY) exchange rate series. We draw conclusions in Section 5.

2. CUSUM test statistic

Consider the observation of a stochastic phenomenon $(y_1, y_2, \ldots, y_n)$ known to be generated by the square of an ARMA(P,Q) model with GARCH (p,q) innovations process. The model is formally defined as

$$y_t = X_t^2, \tag{2.1}$$

where

$$X_t - \sum_{i=1}^{P} a_i X_{t-i} = u_t - \sum_{i=1}^{Q} b_i u_{t-i} \tag{2.2}$$

with the polynomials $A(z) = 1 - a_1 z - \cdots - a_P z^P$ and $B(z) = 1 - b_1 z - \cdots - b_Q z^Q$ having no common factors and having roots outside the unit disk, with $a_P b_Q \neq 0$ and where (u_t) is a GARCH white noise of the form

$$u_t = \sigma_t \varepsilon_t, \tag{2.3}$$

such that $\mathbb{E}u_t^4 < \infty$ and

$$\mathbb{E}(\varepsilon_t) = 0,\ \mathbb{E}(\varepsilon_t^2) = 1,$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i u_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2, \; t = 1, 2, \dots \tag{2.4}$$

Here, α_i for $i = 1, 2, \dots, q$ and β_j for $j = 1, 2, \dots, p$ are nonnegative constants and α_0 is a (strictly) positive constant. This ensures the strict positivity of the conditional variance of the process (u_t), and

$$\sum_{i=1}^{q} \alpha_i + \sum_{j=1}^{p} \beta_j < 1, \tag{2.5}$$

guarantees the stationarity of the process (u_t).

Assuming that the polynomials $A(z)$ have roots outside the unit disk, we can say that the process (X_t) is a causal function of (u_t). Therefore, there exist constants (c_j) such that $\sum_{j=0}^{\infty} |c_j| < \infty$ and

$$X_t = \sum_{j=0}^{\infty} c_j u_{t-j} = C(L) u_t, \text{ for all } t, \tag{2.6}$$

where L denotes the lag-operator ($L^j X_t = X_{t-j}$) and

$$C(L) = \sum_{j=0}^{\infty} c_j L^j, \tag{2.7}$$

with $0 < |C(1)| < \infty$ and $\sum_{j=0}^{\infty} c_j^2 < \infty$.
The sequence (c_j), in (2.6) is determined by the relation

$$A^{-1}(z) B(z) = 1 + \sum_{j=1}^{\infty} c_j z^j. \tag{2.8}$$

Let's now present the innovation corresponding to the square of the process (u_t): $\nu_t = u_t^2 - \sigma_t^2$. Replacing σ_{t-j}^2 by $u_{t-j}^2 - \nu_{t-j}$ in (2.4), we obtain (see, [13])

$$u_t^2 = \omega + \nu_t + \sum_{j=1}^{\infty} \lambda_j \nu_{t-j}, \tag{2.9}$$

with

$$\omega = \left(1 - \sum_{i=1}^{\max(p,q)} \theta_i \right)^{-1} \alpha_0$$

and the $\lambda_j, \; j \geqslant 1$ are given by the relationship

$$\left(1 - \sum_{i=1}^{\max(p,q)} \theta_i x^i \right)^{-1} \left(1 - \sum_{j=1}^{p} \beta_j x^j \right) = 1 + \sum_{j=1}^{\infty} \lambda_j x^j.$$

The question at hand is whether the phenomenon $y_t = X_t^2$ has undergone a sudden shift at an unknown point in time t^*, which may be understood as a sudden modification in the variance of the process (X_t).

Here and in the sequel, we denote by $\gamma_i = \mathbb{E}(X_0 X_i)$ the autocovariance function of X_t, then we have $\gamma_i = \omega \bar{\gamma}_i$, where $\bar{\gamma}_i = \sum_{j=0}^{\infty} c_j c_{j+i}$, and ω is the unconditional variance of u_t (see, [7]).

Let's formulate the change-point problem as follows:

$$\mathbb{E}(y_t) = \begin{cases} \gamma_0 & \text{if } t < t^* \\ \gamma_0 + \delta & \text{if } t \geq t^*. \end{cases}$$

We next address the testing problem pertaining to the null hypothesis stated as follows

$$H_0 : \ \mathbb{E}(y_t) = \gamma_0,$$

as opposed to the alternative hypothesis

$$H_1 : \ \mathbb{E}(y_t) = \gamma_0 + \delta,$$

where δ denotes a real number that is not equal to zero.

The deviations and cumulative sum of the deviations are the two statistics provided below:

$$d_r = y_r - \overline{y},$$

where $\overline{y} = \dfrac{1}{n} \sum_{i=1}^{n} y_i = \dfrac{1}{n} \sum_{i=1}^{n} X_i^2$ and

$$S_r = \sum_{i=1}^{r} d_i.$$

Under the null hypothesis H_0, the sums S_r exhibit a mean of zero. However, a sudden change in the mean of y_t (which is equivalent to a sudden change in the variance of the process X_t) introduces bias after t^*. It is demonstrated that

$$E(S_r) = \frac{1}{n}(t^* - 1)(r - n)\delta.$$

This indicates how responsive the cumulative sum is to a potential sudden change. Thus we define the test statistic, as in [13], by

$$K_n = \sup_{0 \leqslant t \leqslant 1} |Y_n(t)|, \tag{2.10}$$

where

$$Y_n(t) = \frac{1}{\sqrt{n}} S_{[nt]}$$

and $[x]$ represents the greatest integer less than or equal to x.

Let $D[0;1]$ the space of functions defined on $[0;1]$ continuous on the left and admitting a limit on the right endowed with the topology of Skorokhod. According to [6], Y_n is a random element of $D[0;1]$.

Before giving the main results, let us introduce a simple polynomial decomposition.

We have

$$y_t = X_t^2 = (C(L)u_t)^2 = \left(\sum_{j=0}^{\infty} c_j u_{t-j}\right)^2$$
$$= \sum_{j=0}^{\infty} c_j^2 u_{t-j}^2 + 2\sum_{i=1}^{\infty}\sum_{j=0}^{\infty} c_j c_{j+i} u_{t-j} u_{t-j-i}$$
$$= X_t^{(1)} + 2X_t^{(2)}, \tag{2.11}$$

where
$X_t^{(1)} = \sum_{j=0}^{\infty} c_j^2 u_{t-j}^2 = f_0(L) u_t^2$
and
$X_t^{(2)} = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} c_j c_{j+i} u_{t-j} u_{t-j-i} = \sum_{i=1}^{\infty} f_i(L) u_t u_{t-i}$,
with

$$f_i(L) = \sum_{j=0}^{\infty} c_j c_{j+i} L^j = \sum_{j=0}^{\infty} f_{ij} L^j. \tag{2.12}$$

Applying the Beveridge-Nelson decomposition to the lag polynomial $f_i(L)$ (see, [22]), we have

$$f_i(L) = f_i(1) - (1-L)\tilde{f}_i(L) \tag{2.13}$$

with

$$\tilde{f}_i(L) = \sum_{j=0}^{\infty} \tilde{f}_{ij} L^j,\ \ \tilde{f}_{ij} = \sum_{k=j+1}^{\infty} f_{ik} = \sum_{k=j+1}^{\infty} c_k c_{k+i}.$$

Using the decomposition (2.13) on both components of (2.11), we have

$$X_t^{(1)} = f_0(1)u_t^2 - (1-L)\tilde{f}_0(L)u_t^2 = f_0(1)u_t^2 - (1-L)\tilde{X}_t^{(1)} \tag{2.14}$$

and

$$X_t^{(2)} = \sum_{i=1}^{\infty} f_i(1) u_t u_{t-i} - (1-L)\sum_{i=1}^{\infty} \tilde{f}_i(L) u_t u_{t-i} = \sum_{i=1}^{\infty} \bar{\gamma}_i u_t u_{t-i} - (1-L)\tilde{X}_t^{(2)}, \tag{2.15}$$

where

$$\tilde{X}_t^{(1)} = \tilde{f}_0(L)u_t^2,\ \ \tilde{f}_0(L) = \sum_{j=0}^{\infty} \tilde{f}_{0j} L^j,\ \ \tilde{f}_{0j} = \sum_{k=j+1}^{\infty} f_{0k} = \sum_{k=j+1}^{\infty} c_k^2,$$

$$\bar{\gamma}_i = f_i(1) = \sum_{j=0}^{\infty} c_j c_{j+i}$$

and

$$\tilde{X}_t^{(2)} = \sum_{i=1}^{\infty} \tilde{f}_i(L) u_t u_{t-i}.$$

Let

$$z_r = y_r - \mathbb{E}(y_r) = X_r^2 - \gamma_0,$$

then we have

$$Y_n(t) = \frac{1}{\sqrt{n}} \sum_{r=1}^{[nt]} (z_r - \bar{z}) = Y_n^{(1)}(t) - Y_n^{(2)}(t),$$

where

$$Y_n^{(1)}(t) = \frac{1}{\sqrt{n}} \sum_{r=1}^{[nt]} z_r, \; Y_n^{(2)}(t) = \frac{1}{\sqrt{n}} \sum_{r=1}^{[nt]} \bar{z}.$$

From (2.11), (2.14) and (2.15), we have the decomposition

$$\begin{aligned}
Y_n^{(1)}(t) &= \frac{1}{\sqrt{n}} \sum_{r=1}^{[nt]} (X_r^2 - \gamma_0) \\
&= \frac{1}{\sqrt{n}} \sum_{r=1}^{[nt]} \left(\bar{\gamma}_0 (u_r^2 - \omega) + 2 \sum_{i=1}^{\infty} f_i(1) u_r u_{r-i} \right) \\
&+ \frac{1}{\sqrt{n}} \sum_{r=1}^{[nt]} \left(\tilde{X}_{r-1}^{(1)} - \tilde{X}_r^{(1)} \right) + \frac{2}{\sqrt{n}} \sum_{r=1}^{[nt]} \left(\tilde{X}_{r-1}^{(2)} - \tilde{X}_r^{(2)} \right) \\
&= \frac{1}{\sqrt{n}} \sum_{r=1}^{[nt]} \left(\bar{\gamma}_0 (u_r^2 - \omega) + 2 \sum_{i=1}^{\infty} f_i(1) u_r u_{r-i} \right) \\
&+ \frac{1}{\sqrt{n}} \left(\tilde{X}_0^{(1)} - \tilde{X}_{[nt]}^{(1)} \right) + \frac{2}{\sqrt{n}} \left(\tilde{X}_0^{(2)} - \tilde{X}_{[nt]}^{(2)} \right) \\
&= \frac{1}{\sqrt{n}} \sum_{r=1}^{[nt]} Z_r + \frac{1}{\sqrt{n}} \left(\tilde{X}_0^{(1)} - \tilde{X}_{[nt]}^{(1)} \right) + \frac{2}{\sqrt{n}} \left(\tilde{X}_0^{(2)} - \tilde{X}_{[nt]}^{(2)} \right), \qquad (2.16)
\end{aligned}$$

where

$$\begin{aligned}
Z_r &= \bar{\gamma}_0 (u_r^2 - \omega) + 2 \sum_{i=1}^{\infty} f_i(1) u_r u_{r-i} \\
&= \bar{\gamma}_0 (u_r^2 - \omega) + 2 u_r \sum_{i=1}^{\infty} \bar{\gamma}_i u_{r-i},
\end{aligned}$$

and $\gamma_0 = \mathbb{E}(X_r^2) = \omega \bar{\gamma}_0$.

In the following section, we analyze the asymptotic properties of the test statistic K_n.

3. Asymptotic distribution of K_n

From this point forward, we will adopt the following assumptions:

A.1 $\quad \mathbb{E}\left(u_t^4\right) < \infty$.

A.2 i) $\sum_{j=0}^{\infty} |\bar{\gamma}_j| < \infty$.
ii) $\sum_{j=0}^{\infty} |\tilde{f}_{ij}| < \infty, \ \forall i \geqslant 0$.

A.3 i) (Z_t) is uniformly integrable with a dominating random variable Y satisfying
$\mathbb{E}Y^2 < +\infty$ and such that
ii) $\mathbb{P}\left[|Z_t| \geqslant x\right] \leqslant c\mathbb{P}\left[|Y| \geqslant x\right]$, for each $x \geqslant 0$, $t \geqslant 1$ and for some nonnegative constant c.

A.4 $\sup_k \mathbb{E}\left((u_k^2)^{2+\eta}\right) < \infty, \ \forall \eta > 0$.

In the following, we shall assume that $\sigma_Z \neq 0$.

Theorem 3.1. *If Assumptions A.1-A.4 hold then under H_0, in $D[0;1]$,*

$$\frac{1}{\sigma_Z} \sup_{0 \leqslant t \leqslant 1} Y_n(t) \xrightarrow{d} \sup_{0 \leqslant t \leqslant 1} W^0(t)$$

and

$$\frac{1}{\sigma_Z} K_n \xrightarrow{d} \sup_{0 \leqslant t \leqslant 1} |W^0(t)|,$$

where $\sigma_Z^2 = \bar{\gamma}_0^2 \left(\mathbb{E}(u_r^4) - \omega^2\right) + 4\omega^2 \sum_{i=1}^{\infty} \bar{\gamma}_i^2$, $W^0(t)$ is a standard Brownian bridge and $\xrightarrow{d}$ denotes the convergence in distribution.

Proof. The demonstration of Theorem 3.1 relies on the two lemmas provided below.

Lemma 3.2. *If Assumptions A.1-A.4 hold, then under H_0, in $D[0;1]$, we have*

$$Y_n^{(1)}(t) \xrightarrow{d} \sigma_Z W(t),$$

where $W(t)$ is a standard Brownian motion.

Proof. We first have to prove that

$$\frac{1}{\sqrt{n}} \sum_{r=1}^{[nt]} Z_r \xrightarrow{\mathrm{d}} \sigma_Z W(t). \tag{3.1}$$

By construction, (Z_r) is a martingale difference sequence (mds). In fact, representing by $u_{\underline{i}}$ the $\sigma-$field generated by $u_{i'}$, $i' \leqslant i$, it's clear that $Z_{\underline{r-1}} \subset u_{\underline{r-1}}$ and that the sequence (Z_r) is adapted to the filtration $(Z_{\underline{r}})$. By Assumption A.3 i), (Z_r) is uniformly integrable hence it is bounded in L^1 i.e. $\mathbb{E}|Z_r| < \infty$, for any r.

Moreover we have

$$\begin{aligned}
\mathbb{E}\left(Z_r|Z_{\underline{r-1}}\right) &= \mathbb{E}\left(\mathbb{E}(Z_r|u_{\underline{r-1}})|Z_{\underline{r-1}}\right) \\
&= \mathbb{E}\left(\left\{\mathbb{E}\left((\bar{\gamma}_0(u_r^2-\omega)+2u_r\sum_{i=1}^{\infty}\bar{\gamma}_i u_{r-i}\right)|u_{\underline{r-1}}\right\}|Z_{\underline{r-1}}\right) \\
&= \mathbb{E}\left(\left\{\bar{\gamma}_0\mathbb{E}(u_r^2|u_{\underline{r-1}}-\omega)+2\left(\sum_{i=1}^{\infty}\bar{\gamma}_i u_{r-i}\right)\underbrace{\mathbb{E}(u_r|u_{\underline{r-1}})}_{=0}\right\}|Z_{\underline{r-1}}\right) \\
&= \mathbb{E}\left(\left\{\bar{\gamma}_0\mathbb{E}(u_r^2|u_{\underline{r-1}}-\omega)\right\}|Z_{\underline{r-1}}\right) \\
&= \bar{\gamma}_0\mathbb{E}\left(\left\{\mathbb{E}(u_r^2|u_{\underline{r-1}})\right\}|Z_{\underline{r-1}}\right)-\bar{\gamma}_0\omega \\
&= \bar{\gamma}_0\mathbb{E}\left(\sigma_r^2|Z_{\underline{r-1}}\right)-\bar{\gamma}_0\omega \\
&= \bar{\gamma}_0\mathbb{E}\left(\mathbb{E}(\sigma_r^2|u_{\underline{r-1}})|Z_{\underline{r-1}}\right)-\bar{\gamma}_0\omega \\
&= \bar{\gamma}_0\mathbb{E}\left(\omega|Z_{\underline{r-1}}\right)-\bar{\gamma}_0\omega \\
&= 0.
\end{aligned}$$

Using Theorem 2.3 in [22], we have

$$\frac{1}{n}\sum_{r=1}^{n}\left[Z_r^2-\mathbb{E}(Z_r^2|Z_{\underline{r-1}})\right]\xrightarrow[n\to+\infty]{\mathbb{P}}0,$$

where $\xrightarrow{\mathbb{P}}$ denotes convergence in probability.
On the other hand, we have:

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(Z_i^2|Z_{\underline{i-1}})\xrightarrow{\mathbb{P}}\sigma_Z^2 \quad \text{as} \quad n\to\infty.$$

In fact, consider $S_n=\sum_{r=1}^{n}(\mathbb{E}(Z_r^2|Z_{\underline{r-1}})-\sigma_Z^2)$. S_n is a martingale with respect to $Z_{\underline{n}}$:

$$\mathbb{E}(S_{n+1}|Z_{\underline{n}})=S_n+\mathbb{E}(\mathbb{E}(Z_{n+1}^2|Z_{\underline{n}})-\sigma_Z^2|Z_{\underline{n}})=S_n.$$

By the strong law of large numbers for martingales:

$$\frac{S_n}{n}=\frac{1}{n}\sum_{r=1}^{n}(\mathbb{E}(Z_r^2|Z_{\underline{r-1}})-\sigma_Z^2)\xrightarrow{\text{a.s.}}0.$$

Thus,

$$\frac{1}{n}\sum_{r=1}^{n}\mathbb{E}(Z_r^2|Z_{\underline{r-1}})\xrightarrow{\text{a.s.}}\sigma_Z^2.$$

Since almost sure (a.s.) convergence implies convergence in probability, we conclude:

$$\frac{1}{n}\sum_{r=1}^{n}\mathbb{E}(Z_r^2|Z_{\underline{r-1}})\xrightarrow{\mathbb{P}}\sigma_Z^2.$$

Then, we have

$$\frac{1}{n}\sum_{r=1}^{n} Z_r^2 \xrightarrow[n\to+\infty]{\mathbb{P}} \sigma_Z^2. \tag{3.2}$$

According to the invariance principle in [10], to end the proof of (3.1), we have to prove the Lindeberg condition, namely

$$\frac{1}{n}\sum_{r=1}^{n} \mathbb{E}\left(Z_r^2 \mathbb{1}_{\{Z_r^2>n\varepsilon\}}\right) \xrightarrow[n\to+\infty]{} 0,\ \varepsilon>0. \tag{3.3}$$

In accordance to the Formula (3) in [6], Page 223, and by Assumption A.3, we have, for all $\varepsilon>0$

$$\begin{aligned}
\mathbb{E}\left(Z_r^2 \mathbb{1}_{\{Z_r^2>n\varepsilon\}}\right) &= \int_{\{Z_r^2>n\varepsilon\}} Z_r^2 d\mathbb{P} \\
&= n\varepsilon\mathbb{P}\left[Z_r^2>n\varepsilon\right] + \int_{n\varepsilon}^{\infty}\mathbb{P}\left[Z_r^2>t\right]dt \\
&\leqslant c\left[n\varepsilon\mathbb{P}\left[Y^2>n\varepsilon\right] + \int_{n\varepsilon}^{\infty}\mathbb{P}\left[Y^2>t\right]dt\right] \\
&= c\mathbb{E}\left(Y^2\mathbb{1}_{\{Y^2>n\varepsilon\}}\right) \leqslant c\mathbb{E}\left(Y^2\right) < \infty.
\end{aligned}$$

Moreover $Z_r^2\mathbb{1}_{\{Z_r^2>n\varepsilon\}} \to 0$ since Z_r^2 is bounded by A.1 and A.3. Thus by the theorem of the dominated convergence (TDC), $\mathbb{E}\left(Z_r^2\mathbb{1}_{\{Z_r^2>n\varepsilon\}}\right) \xrightarrow[n\to+\infty]{} 0$ and by the lemma of Cesàro, we have (3.3).

It is clear that

$$\sup_{0\leqslant t\leqslant 1}\left|Y_n^{(1)}(t) - \frac{1}{\sqrt{n}}\sum_{r=1}^{[nt]} Z_r\right| \leqslant \frac{\tilde{X}_0^{(1)}}{\sqrt{n}} + 2\frac{\tilde{X}_0^{(2)}}{\sqrt{n}} + \sup_{0\leqslant t\leqslant 1}\left|\frac{\tilde{X}_{[nt]}^{(1)}}{\sqrt{n}}\right| + 2\sup_{0\leqslant t\leqslant 1}\left|\frac{\tilde{X}_{[nt]}^{(2)}}{\sqrt{n}}\right|.$$

Then, to end the proof of Lemma 3.2, by using Theorem 4.1, Page 25 in [6], it is sufficient to prove that

$$\frac{\tilde{X}_0^{(1)}}{\sqrt{n}} + 2\frac{\tilde{X}_0^{(2)}}{\sqrt{n}} + \sup_{0\leqslant t\leqslant 1}\left|\frac{\tilde{X}_{[nt]}^{(1)}}{\sqrt{n}}\right| + 2\sup_{0\leqslant t\leqslant 1}\left|\frac{\tilde{X}_{[nt]}^{(2)}}{\sqrt{n}}\right| \xrightarrow{\mathbb{P}} 0, \tag{3.4}$$

which holds if

$$\max_{1\leqslant k\leqslant n}\left\{\frac{1}{n}\left(\tilde{X}_k^{(1)}\right)^2\right\} \xrightarrow{\mathbb{P}} 0 \tag{3.5}$$

and

$$\max_{1\leqslant k\leqslant n}\left\{\frac{1}{n}\left(\tilde{X}_k^{(2)}\right)^2\right\} \xrightarrow{\mathbb{P}} 0. \tag{3.6}$$

By [10], Page 53, we have

$$\mathbb{P}\left[\max_{1\leqslant k\leqslant n}\left\{\frac{1}{n}\left(\tilde{X}_k^{(1)}\right)^2\right\} > \varepsilon\right] = \mathbb{P}\left[\left\{\frac{1}{n}\sum_{k=1}^{n}\left(\left(\tilde{X}_k^{(1)}\right)^2\mathbb{1}_{\left\{\left(\tilde{X}_k^{(1)}\right)^2>n\delta\right\}}\right)\right\} > \varepsilon^2\right],\ \forall\delta>0$$

and applying Markov's inequality, we have

$$\mathbb{P}\left[\left\{\frac{1}{n}\sum_{k=1}^{n}\left(\left(\tilde{X}_k^{(1)}\right)^2 \mathbb{1}_{\left\{\left(\tilde{X}_k^{(1)}\right)^2>n\delta\right\}}\right)\right\} > \varepsilon^2\right] \leqslant \frac{\frac{1}{n}\sum_{k=1}^{n}\mathbb{E}\left(\left(\tilde{X}_k^{(1)}\right)^2 \mathbb{1}_{\left\{\left(\tilde{X}_k^{(1)}\right)^2>n\delta\right\}}\right)}{\varepsilon^2}, \ \forall\delta > 0.$$

Thus, to prove (3.5), it is enough to show that

$$\frac{1}{n}\sum_{k=1}^{n}\mathbb{E}\left(\left(\tilde{X}_k^{(1)}\right)^2 \mathbb{1}_{\left\{\left(\tilde{X}_k^{(1)}\right)^2>n\delta\right\}}\right) \xrightarrow[n\to+\infty]{} 0,$$

which holds by Cesàro's lemma if $\left(\tilde{X}_k^{(1)}\right)^2$ is uniformly integrable. Let us prove that $\left(\tilde{X}_k^{(1)}\right)^2$ is uniformly integrable. Under A.2 and A.4, we have for all $\eta > 0$,

$$\begin{aligned}
\left(\mathbb{E}\left|\tilde{X}_k^{(1)}\right|^{2+\eta}\right)^{\frac{1}{2+\eta}} &= \left(\mathbb{E}\left|\sum_{j=0}^{\infty}\tilde{f}_{0j}u_{k-j}^2\right|^{2+\eta}\right)^{\frac{1}{2+\eta}} \\
&\leqslant \sum_{j=0}^{\infty}\left(\mathbb{E}\left|\tilde{f}_{0j}u_{k-j}^2\right|^{2+\eta}\right)^{\frac{1}{2+\eta}} \quad (\text{ from Minkowski's inequality of infinite sum}) \\
&\leqslant \left(\sup_k \mathbb{E}(u_k^2)^{2+\eta}\right)^{\frac{1}{2+\eta}}\sum_{j=0}^{\infty}\left|\tilde{f}_{0j}\right| \\
&< \infty.
\end{aligned}$$

Hence, following [6], Page 32, we have

$$\int_{\left\{\left(\tilde{X}_k^{(1)}\right)^2>n\delta\right\}}\left(\tilde{X}_k^{(1)}\right)^2 d\mathbb{P} < \frac{1}{(n\delta)^{1+\eta}}\mathbb{E}\left[\left\{\left(\tilde{X}_k^{(1)}\right)^2\right\}^{2+\eta}\right] \xrightarrow[n\to+\infty]{} 0,$$

which yields the uniform integrability of $\left(\tilde{X}_k^{(1)}\right)^2$ and ends the proof of (3.5).

The same arguments are used for the proof of (3.6) and so the proof of Lemma 3.2 is established. □

Lemma 3.3. *If the assumptions of Lemma 3.2 hold, then, in* $D[0;1]$,

$$Y_n^{(2)}(t) \xrightarrow{d} \sigma_Z tW(1).$$

Proof. We have

$$\frac{1}{\sqrt{n}}\sum_{r=1}^{[nt]}\overline{z} = \frac{1}{n}\sum_{r=1}^{[nt]}\frac{1}{\sqrt{n}}\sum_{s=1}^{n}z_s = \frac{[nt]}{n}\left(\frac{1}{\sqrt{n}}\sum_{s=1}^{n}z_s\right)$$

and Lemma 3.3 is derived from Lemma 3.2 and the result that $\dfrac{[nt]}{n} \longrightarrow t$ as $n \longrightarrow +\infty$. □

End of the proof of Theorem 3.1:

Under the null hypothesis, $Y_n(t)$ follows the invariance principle in $D[0;1]$. Therefore, following [17], Page 183, the asymptotic theory for the standard statistic $\sup_{0\leqslant t\leqslant 1}|Y_n(t)|$ and its modifications unfolds automatically. This concludes the proof of the Theorem 3.1. □

4. Simulations and applications

In this section, we assess the performance of the test statistic K_n using Monte Carlo simulations. In practical terms, a high value of K_n indicates a variance change. At the significance level $\alpha = 0.05$, we reject the null hypothesis, which assumes no variance change, if $(K_n/\sigma_Z) > 1.358$ (see, [11]). Additionally, it is worth noting that with $t = \dfrac{k}{n}$ we have $t^* = \dfrac{k^*}{n}$ and $\dfrac{K_n}{\sigma_Z} = \max_{1\leqslant k\leqslant n} C_k$, where C_k are the corresponding curves of the values taken by the statistic $\left|Y_n\left(\dfrac{k}{n}\right)\right|$, $k = 1, 2, \cdots$.

4.1. **Simulation study.** Now, to evaluate the performance of the test statistic K_n, we consider the following ARMA(1,1)-GARCH(1,1) model:

$$\begin{aligned}
X_t &= aX_{t-1} + u_t - bu_{t-1},\\
u_t &= \sigma_t\varepsilon_t,\\
\sigma_t^2 &= \alpha_0 + \alpha u_{t-1}^2 + \beta\sigma_{t-1}^2.
\end{aligned}$$

To utilize the asymptotic outcome of Theorem 3.1, we need to estimate σ_Z^2, considering that y_t has an MA(q) approximation. For a MA(q), $\gamma_h = 0$ if $h > q$. We estimate σ_Z^2 following the approach of [4], by

$$\hat{\sigma}_{nZ}^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \overline{y})^2 + 2\sum_{j=1}^{q}\left(1 - \frac{j}{q+1}\right)\frac{1}{n-j}\sum_{i=1}^{n-j}(y_i - \overline{y})(y_{i+j} - \overline{y})$$

it has been demonstrated in [9] that

$$|\hat{\sigma}_{nZ}^2 - \sigma_Z^2| = o_p(1).$$

As per [17], when $q(n)$ is a numerical sequence such that $q(n) \to \infty$, and $\dfrac{q(n)}{n} \xrightarrow[n\to\infty]{} 0$, substituting σ_Z with $\hat{\sigma}_{nZ}$, Theorem 3.1 remains valid.

We denote by $\theta = (a, b, \alpha_0, \alpha, \beta)$ the parameter and by γ_0 the variance of the ARMA(1,1)-GARCH(1,1) model considered obtained by

$$\gamma_0 = \omega\bar{\gamma}_0 = \omega\sum_{j=0}^{+\infty}c_j^2 = \frac{\alpha_0}{1-\alpha-\beta}\left(1 + \frac{(a-b)^2}{1-a^2}\right).$$

The real numbers $\gamma_0^{(0)}$ and $\gamma_0^{(1)}$ correspond to the variances under the null hypothesis (no change) and the alternative hypothesis, respectively. We will take

$q(n) = \left[(\ln(n))^{(3/2)}\right]$ as in [18].

Figure 1 depicts two scenarios of the ARMA(1,1)-GARCH(1,1) model: one without a change (Figure 1(a) and Figure 1(c)) and the same series with a change at $k^* = 500$ (Figure 1(b) and Figure 1(d)). Figure 1(a) displays a sample of the ARMA(1,1)-GARCH(1,1) model with a size of 1000 for $\gamma_0^{(0)} = 4.482$ ($\theta_0 = (0.50, 0.30, 2.00, 0.03, 0.50)$) (without change), while Figure 1(b) shows the same series with a variance change at $k^* = 500$ from $\gamma_0^{(0)} = 4.482$ to $\gamma_0^{(1)} = 7.022$ ($\theta_1 = (0.50, 0.30, 2.00, 0.20, 0.50)$).

On the plot of C_k (Figure 1(c)), all values (including the maximum) of C_k are below the critical region limit represented by the horizontal dashed line, confirming no change. Conversely, in Figure 1(d), around the point of change, C_k surpasses the critical value, with the maximum occurring at the time k^* where the change happens. Given the large value of the maximum of C_k (our test statistic), we conclude a change occurs and reject the null hypothesis. Furthermore, in Figure 1(b), where the change is not visually apparent, our proposed test statistic successfully detects it, affirming its performance.

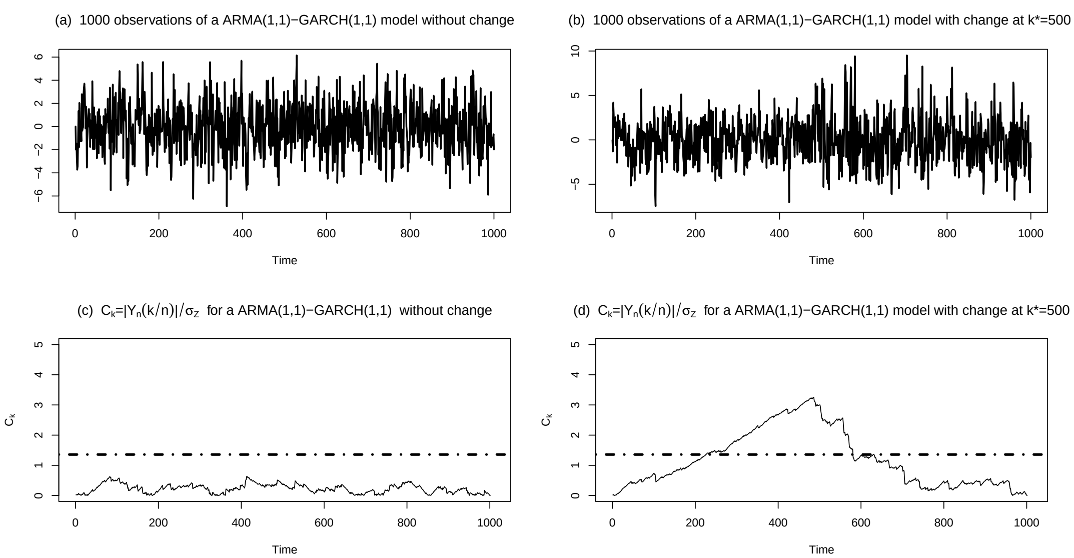

FIGURE 1. Realisation of 1000 obsevations of a ARMA(1,1)-GARCH(1,1) process and the corresponding curve $C_k = \dfrac{1}{\hat{\sigma}_{nZ}} |Y_n(k/n)|$. **(a)** is a ARMA$(1,1)$-GARCH$(1,1)$ without change-point, where the variance $\gamma_0^{(0)} = 4.482$. **(b)** is a ARMA(1,1)-GARCH(1,1) with change-point where the variance $\gamma_0^{(0)} = 4.482$ changes to $\gamma_0^{(1)} = 7.022$ at $k^* = 500$. Their corresponding curves C_k are represented by **(c)** and **(d)**. The horizontal dashed line represents the limit of the critical region of the test.

Figure 2 presents a scenario of the ARMA(1,1)-GARCH(1,1) model with a variance change, transitioning from $\gamma_0^{(0)} = 10.533$ ($\theta_0 = (0.50, 0.30, 1.00, 0.10, 0.80)$)

to $\gamma_0^{(1)} = 3.511$ ($\theta_1 = (0.50, 0.30, 1.00, 0.10, 0.60)$) at two distinct times. The first change occurs at $k^* = 300$ (Figure 2, (a) and (c)), and the second at $k^* = 700$ (Figure 2, (b) and (d)). In Figure 2(b) and Figure 2(d), we observe a similar pattern as in Figure 1(d): the maximum value of C_k, our test statistic, occurs precisely at the change-point. This value surpasses the critical value of the test, confirming the occurrence of a change at these different points in time.

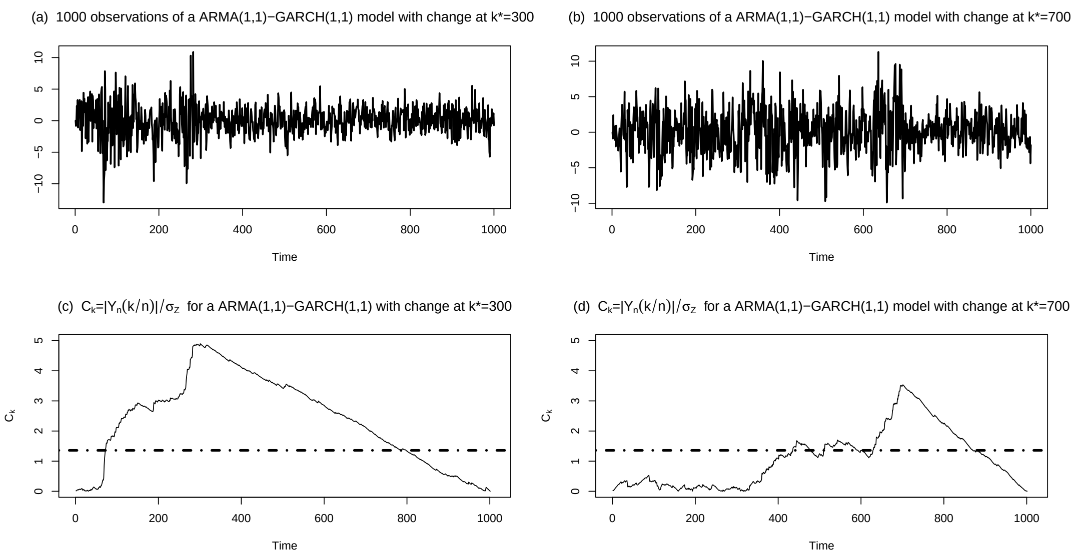

FIGURE 2. Realisation of 1000 obsevations of a ARMA(1,1)-GARCH(1,1) with change-point and their corresponding curve $C_k = \dfrac{1}{\hat{\sigma}_{nZ}} |Y_n(k/n)|$. **(a)** curve with change-point where the variance $\gamma_0^{(0)} = 10.533$ changes to $\gamma_0^{(1)} = 3.510$ at $k^* = 300$. **(b)** curve with change-point where the variance $\gamma_0^{(0)} = 10.533$ changes to $\gamma_0^{(1)} = 3.510$ at $k^* = 700$. **(c)** and **(d)** are their corresponding curves C_k. The horizontal dashed line represents the limit of the critical region of the test.

Table 1 displays the outcomes of a simulation experiment designed to assess the effectiveness of change-point detection in variance. The empirical levels are computed when the variance is γ_0 (no change), and the empirical powers are calculated when the variance transitions from $\gamma_0^{(0)}$ to $\gamma_0^{(1)}$ at $k^* = 0.3n, 0.5n, 0.7n$, where $n = 500,\ 1000$ represents the sample size. The findings of empirical levels and power calculations are based on 200 replications. For the powers, we considered various data-generating models labeled $S_i,\ i = 1, 2, 3, 4$ of ARMA$(1,1)$-GARCH(1,1).

S_1: ARMA(1,1)-GARCH(1,1) where the variance remains constant ($\gamma_0 = 0.181$)

$$\begin{aligned} \gamma_0^{(0)} &= 0.181 \quad \text{for } \theta_0 = (0.010, 0.020, 0.100, 0.050, 0.400) \\ \gamma_0^{(1)} &= 0.181 \quad \text{for } \theta_1 = (0.010, 0.020, 0.150, 0.030, 0.145). \end{aligned}$$

S_2 : ARMA(1,1)-GARCH(1,1) where the variance changes (small change) from $\gamma_0^{(0)} = 0.234$ (for $\theta_0 = (0.50, 0.30, 0.10, 0.05, 0.50)$) to $\gamma_0^{(1)} = 0.421$ (for $\theta_1 = (0.50, 0.30, 0.10, 0.05, 0.70)$).

S_3 : ARMA(1,1)-GARCH(1,1) where the variance changes (large change) from $\gamma_0^{(0)} = 4.482$ (for $\theta_0 = (0.50, 0.30, 2.00, 0.03, 0.50)$) to $\gamma_0^{(1)} = 2.241$ (for $\theta_1 = (0.50, 0.30, 1.00, 0.03, 0.50)$).

S_4 : ARMA(1,1)-GARCH(1,1) where the variance changes (large change) from $\gamma_0^{(0)} = 4.482$ for ($\theta_0 = (0.50, 0.30, 2.00, 0.03, 0.50)$) to $\gamma_0^{(1)} = 7.022$ (for $\theta_1 = (0.50, 0.30, 2.00, 0.2, 0.50)$).

TABLE 1. Testing variance changes in ARMA(1,1)-GARCH(1,1) model with one change-point: empirical levels and powers at the nominal level of 0.05.

				$n = 500$	$n = 1000$
Empirical levels :					
	$\gamma_0 = 0.181$			0.090	0.055
	$\gamma_0 = 4.482$			0.080	0.050
Empirical powers :					
	$\gamma_0^{(0)} = 0.181$;	$\gamma_0^{(1)} = 0.181$;	$k^* = 0.3n$	0.045	0.080
			$k^* = 0.5n$	0.050	0.050
			$k^* = 0.7n$	0.060	0.060
	$\gamma_0^{(0)} = 0.234$;	$\gamma_0^{(1)} = 0.421$;	$k^* = 0.3n$	0.940	1.000
			$k^* = 0.5n$	0.985	1.000
			$k^* = 0.7n$	0.930	1.000
	$\gamma_0^{(0)} = 4.482$;	$\gamma_0^{(1)} = 2.240$;	$k^* = 0.3n$	0.990	1.000
			$k^* = 0.5n$	0.995	1.000
			$k^* = 0.7n$	0.975	1.000
	$\gamma_0^{(0)} = 4.482$;	$\gamma_0^{(1)} = 7.022$;	$k^* = 0.3n$	0.680	0.910
			$k^* = 0.5n$	0.805	0.975
			$k^* = 0.7n$	0.640	0.895

Table 2 showcases the findings from a simulation study aimed at comparing the effectiveness of our technique for detecting change-point to that of [24] (a method based on the Residual CUSUM test). Although their approach is not directly centered on detecting changes in the model's variance but rather on identifying changes in the parameters of the model, we evaluate their method's performance in this context. The results of empirical power calculations are based on 5000 replications. The values obtained using our method are highlighted in bold in the table, while those from [24] are presented in brackets. Our method outperforms the approach described in [24], and notably, our square-based CUSUM test excels in comparison to their residual-based CUSUM test when testing for ARMA parameter changes.

4.2. **Real data analysis.** In this subsection, we validate the effectiveness of our method in practical applications. To accomplish this, we analyze the exchange

TABLE 2. ARMA(1,1)-GARCH(1,1) process with change at the midpoint; the variance $\gamma_0^{(0)} = 0.1053$ (for $\theta_0 = (0.5, 0.3, 0.02, 0.1, 0.7)$) changes to $\gamma_0^{(1)}$ (for θ_1)

	$n = 500$	$n = 1000$	$n = 2000$	$n = 3000$
Empirical powers				
$\theta_1 = (0.7, 0.3, 0.02, 0.1, 0.7)$ $\gamma_0^{(1)} = 0.1313$	**0.535** (0.027)	**0.684** (0.034)	**0.836** (0.039)	**0.919** (0.043)
$\theta_1 = (0.5, 0.5, 0.02, 0.1, 0.7)$ $\gamma_0^{(1)} = 0.1000$	**0.264** (0.031)	**0.304** (0.037)	**0.352** (0.041)	**0.381** (0.042)
$\theta_1 = (0.5, 0.3, 0.04, 0.1, 0.7)$ $\gamma_0^{(1)} = 0.2106$	**0.983** (0.473)	**0.996** (0.812)	**1.000** (0.937)	**1.000** (0.956)
$\theta_1 = (0.5, 0.3, 0.02, 0.17, 0.7)$ $\gamma_0^{(1)} = 0.1620$	**0.779** (0.140)	**0.921** (0.411)	**0.988** (0.782)	**0.998** (0.932)
$\theta_1 = (0.5, 0.3, 0.02, 0.1, 0.77)$ $\gamma_0^{(1)} = 0.1620$	**0.997** (0.211)	**1.000** (0.612)	**1.000** (0.950)	**1.000** (0.995)

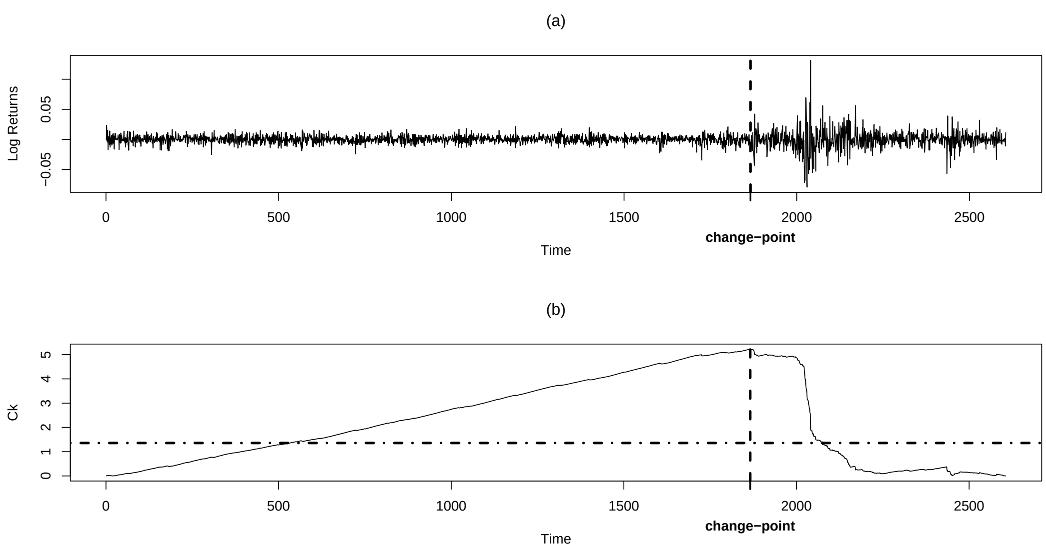

FIGURE 3. **(a)** is a KRW/JPY exchange rate series from January 2001 to December 2010. **(b)** is the corresponding curve $C_k = \dfrac{1}{\hat{\sigma_{Z_n}}} |Y_n(k/n)|$. The horizontal dashed line denotes the boundary of the test's critical region. The vertical dashed line represents the time where the change occurs.

rate of the Korean won to the Japanese yen (KRW/JPY) from January 2, 2001, to December 30, 2010. This dataset has also been recently examined by [24]. Our procedure identifies a change at the vertical dashed line in Figure 3, corresponding to a volatility shift that occurred in November 2007. It is noteworthy that this volatility change coincides with the global financial crisis of 2007-2008. These findings are consistent with the results obtained by [24].

5. Conclusion

In this paper, we introduced a CUSUM-type test utilizing the square of the ARMA-GARCH model to identify a variance change within this framework. Leveraging the Beveridge-Nelson decomposition with a lag polynomial, we established the convergence of our test statistic to the supremum of a standard Brownian bridge. Simulation results provided empirical evidence for the effectiveness of our approach. In addition our proposed square-based CUSUM test outperforms residual-based CUSUM test proposed by [24] in testing for ARMA parameter change. Furthermore, we applied this method to real-world data concerning the KRW/JPY exchange rate spanning from January 2001 to December 2010, successfully detecting a change corresponding to the global financial crisis of 2007-2008.

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