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L_q-NORM INEQUALITIES FOR THE GENERALIZED DERIVATIVE OF A POLYNOMIAL

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Abstract. In this paper, we aim to present some Turán-type inequalities in L_q-norm for the generalized derivative of a polynomial. We further extend these results to the generalized polar derivative. Our results besides generalizing various known results also include other inequalities as special cases.

1. Introduction and preliminaries

For a positive integer n, we denote by $\mathcal{P}_n$ the set of all polynomials of degree n over the field $\mathbb{C}$ of complex numbers, $S = \{\gamma : \gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n), \gamma_j \geq 0 \ \forall \ j = 1, 2, \cdots, n\}$ and $\Lambda = \sum_{j=1}^{n} \gamma_j$.

Inequalities governing polynomials and their derivatives are fundamental in various branches of mathematics and applied sciences, particularly in approximation theory. The origin of these concepts can be traced back to a problem posed and solved by chemist Dmitri Mendeleev [15]. While investigating the specific gravity of solutions as a function of percentage of dissolved substance, Mendeleev came up with a problem that for a quadratic polynomial $P(x)$ such that $|P(x)| \leq 1$ for $x \in [-1, 1]$, what limit exists on the magnitude of its derivative ($|P'(x)|$) over this interval? Mandleev himself was able to solve this problem and found that if $P(x)$ is a quadratic polynomial with $|P(x)| \leq 1$ on $[-1, 1]$, then $|P'(x)| \leq 4$ on the same interval. Later A. A. Markov [14] generalized this result for the polynomials of degree n with real coefficients. He proved that if $P(x) = \sum_{j=1}^{1} a_j x^j$ is a real polynomial of degree n, then

$$\max_{-1 \leq x \leq 1} |P'(x)| \leq n^2 \max_{-1 \leq x \leq 1} |P(x)|.$$

After about twenty years, S. Bernstein [4] developed a version of Markovs result for the unit disc in the complex plane by proving that if $P \in \mathcal{P}_n$, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}$$

This estimation of the upper bound for the maximum modulus of $P'(z)$ in terms of maximum modulus of $P(z)$ has intrigued many researchers and inspired them

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to develop generalizations, extensions and improvements of above inequalities, see [2, 16, 20] for refrences. In this spirit, Paul Turán [23] was the first to estimate the lower bound for the maximum modulus of $P'(z)$ in terms of maximum modulus of underlying polynomial $P(z)$ by proving that if $P \in \mathcal{P}_n$ does not vanish in $|z| > 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.2}$$

The above result is best possible with equality holding for the polynomials whose all zeros lie on the boundary of the unit circle.
Later, the extension of (1.2) was obtained Malik [11] by proving that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k,\ k \leq 1$, then

$$(1+k) \max_{|z|=1} |P'(z)| \geq n \max_{|z|=1} |P(z)|, \tag{1.3}$$

with equality holding for $P(z) = (z+k)^n$.
Similarly, Govil [9] considered the case $k \geq 1$ and proved that if $P \in \mathcal{P}_n$ has all zeros in $|z| \leq k,\ k \geq 1$, then

$$(1+k^n) \max_{|z|=1} |P'(z)| \geq n \max_{|z|=1} |P(z)|. \tag{1.4}$$

Inequality (1.4) sharp and the extremal polynomial is $P(z) = z^n + k^n$.
These inequalities, where the lower bound of maximum modulus of a polynomial's is estimated based on the the maximum modulus of polynomial itself, are usually reffered as "Turán type inequalities" in the literature. The researcher's have obtained various variants of these type of inequalities which provide the improvements, generalizations and extensions of aforementioned inequalities, (for refrences see [5, 21, 24]). One such generalization is replacing the sup-norm with a factor involving integral mean. For $P \in \mathcal{P}_n$, we define

$$\|P(z)\|_q := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right\}^{\frac{1}{q}}, \quad 0 < q < \infty, \text{ and}$$
$$\|P(z)\|_\infty := \max_{|z|=1} |P(z)|.$$

Zygmund [25] obtained the generalization of Bernstein's inequality (1.1) to L_q norm by proving that if $P \in \mathcal{P}_n$, then for $q \geq 1$

$$\|P'(z)\|_q \leq n \|P(z)\|_q. \tag{1.5}$$

By letting $q \to \infty$ in (1.5), we obtain inequality (1.1). Later Aerstov [1] confirmed that (1.1) remains true for $0 \leq q < 1$ as well. The generalization of inequality (1.2) to L_q-norm was obtained by Malik [12]. In fact, he proved that if all the zeros of $P \in \mathcal{P}_n$ lie in $|z| \leq 1$, then for each $q > 0$

$$\|1+z\|_q \|P'(z)\|_\infty \geq n \|P(z)\|_q. \tag{1.6}$$

The correspounding extensions of inequalities (1.3) and (1.4), was obtained by Aziz [3] who proved that, if $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k$, then for each

$q > 0$

$$n\left\|\frac{P(z)}{P'(z)}\right\|_q \leq \|1 + kz\|_q \qquad \text{for} \quad k \leq 1, \tag{1.7}$$

and

$$n\|P(z)\|_q \leq \|1 + k^n z\|_q \|P'(z)\|_\infty \qquad \text{for} \quad k \geq 1. \tag{1.8}$$

Recently Laishangbam et al. [10] provided the improvement of inequality (1.8) by involving the coefficients of $P(z)$ and proved that, if $P(z) = \sum_{j=0}^{n} a_j z^j = a_n \prod_{j=1}^{n}(z - z_j)$ is a polynomial of degree n having all its zeros in $|z| \leq k$ where $k \geq 1$, then for each $q > 0$

$$\|P'(z)\|_\infty \geq \left(\sum_{j=1}^{n} \frac{k}{k + |z_j|}\right) \frac{\left\|\frac{|a_n|k^{n+1}+|a_0|}{|a_n|k^n+k|a_0|} + z\right\|_q}{\|1 + zk^n\|_q} \|P(z)\|_q. \tag{1.9}$$

Before preceding to some other results, let's introduce the concepts of polar and generalized derivative of a polynomial.

Definition 1.1. (**Polar derivative of a polynomial**) The polar derivative of a polynomial $P(z) \in \mathcal{P}_n$, with respect to a complex number α is the polynomial $nP(z) + (\alpha - z)P'(z)$, often represented as $D_\alpha P(z)$. The derivative

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z) \tag{1.10}$$

generalizes the ordinary derivative in the sense that as $\alpha \to \infty$, the ratio $\frac{D_\alpha P(z)}{\alpha}$ approaches the standard derivative $P'(z)$ uniformly for $|z| \leq R$, $R > 0$.

The concept of generalized derivative was first introduced by S. Z. Nagy [18] by defining the generalized derivative as:

Definition 1.2. (**Generalized derivative of a polynomial**) If $P(z) = c\prod_{j=1}^{n}(z - z_j)$ is a polynomial of degree n and $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ is an n-tuple of non negative real numbers with $\gamma_j \neq 0$ for atleast one j. Then the generalized derivative of polynomial $P(z)$ is defined as the linear combination

$$P^\gamma(z) = P(z)\sum_{j=1}^{n} \frac{\gamma_j}{z - z_j} = \sum_{j=1}^{n} \gamma_j P_j(z), \tag{1.11}$$

where $P_j(z) = c\prod_{i=1, i\neq j}^{n}(z - z_i)$ for $1 \leq j \leq n$. The ordinary derivative $P'(z)$ of $P(z)$ can be derived from $P^\gamma(z)$ by choosing $\gamma = (1, 1, \cdots, 1)$. To clearly illustrate the concept of generalized derivative, we present the following example.

Example 1.3. Consider the polynomial $P(z) = (z - 2)(z + 3i)(z - 4)$ with $\gamma_1 = \frac{1}{2}$, $\gamma_2 = 1$ and $\gamma_3 = 2$, then

$$\begin{aligned} P^{(1/2,1,2)}(z) &= \frac{1}{2}(z + 3i)(z - 4) + 1(z - 2)(z - 4) + 2(z - 2)(z + 3i) \\ &= \frac{7}{2}z^2 - \left(12 - \frac{15}{2}i\right)z + 8 - 18i. \end{aligned}$$

Following the above definitions, the concept of generalized polar derivative of $P(z)$ can be defined analogously to (1.10) as

$$D_\alpha^\gamma P(z) \ := \Lambda P(z) + (\alpha - z)P^\gamma(z), \qquad \alpha \in \mathbb{C}.$$

By taking $\gamma_j = 1, \ \forall j = 1, 2, \cdots, n$, we observe $D_\alpha^\gamma P(z) = D_\alpha P(z)$.

Various researchers have extended the concept of aforementioned inequalities (1.1), (1.2), (1.3) and (1.4) beyond the ordinary derivative of a polynomial. They have explored domains where classical derivative is replaced with generalized and polar derivatives. For the references, their results can be seen in [5, 13, 16, 21]. Among these we mention the following extenions of inequalities (1.3) and (1.4) to the generalized derivative by Bhat et al. [6].

If $P \in \mathcal{P}_n$ does not vanish in $|z| > k$, then for $0 \neq \gamma \in S$ and $\Lambda = \sum_{j=1}^n \gamma_j$

$$(1+k) \max_{|z|=1} |P^\gamma(z)| \geq \Lambda \max_{|z|=1} |P(z)| \ for \ k \leq 1, \tag{1.12}$$

and

$$(1+k^n) \max_{|z|=1} |P^\gamma(z)| \geq \Lambda \max_{|z|=1} |P(z)|, \ for \ k \geq 1. \tag{1.13}$$

Recently Rather et al. [21] provided the corresponding extensions of inequalities (1.12) and (1.13) to the generalized polar derivative by proving the following.

If $P \in \mathcal{P}_n$ does not vanish in $|z| > k$, then for $0 \neq \gamma \in S$, $\Lambda = \sum_{j=1}^n \gamma_j$ and $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$,

$$\|D_\alpha^\gamma P(z)\|_\infty \geq \frac{\Lambda}{1+k}(|\alpha| - k) \ |P(z)\|_\infty \ for \ k \leq 1, \tag{1.14}$$

and

$$\|D_\alpha^\gamma P(z)\|_\infty \geq \frac{\Lambda}{1+k^n}(|\alpha| - k)\|P(z)\|_\infty, \ for \ k \geq 1. \tag{1.15}$$

2. Lemmas

For the proofs of main results, we need the following lemmas.

The first lemma is due to Singha and Chanam [22].

Lemma 2.1. *If all the zeros of the polynomial $P(z) = \sum_{j=0}^n a_j z^j, \ a_n \neq 0$ lie in $|z| \leq k, \ k \geq 1$ then for $q > 0$*

$$\|P(kz)\|_q \geq k^n \frac{\left\| \frac{|a_n|k^{n+1} + |a_0|}{|a_n|k^n + k|a_0|} + z \right\|_q}{\|1 + zk^n\|_q} \|P(z)\|_q.$$

The following lemma is a simple consequence of maximum modulus principle (see [19] Vol. I, p. 137).

Lemma 2.2. *If $P \in \mathcal{P}_n$, then for $R \geq 1$*

$$\|P(Rz)\|_\infty \leq R^n \|P(z)\|_\infty.$$

The following lemma is due to Frappier et al. [7].

Lemma 2.3. *If $P \in \mathcal{P}_n$, then for $k \geq 1$*

$$\max_{|z|=k} |P(z)| \leq k^n \|P(z)\|_\infty - \Phi(k)|P(0)|,$$

where

$$\Phi(k) = \begin{cases} k^n - k^{n-1} & \text{if} \quad n > 1 \\ k - 1 & \text{if} \quad n = 1. \end{cases} \tag{2.1}$$

The next Lemma is due to Rather et al. [21].

Lemma 2.4. *If $P \in \mathcal{P}_n$ has all zeros in $|z| \leq 1$, then for any $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$,*

$$\|D_\alpha^\gamma P(z)\|_\infty \geq \frac{\Lambda}{2}(|\alpha| - 1)\|P(z)\|_\infty.$$

3. Main results

In this section, first we present improved integral extension of inequality (1.9) to the generalized derivative of a polynomial. More precisely, we prove

Theorem 3.1. *If $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = a_n \prod_{j=1}^n (z - z_j)$ is a polynomial of degree n having all its zeros $z_1, z_2, ..., z_n$ in the disc $|z| \leq k$, $k \geq 1$. Then for $0 \neq \gamma \in S$ and $q > 0$,*

$$\|P^\gamma(z)\|_\infty \geq \left(\sum_{j=1}^n \frac{k\gamma_j}{k + |z_j|}\right) \frac{\left\| \frac{|a_n|k^{n+1} + |a_0|}{|a_n|k^n + k|a_0|} + z \right\|_q}{\|1 + zk^n\|_q} \|P(z)\|_q + \frac{\Phi(k)}{k^{n-1}}|P^\gamma(0)|, \tag{3.1}$$

where $\Phi(k)$ is defined as in Lemma 2.3.

The following refinement of inequality (1.9) can be obtained by taking $\gamma_j = 1, \forall,\ j = 1, 2, \cdots, n$ in Theorem 3.1.

Corollary 3.2. *If $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = a_n \prod_{j=1}^n (z - z_j)$ is a polynomial of degree n having all its zeros $z_1, z_2, ..., z_n$ in the disc $|z| \leq k$, $k \geq 1$. Then for $q > 0$,*

$$\|P'(z)\|_\infty \geq \left(\sum_{j=1}^n \frac{k}{k + |z_j|}\right) \frac{\left\| \frac{|a_n|k^{n+1} + |a_0|}{|a_n|k^n + k|a_0|} + z \right\|_q}{\|1 + zk^n\|_q} \|P(z)\|_q + \frac{\Phi(k)}{k^{n-1}}|P'(0)|. \tag{3.2}$$

In particular for $k = 1$ inequality (3.2) reduces to

$$\|P'(z)\|_\infty \geq \left(\sum_{j=1}^n \frac{1}{1 + |z_j|}\right) \|P(z)\|_q.$$

By choosing $\gamma = (0, 0, \cdots, \gamma_j, \cdots, 0)$ with $\gamma_j = 1$ for some $j = 1, 2, \cdots, n$ in (3.1) we obtain the following result.

Corollary 3.3. *If $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = a_n \prod_{j=1}^{n}(z - z_j)$ is a polynomial of degree n having all its zeros $z_1, z_2, ..., z_n$ in the disc $|z| \leq k$, $k \geq 1$. Then for each $j = 1, 2, \cdots, n$ and $q > 0$,*

$$\left\|\frac{P(z)}{z - z_j}\right\|_\infty \geq \frac{k}{k + |z_j|} \frac{\left\|\frac{|a_n|k^{n+1} + |a_0|}{|a_n|k^n + k|a_0|} + z\right\|_q}{\|1 + zk^n\|_q} \|P(z)\|_q + \frac{\Phi(k)}{k^{n-1}} \left|\frac{a_0}{z_j}\right|.$$

In particular, taking $k = 1$ and then letting $q \to \infty$ in corollary 3.3, we obtain the following inequality due to Bhat et al. ([5], corollary 1).

$$\left\|\frac{P(z)}{z - z_j}\right\|_\infty \geq \frac{1}{1 + |z_j|} \|P(z)\|_\infty.$$

Next we provide the improvement of inequality (1.15) in L_q-norm by including the coefficients of underlying polynomial.

Theorem 3.4. *If $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = a_n \prod_{j=1}^{n}(z - z_j)$ is a polynomial of degree n having all its zeros $z_1, z_2, ..., z_n$ in the disc $|z| \leq k$, $k \geq 1$. Then for any $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$, $0 \neq \gamma \in S$, and $q > 0$,*

$$\|D_\alpha^\gamma P(z)\|_\infty \geq \frac{\Lambda}{2}(|\alpha| - k) \frac{\left\|\frac{|a_n|k^{n+1} + |a_0|}{|a_n|k^n + k|a_0|} + z\right\|_q}{\|1 + zk^n\|_q} \|P(z)\|_q. \tag{3.3}$$

On dividing both sides of inequality (3.3) by $|\alpha|$ and let $|\alpha| \to \infty$, we obtain the following result.

Corollary 3.5. *If $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = a_n \prod_{j=1}^{n}(z - z_j)$ is a polynomial of degree n having all its zeros $z_1, z_2, ..., z_n$ in the disc $|z| \leq k$, $k \geq 1$. Then for $0 \neq \gamma \in S$, and $q > 0$,*

$$\|P^\gamma(z)\|_\infty \geq \left(\frac{\Lambda}{2}\right) \frac{\left\|\frac{|a_n|k^{n+1} + |a_0|}{|a_n|k^n + k|a_0|} + z\right\|_q}{\|1 + zk^n\|_q} \|P(z)\|_q.$$

If we take $(\gamma_1, \gamma_2, \cdots, \gamma_n) = (1, 1, \cdots, 1)$ in corollary 3.5 and then let $q \to \infty$, we get the following result.

Corollary 3.6. *If $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = a_n \prod_{j=1}^{n}(z - z_j)$ is a polynomial of degree n having all its zeros $z_1, z_2, ..., z_n$ in the disc $|z| \leq k$, $k \geq 1$, then*

$$\|P'(z)\|_\infty \geq \frac{n}{2} \left(\frac{\left|\frac{|a_n|k^{n+1} + |a_0|}{|a_n|k^n + k|a_0|}\right| + 1}{1 + k^n}\right) \|P(z)\|_\infty.$$

4. Proofs of main results

Proof of Theorem 3.1. By hypothesis $P(z) = a_n \prod_{j=1}^{n}(z - z_j)$ with $|z_j| \leq k$, $j = 1, 2, ..., n$, $k \geq 1$, therefore all the zeros of t the polynomial $F(z) = P(kz)$

lie in $|z| \leq 1$. Thus for $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n) \neq (0, 0, \cdots, 0)$, we have

$$F^{\gamma}(z) = F(z) \sum_{j=1}^{n} \frac{\gamma_j}{z - w_j} \quad \text{where} \quad w_j = z_j/k, \quad |w_j| \leq 1, \quad \text{for } 1 \leq j \leq n.$$

This implies for the points $e^{i\theta}, 0 \leq \theta < 2\pi$, which are not the zeros of $F(z)$

$$\frac{e^{i\theta} F^{\gamma}(e^{i\theta})}{F(e^{i\theta})} = \sum_{j=1}^{n} \frac{e^{i\theta}\gamma_j}{e^{i\theta} - k_j e^{i\theta_j}}, \quad w_j = k_j e^{i\theta_j}, k_j \leq 1, \ 0 \leq \theta_j < 2\pi \quad \text{for } 1 \leq j \leq n.$$

Therefore

$$\begin{aligned}
\mathrm{Re}\frac{e^{i\theta} F^{\gamma}(e^{i\theta})}{F(e^{i\theta})} &= \sum_{j=1}^{n} \mathrm{Re}\frac{e^{i\theta}\gamma_j}{e^{i\theta} - k_j e^{i\theta_j}} \\
&= \sum_{j=1}^{n} \mathrm{Re}\frac{\gamma_j}{1 - k_j e^{i(\theta_j - \theta)}} \\
&= \sum_{j=1}^{n} \gamma_j \mathrm{Re}\frac{1 - k_j e^{-i(\theta_j - \theta)}}{\left|1 - k_j e^{i(\theta_j - \theta}\right|^2} \\
&= \sum_{j=1}^{n} \gamma_j \frac{1 - k_j \cos(\theta_j - \theta)}{1 + k_j^2 - 2k_j \cos(\theta_j - \theta)} \\
&\geq \sum_{j=1}^{n} \frac{\gamma_j}{1 + k_j}
\end{aligned}$$

for the points $e^{i\theta}, 0 \leq \theta < 2\pi$, which are not the zeros of $F(z)$. This implies

$$\begin{aligned}
\left|\frac{e^{i\theta} F^{\gamma}(e^{i\theta})}{F(e^{i\theta})}\right| &\geq \mathrm{Re}\frac{e^{i\theta} F^{\gamma}(e^{i\theta})}{F(e^{i\theta})} \\
&\geq \sum_{j=1}^{n} \frac{\gamma_j}{1 + |w_j|}
\end{aligned}$$

for the points $e^{i\theta}, 0 \leq \theta \leq 2\pi$, which are not the zeros of $F(z)$.
Equivalently

$$\left|F^{\gamma}(e^{i\theta})\right| \geq \sum_{j=1}^{n} \frac{\gamma_j}{1 + |w_j|} \left|F(e^{i\theta})\right| \tag{4.1}$$

for the points $e^{i\theta}, 0 \leq \theta \leq 2\pi$, which are not the zeros of $F(z)$. Since the inequality (4.1) is already true for the points $e^{i\theta}, 0 \leq \theta \leq 2\pi$ which are the zeros of $F(z)$, therefore it follows that

$$\left|F^{\gamma}(e^{i\theta})\right| \geq \sum_{j=1}^{n} \frac{\gamma_j}{1 + |w_j|} \left|F(e^{i\theta})\right| \quad \textit{for all } z \textit{ on} \quad |z| = 1.$$

This gives for $|z| = 1$

$$\begin{aligned}\|F^\gamma(z)\|_\infty &\geq \sum_{j=1}^{n} \frac{k\gamma_j}{k+|z_j|}\|F(z)\|_\infty \\ &\geq \sum_{j=1}^{n} \frac{k\gamma_j}{k+|z_j|}|P(kz)|,\end{aligned}$$

which is equivalent to

$$\max_{|z|=k} |P^\gamma(z)| \geq \sum_{j=1}^{n} \frac{\gamma_j}{k+|z_j|}|P(kz)|.$$

Therefore for any $q > 0$ and $0 \leq \theta < 2\pi$, we have

$$\left\{\max_{|z|=k} |P^\gamma(e^{i\theta})|\right\}^q \geq \left\{\sum_{j=1}^{n} \frac{\gamma_j}{k+|z_j|}\right\}^q |P(ke^{i\theta})|^q,$$

or equivalently,

$$\max_{|z|=k} |P^\gamma(z)| \geq \sum_{j=1}^{n} \frac{\gamma_j}{k+|z_j|} \left\{\frac{1}{2\pi}\int_0^{2\pi} |P(ke^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}.$$

Applying Lemma 2.1, we get

$$\max_{|z|=k} |P^\gamma(z)| \geq k^n \left(\sum_{j=1}^{n} \frac{\gamma_j}{k+|z_j|}\right) \frac{\left\|\frac{|a_n|k^{n+1}+|a_0|}{|a_n|k^n+k|a_0|}+z\right\|_q}{\|1+zk^n\|_q}\|P(z)\|_q. \tag{4.2}$$

Since $P^\gamma(z)$ is a polynomial of degree at most $n-1$, therefore by using Lemma 2.3, we get

$$\max_{|z|=k} |P^\gamma(z)| \leq k^{n-1}\|P^\gamma(z)\|_\infty - \Phi(k)|P^\gamma(0)|, \tag{4.3}$$

where

$$\Phi(k) = \begin{cases} k^{n-1} - k^{n-2} & \text{if} \quad n > 1 \\ k - 1 & \text{if} \quad n = 1. \end{cases} \tag{4.4}$$

Using (4.2) in (4.3), we obtain

$$\begin{aligned}&k^{n-1}\|P^\gamma(z)\|_\infty - \Phi(k)|P^\gamma(0)| \\ &\qquad \geq k^n \left(\sum_{j=1}^{n} \frac{\gamma_j}{k+|z_j|}\right) \frac{\left\|\frac{|a_n|k^{n+1}+|a_0|}{|a_n|k^n+k|a_0|}+z\right\|_q}{\|1+zk^n\|_q}\|P(z)\|_q.\end{aligned}$$

Equivalently

$$\|P^\gamma(z)\|_\infty \geq \left(\sum_{j=1}^{n} \frac{k\gamma_j}{k+|z_j|}\right) \frac{\left\|\frac{|a_n|k^{n+1}+|a_0|}{|a_n|k^n+k|a_0|}+z\right\|_q}{\|1+zk^n\|_q}\|P(z)\|_q + \frac{\Phi(k)}{k^{n-1}}|P^\gamma(0)|.$$

This completes the proof of Theorem 3.1. $\square$

Proof of Theorem 3.4. Since all the zeros of $P(z)$ lie in $|z| \leq k,\ k \geq 1$. Therefore the polynomial $F(z) = P(kz)$ has all its zeros in $|z| \leq 1$. Thus applying lemma 2.4 to the polynomial $F(z)$ while also considering that $\left|\frac{\alpha}{k}\right| \geq 1$, we obtain

$$\|D^{\gamma}_{\frac{\alpha}{k}} F(z)\|_{\infty} \geq \frac{\Lambda}{2}\left(\frac{|\alpha|}{k} - 1\right)\|F(z)\|_{\infty}.$$

Or equivalently

$$\|D^{\gamma}_{\frac{\alpha}{k}} P(kz)\|_{\infty} \geq \frac{\Lambda}{2}\left(\frac{|\alpha|}{k} - 1\right)\|P(kz)\|_{\infty}. \tag{4.5}$$

Now

$$\begin{aligned}
\|D^{\gamma}_{\frac{\alpha}{k}} P(kz)\|_{\infty} &= \left\|\Lambda P(kz) + \left(\frac{\alpha}{k} - z\right) p^{\gamma}(kz)\right\|_{\infty} \\
&= \left\|\Lambda P(kz) + \left(\frac{\alpha - kz}{k}\right) P(kz)\sum_{j=1}^{n}\frac{\gamma_j}{z - \frac{z_j}{k}}\right\|_{\infty} \\
&= \left\|\Lambda P(kz) + \left(\frac{\alpha - kz}{k}\right) kP(kz)\sum_{j=1}^{n}\frac{\gamma_j}{kz - z_j}\right\|_{\infty} \\
&= \left\|\Lambda P(kz) + (\alpha - kz)P(kz)\sum_{j=1}^{n}\frac{\gamma_j}{kz - z_j}\right\|_{\infty} \\
&= \|H(kz)\|_{\infty} \\
&= \max_{|z|=k}|H(z)|,
\end{aligned}$$

where $H(z) = \Lambda P(z) + (\alpha - z)P(z)\sum_{j=1}^{n}\frac{\gamma_j}{z - z_j}$ is a polynomial of degree at most $n-1$. This gives by using Lemma 2.2

$$\begin{aligned}
\|D^{\gamma}_{\frac{\alpha}{k}} P(kz)\|_{\infty} &= \max_{|z|=k}|H(z)| \\
&\leq k^{n-1}\|H(z)\|_{\infty} \\
&= k^{n-1}\left\|P(z) + (\alpha - z)P(z)\sum_{j=1}^{n}\frac{\gamma_j}{z - z_j}\right\|_{\infty} \\
&= k^{n-1}\|D^{\gamma}_{\alpha} P(z)\|_{\infty}.
\end{aligned}$$

Thus

$$\|D^{\gamma}_{\frac{\alpha}{k}} P(kz)\|_{\infty} \leq k^{n-1}\|D^{\gamma}_{\alpha} P(z)\|_{\infty}. \tag{4.6}$$

Combining inequalities (4.5) and (4.6), we get

$$k^{n-1}\|D^{\gamma}_{\alpha} P(z)\|_{\infty} \geq \frac{\Lambda}{2}\left(\frac{|\alpha|}{k} - 1\right)|P(kz)|$$

Thus for $q > 0$, we have

$$\left\{k^{n-1}\|D_\alpha^\gamma P(z)\|_\infty\right\}^q \geq \left\{\frac{\Lambda}{2}\left(\frac{|\alpha|}{k}-1\right)\right\}^q |P(kz)|^q.$$

This gives for $0 \leq \theta < 2\pi$

$$k^{n-1}\|D_\alpha^\gamma P(z)\|_\infty \geq \frac{\Lambda}{2}\left(\frac{|\alpha|-k}{k}\right)\left\{\frac{1}{2\pi}\int_0^{2\pi}|P(ke^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}.$$

Invoking Lemma 2.1, it follows that

$$\|D_\alpha^\gamma P(z)\|_\infty \geq \frac{\Lambda(|\alpha|-1)}{2}\frac{\left\|\frac{|a_n|k^{n+1}+|a_0|}{|a_n|k^n+k|a_0|}+z\right\|_q}{\|1+zk^n\|_q}\|P(z)\|_q.$$

This completes the proof of Theorem 3.4. □

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