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GONALITY OF CURVES AND EXTENSIONS OF THE BASE FIELD

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Abstract. Let K be a perfect field and C a geometrically connected smooth curve of genus $g \geq 2$ defined over K. Here we gives some upper bounds for the K-gonality $\gamma_K(C)$ of C in terms of the gonality d of $C_{\overline{K}}$. In some cases, this is obtained giving an upper bound for g, sometimes $g \leq 5d^2/8 + 1$, sometimes $g \leq (d-2)^2/2$.

1. Introduction

Almost all the content of this note is due to a referee of a first version of a paper of mine with the same title. In particular he/she pointed out part (a) of Theorem 1.1 and suggested the use of [1] (which gives part (c) of Theorem 1.1). Let K be a perfect field and $\overline{K}$ its algebraic closure. In this note a curve over K is a smooth and geometrically integral projective scheme C of dimension one. We recall that the K-gonality $\gamma_K(C)$ of the curve C is the minimal degree of a non-constant rational map from C onto $\mathbb{P}^1$ defined over K ([4]). Since K is perfect, $\gamma_K(C)$ is the minimal degree of a non-constant separable morphism $C \to \mathbb{P}^1$. Set $\gamma(C) := \gamma_{\overline{K}}(C)$. We have $\gamma_K(C) \geq \gamma_F(C) \geq \gamma(C)$ for any field $F \supseteq K$ ([4, Appendix A]). If $C(K) \neq \emptyset$, then $\gamma_K(C)$ is the minimal degree of a line bundle R on C defined over K and with $h^0(X, R) \geq 2$ (or $h^0(X, R) = 2$, if you prefer). Let $g(X)$ denote the genus of X. We have $g(C) = g(C_{\overline{K}})$ ([5, Theorem 3.6.3 (b)]). In this note we prove the following result.

Theorem 1.1. *Set $d := \gamma(C)$. Let $G = G(\overline{K}/K)$ denote the absolute Galois group of K. Take $f : C(\overline{K}) \to \mathbb{P}^1$ evincing $\gamma(C)$ and let $\{f_1, \dots, f_k\}$ be the different elements of $\overline{K}(C)$ in the set $\{f^\sigma\}_{\sigma \in G}$. If $k = 1$, then $\gamma_K(C) = \gamma(C)$. Assume $k \geq 2$.*

(a) *We have $\gamma_K(C) \leq kd$. If $C(K) \neq \emptyset$, then $\gamma_K(C) \leq kd - k$. If there is a field extension L of K such that $t := \deg(L/K) \leq k$ and $C(L) \neq \emptyset$, then there is a line bundle A on C defined over K such that $\deg(A) = kd - k$ and $h^0(A) \geq 2$.*

(b) *Assume $C(K) \neq \emptyset$ and the existence of $m_1 : C \to C'$ and $m_2 : C' \to \mathbb{P}^1$, with C' and m_1 defined over K, m_2 defined over $\overline{K}$, $f = m_2 \circ m_1$ and*

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$e := \deg(m_2) \in \{2, \dots, d-1\}$. *Then* $e|d$, $e \le d/2$, $g(C') \le (e-1)^2$ *and* $\gamma_K(C) \le d(e-1)^2/e$. *Hence* $\gamma_K(C) \le (d-2)^2/2$ *in this case.*

(c) *Assume that* $C(\overline{K})$ *is not a multiple covering (over* $\overline{K}$*) of a curve of positive genus defined over* $\overline{K}$. *We have* $g \le \frac{d^2k}{2k-2} + 1$. *Hence if* $k \ge 5$, *then* $g \le 5d^2/8 + 1$.

(d) *If* C *is neither as in case (b) nor as in case (c), then* $g \le 5d^2/8 + 1$.

Remark 1.2. If $C(K) \neq \emptyset$, then $\gamma_K(C) \le g$ ([4, Proposition A.1 (iv)]). If $C(K) = \emptyset$, we have $\gamma_K(C) \le 2g-2$, because ω_C is defined over K and we assumed $g > 1$ ([4, Proposition A.1 (iii)]). Hence parts (a), (c) and (d) of Theorem 1.1 give a very good upper bound of $\gamma_K(C)$ in terms of the integer $d := \gamma(C)$.

Remark 1.3. Case (d) is dealt with in step (c) of the proof of Theorem 1.1. In that part of the proof of Theorem 1.1 for (C, K) not as in (b) we introduce an integer $m \ge 2$. The case $m = 2$ corresponds to case (c) of Theorem 1.1. If $m \ge 3$ we always prove that $g \le 5d^2/8+1$ and (except in step (c4)) that $g \le (d-2)^2/2$. In most cases a better upper bound for g is obtained. To prove more (except in a few cases) one should distinguish the case $m = 3$ from the case $m \ge 4$ and then the case $m = 4$ from the case $m \ge 5$ and so on. Notice that the condition " $m \ge 3$ " imposes some restrictions to d and very strong restrictions to the geometry of $C(\overline{K})$.

2. Lemmas

In characteristic zero the following lemma is the case $s = k$, $r_i = l_i = 1$ for all i of [1, Theorem 4.1].

Lemma 2.1. *Let* C *be a smooth and connected curve defined over an algebraically closed field* $\mathbb{K}$. *Let* $R_1, \dots, R_k$, $k \ge 2$, *be pairwise distinct spanned line bundles on* C *with* $h^0(C, R_i) = 2$ *for all* i. *Let* $f_i : C \to \mathbb{P}^1$ *be the morphism associated to* $|R_i|$. *Assume that each morphism* $(f_i, f_j) : C \to \mathbb{P}^1 \times \mathbb{P}^1$, $1 \le i < j \le k$, *is birational onto its image. Then* $h^0(C, R_1 \otimes \cdots \otimes R_k) \ge 1 + k(k+1)/2$. *Fix a general* $D \in |R_1 \otimes \cdots \otimes R_{k-1}|$. *Then* $\deg(D \cap B) \le 1$ *for every* $B \in |R_k|$.

Proof. Since $k \ge 2$, C has genus > 0. Since $\dim(|R_i|) = 1$, the map f_i is not composed with a Frobenius and hence it is separable. We may use induction on k, starting the induction with the case $k = 1$. Since each f_i is separable, D is reduced, say $D = P_1 + \cdots + P_x$ with $P_i \neq P_j$ for all $i \neq j$, and we may see each P_j as a general point of $C(\mathbb{K})$. Let A_j be the only divisor of $|R_k|$ containing P_j. Since P_j is general and f_k is separable, each A_j is reduced. Since (f_1, f_k) is birational onto its image and D is general, we have $f_k(P_i) \neq f_k(P_j)$ for all $i \neq j$. Since each A_j is a fiber of f_k, we have $A_i \cap A_j = \emptyset$ for all $i \neq j$. Let E be a general element of $|R_{k-1} \otimes R_k|$ containing P_k. Since (f_{k-1}, f_k) is birational onto its image and P_k is general, we have $E \cap A_k = \{P_k\}$ (as schemes). Set $D' := A_1 + \cdots + A_{k-2} + E$. By construction we have $D' \cap A_k = \{P_k\}$ (as schemes). Since $D' \cap A_k = \{P_k\}$ we get that for a general $D'' \in |R_1 \otimes \cdots \otimes R_{k-1}|$ there is at least one point, P, in the support of D'' with $\deg(B \cap D'') = 1$, where B is the only divisor of $|R_k|$ containing B. Since $C(\mathbb{K})$ is irreducible, this is true for every point of the

support of a general element of $|R_1 \otimes \cdots \otimes R_{k-1}|$, proving this particular case of [1, Theorem 4.1] in arbitrary characteristic. Since $\deg(D \cap B) \le 1$ for every $B \in |R_k|$, the case $r_j = l_j = 1$ for all j of [1, Lemma 4.2] works verbatim and gives that $h^0(C, R_1 \otimes \cdots \otimes R_k) \ge 1 + k(k+1)/2$. □

Notation 2.2. For all integers $d \ge 2$, $s \ge 2$, set $d = m(s-1)+q$ with $3-s \le q \le 1$, $R(d,s) := m^2(s^2-s)+ms(2q-2)+(q-1)(q-2)$ and $R'(d,s) := \min_{2\le t\le s} R(d,t)$.

We have $R(d,3) = 3d(d-1)/2+2$ if d is even and $R(d,3) = 3(d-1)^2/2$ if d is odd. We have $R(d,4) = 4d(d-2)/3+2$ if $d \equiv 0 \pmod 3$, $R(d,4) = 4(d-1)^2/3$ if $d \equiv 1 \pmod 3$ and $R(d,4) = 4d(d-2)/3$ if $d \equiv 2 \pmod 3$. Notice that $R'(d,s) \le R(d,s) \le d^2 s/(s-1)+2$.

Using Lemma 2.1 instead of [1, Theorem 4.1] we get in arbitrary characteristic the following particular case of [1, Lemma 4.2 and Theorem 4.3].

Lemma 2.3. *Let C be a smooth and connected curve defined over an algebraically closed field $\mathbb{K}$. Let $R_1, \dots, R_k$, $k \ge 2$, be pairwise distinct spanned line bundles on C with $h^0(C, R_i) = 2$ for all i. Let $f_i : C \to \mathbb{P}^1$ be the morphism associated to $|R_i|$. Assume that each morphism $(f_i, f_j) : C \to \mathbb{P}^1 \times \mathbb{P}^1$, $1 \le i < j \le k$, is birational onto its image. Then:*

(1) $h^0(C, R_1 \otimes \cdots \otimes R_k) \ge 1 + k(k+1)/2$.
(2) *Assume also* $\deg(R_i) = d$ *for all i. Then*

$$2g \le R'(d,k). \tag{2.1}$$

Remark 2.4. In the set-up of Lemma 2.3 assume $k \ge d$. The inequality in [1, Theorem 7.1] says

$$g \le d(d-1)/2.$$

As remarked in [1, lines before Theorem 7.1], this is a consequence of (2.1). Hence $\gamma_K(C) \le d(d-1)/2$ (assuming $C(K) \ne \emptyset$) and $\gamma_K(C) \le d(d-1)-2$ (only assuming $g \ge 2$).

Lemma 2.5. *Let C be a geometrically smooth projective curve defined over an algebraically closed field $\mathbb{K}$ and $R_1, \dots, R_k$, $k \ge 2$, line bundles on C which are pairwise non-isomorphic over $\mathbb{K}$ and such that $h^0(C, R_i) \ge 2$ for all i. Then $h^0(R_1 \otimes \cdots \otimes R_k) \ge k+2$.*

Proof. Taking a subsheaf $R_i' \subseteq R_i$ if necessary we reduce to the case in which each R_i, $i > 1$, is spanned. For each $i \ge 1$ set $L_i := R_1 \otimes \cdots \otimes R_i$. It is sufficient to prove that $h^0(C, L_2) \ge h^0(C, R_1) + 2$ and that $h^0(L_i) > h^0(L_{i-1})$ for all $i > 2$. Since R_i, $i \ge 2$ is spanned, a general pencil of $H^0(R_i)$ spans R_i. Any such pencil induces the following exact sequence

$$0 \to L_{i-1} \otimes R_i^\vee \to L_{i-1}^{\oplus 2} \to L_i \to 0 \tag{2.2}$$

Since $h^0(R_1 \otimes R_2^\vee) = 0$, (2.2) gives $h^0(L_2) \ge 4$. Hence to conclude by induction on i it is sufficient to check that $h^0(L_{i-1} \otimes R_i^\vee) < h^0(L_{i-1})$. Assume that this inequality fails. In particular $h^0(L_{i-1} \otimes R_i^\vee) > 0$. Fix a general $P \in C(\mathbb{K})$. Since P is general and $h^0(L_{i-1}) > 0$, P is not a base point neither of $L_{i-1} \otimes R_i^\vee$ nor of L_{i-1}. Since $h^0(R_i) \ge 2$, there is $\sigma \in H^0(R_i)$ such that $\sigma \ne 0$ and $\sigma(P) = 0$. The

multiplication by σ induces an inclusion $H^0(L_{i-1} \otimes R_i^\vee) \hookrightarrow H^0(\mathcal{I}_P \otimes L_{i-1})$. Since the latter space has dimension $h^0(L_{i-1}) - 1$, we get a contradiction. □

Remark 2.6. Let y be the minimal cardinality of a subset $S \subseteq \{1, \dots, k\}$ such that $|\otimes_{i \in S} R_i|$ maps $C(\overline{K})$ birationally onto its image. If $y < k$, then we may improve Lemma 2.5 using $k-y$ times [1, Lemma 4.2]. We get $h^0(R_1 \otimes \cdots \otimes R_k) \geq 2k-y+2$. If $y < k$, then [1] gives strong informations on the Brill-Noether theory of $C(\overline{K})$.

3. Proof of Theorem 1.1.

Take f evincing $\gamma(C)$ and set $\{f_1, \dots, f_k\} = \{f^\sigma\}_{\sigma \in G}$ with $f_i \neq f_j$ for all $i \neq j$. Let $u_i : C(\overline{K}) \to \mathbb{P}^1(\overline{K})$ be the degree d morphism induced by u_i. Set $R_i := u_i^*(\mathcal{O}_{\mathbb{P}^1}(1)) \in \mathrm{Pic}^d(C)(\overline{K})$. Each R_i is spanned and $|R_i|$ evinces $\gamma(C)$. Hence $h^0(R_i) = 2$. Set $R := R_1 \otimes \cdots \otimes R_k$. Since $h^0(R_i) = 2$ for all i and $f_i \neq f_j$, Lemma 2.5 gives $h^0(C, R) \geq k + 2$. In the set-up of (c) we may also use Lemma 2.3.

(a) Set $v := f_1 \cdots f_k \in \overline{K}(C) \setminus \{0\}$. The rational function v induces a non-constant function $w : C(\overline{K}) \to \mathbb{P}^1(\overline{K})$. Set $M := w^*(\mathcal{O}_{\mathbb{P}^1}(1)) \in \mathrm{Pic}(C)(\overline{K})$.

Since each $\sigma \in G$ permutes the set $\{f_1, \dots, f_k\}$, we have $v \in K(C) \setminus \{0\}$ (take a finite Galois extension L of K, say with Galois group H, containing each f_i and use that $L(C)^H = K(C)$, because $L(C)$ has degree $\sharp(H)$ over $K(C)$ by [5, Proposition 3.6.1 (c)]). We have $\deg(w) \leq kd$ and equality holds (i.e. $R \cong M$ over $\overline{K}$) unless some of the zeros of f_i cancel with some of the poles of f_j for some $j \neq i$. Assume $\deg(M) < kd$ and call D the effective divisor of $C(\overline{K})$ with degree $kd - \deg(M)$ which measures these cancellations. Fix $\sigma \in G$. Since σ permutes the set $\{f_1, \dots, f_k\}$, we have $\sigma(D) = D$. Hence D is defined over K. Since $R \cong M(D)$, R is defined over K. Assume the existence of an extension L of K such that $\deg(L/K) = t \leq k$ and $C(L) \neq \emptyset$. Hence there is an effective degree t divisor D' on C defined over K. Hence $R(-D')$ is a degree $kd - t$ line bundle defined over K. Notice that $h^0(R(-D')) \geq 2$. Hence $\gamma_K(C) \leq kd - k$ if $C(K) \neq \emptyset$.

(b) By [4, Proposition 2.4] there is a triple (e, C', m_1), where $e \in \{1, \dots, d\}$, $d/e \in \mathbb{N}$, C' is a smooth and geometrically connected curve defined over K such that $g(C') \leq (e-1)^2$ and $m_1 : X \to C'$ is a morphism of degree d/e defined over K. Since $g > 0$, we have $e \geq 2$. In this step we assume $e \neq d$. Since $d/e \in \mathbb{N}$, d is not a prime. We have $\gamma_K(X) \leq (d/e) \cdot \gamma_K(C')$ ([4, Proposition A.1 (vi)]). If $C(K) \neq \emptyset$, then $C'(K) \neq \emptyset$, because m_1 is defined over K. Hence $\gamma_K(C) \leq d(e-1)^2/e \leq (d-2)^2/2$. This is the case considered in part (b) of Theorem 1.1.

(c) From now on we assume $e = d$, i.e. we assume that m_1 is an isomorphism. By the proof of [2, Theorem 2.1], this case corresponds to the case in which the map $f_n : C \to C_n$ of [2, page 6], is an isomorphism.

Let $m \geq 2$ be the maximal integer such that there is a chain of subsets $S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_m \subseteq \{1, \dots, k\}$ with $\sharp(S_1) = 1$ and $\overline{K}(f_j, j \in S_i) \subsetneq \overline{K}(f_j, j \in S_{i+1})$. Hence $\overline{K}(f_i, \{i\} = S_1) = \overline{K}(\mathbb{P}^1)$, $\overline{K}(f_i, i \in S_m) = \overline{K}(C)$ and for each $2 \leq i \leq m$ there is a finite map $h_i : C_i \to C_{i-1}$ defined over $\overline{K}$, where $\overline{K}(C_i) = \overline{K}(f_i, i \in S_{i+1})$

and $C_0 = \mathbb{P}^1$. Set $e_i := \deg(h_i)$ and $a_i := \prod_{j=1}^{i} e_j$. By assumptions we have $e_i \geq 2$ for all i and $d = e_1 \cdots e_m$. For each $i \in \{1, \dots, m-1\}$ we have

$$g(C_i) \leq (a_i - 1)^2 \tag{3.1}$$

([2, property (ii) at page 6]). For each $i \in \{2, \dots, m\}$ we have

$$g(C_i) \leq 2e_i g(C_{i-1}) + (e_i - 1)^2 \tag{3.2}$$

(first inequality of [2, eq. (2.3)]).

Until step (d) we assume $m \geq 3$.

Recall that $g(C_{m-2}) \leq (\frac{d}{e_m e_{m-1}} - 1)^2$ by (3.1). Hence (3.2) gives $g(C_{m-1}) \leq 2e_{m-1}(\frac{d}{e_m e_{m-1}} - 1)^2 + (e_{m-1} - 1)^2 = 2(\frac{d}{e_m} - e_{m-1})^2/e_{m-1} + (e_{m-1} - 1)^2$ and $g \leq 4e_m \left(2(\frac{d}{e_m} - e_{m-1})^2/e_{m-1} + (e_{m-1} - 1)^2\right) + (e_m - 1)^2$. Since $4e_m(e_{m-1} - 1)^2 = 4e_m e_{m-1}^2(d - \frac{d}{e_{m-1}})^2/d^2$, we get

$$\begin{aligned} g &\leq 8(d - e_m e_{m-1})^2/e_m e_{m-1} \\ &+ 4e_m e_{m-1}^2(d - \frac{d}{e_{m-1}})^2/d^2 + (e_m - 1)^2 \end{aligned} \tag{3.3}$$

(c1) In this step we assume $e_m e_{m-1} \leq 23$. The set $\{e_m, e_{m-1}\}$ is one of the following sets: $\{2, x\}$, $2 \leq x \leq 11$, $\{3, x\}$, $3 \leq x \leq 7$, $\{4\}$.

First assume $e_m = 2$ and $e_{m-1} = 11$. By (3.1) and (3.2) we have $g(C_{m-2}) \leq (d/22 - 1)^2$, $g(C_{m-1}) \leq 22(d/22 - 1)^2 + 100$ and $g \leq 88(d/22 - 1)^2 + 400 + 1 = 2(d - 11)^2/11 + 401$. Since $d \geq 44$, we get $g \leq (d - 2)^2/2$.

Now assume $e_m = 11$ and $e_{m-1} = 2$. We get $g(C_{m-2}) \leq (d/22 - 1)^2$, $g(C_{m-1}) \leq 4(d/22 - 1)^2 + 1$, $g(C_{m-1}) \leq 88(d/22 - 1)^2 + 22 + 100 = 2(d - 11)^2/11 + 122 < (d - 2)^2/2$.

Now assume $e_m = 3$ and $e_{m-1} = 7$. We get $g(C_{m-2}) \leq (d/21 - 1)^2$, $g(C_{m-1}) \leq 14(d/21 - 1)^2 + 36$ and $g \leq 4(d - 21)^2/21 + 6 \cdot 36 + 4$. Since $d \geq 42$, we get $g \leq (d - 2)^2/2$.

Now assume $e_m = e_{m-1} = 4$. We get $g(C_{m-2}) \leq (d/16 - 1)^2$, $g(C_{m-1}) \leq 8(d/16 - 1)^2 + 9$ and $g \leq (d - 16)^2/6 + 9 \cdot 8 + 9$. Since $d \geq 32$, we get $g < (d - 2)^2/2$.

In all other cases we get $g < g'$, where g' is the upper bound just computed either in the case $(e_m, e_{m-1}) = (2, 11)$ or in the case $(e_m, e_{m-1}) = (3, 7)$ and the inequality $g \leq (d - 2)^2/2$ works just assuming $d = e_m e_{m-1} a_{m-2}$ with $a_{m-2} \geq 2$.

(c2) In this step we assume $e_m e_{m-1} \geq 24$ and $e_m > d/9$. Since $e_m e_{m-1} \geq 24$, we have $d \geq 24 \cdot 2^{m-2}$. Since $d/e_m = e_1 \cdots e_{m-1}$ we are in one of the following cases

(1) $m = 4$, $e_1 = e_2 = e_3 = 2$;
(2) $m = 3$, $\{e_1, e_2\} = \{2, 4\}$;
(3) $m = 3$, $\{e_1, e_2\} = \{2, 3\}$;
(4) $m = 3$, $e_1 = e_2 = 2$.

First assume $m = 4$ and $e_1 = e_2 = e_3 = 2$. From (3.1) and (3.2) we get $g(C_1) \leq 1$, $g(C_2) \leq 3$, $g(C_3) \leq 7$ and $g \leq 2(d/8)g(C_3) + (d/8 - 1)^2 \leq 7d/4 + (d/8 - 1)^2$.

Now assume $m = 3$, $e_1 = 2$ and $e_2 = 4$. From (3.1) and (3.2) we get $g(C_1) \leq 1$, $g(C_2) \leq 17$ and $g \leq 2(d/8) \cdot 17 + (d/8 - 1)^2$. Since $d \geq 17$, we have $17d/4 \leq$

$(d-2)^2/3-1$. Hence $g \le (d-2)^2/2$. Now assume $m=3$, $e_1=4$ and $e_2=2$. Since $e_1=4$, we have $d \ge 96$. We get $g(C_1) \le 9$, $g(C_2) \le 37$ and $g \le 2(d/8)\cdot 37 + (d/8-1)^2 \le (d-2)^2/2$.

Now assume $m=3$, $e_1=2$ and $e_2=3$. We get $g(C_1) \le 1$, $g(C_2) \le 10$ and $g \le 2(d/6)\cdot 10 + (d/6-1)^2$. We get $g \le (d-2)^2/2$, because $17d^2 \ge 156d+32$ for $d \ge 48$.

Now assume $m=3$, $e_1=3$ and $e_2=2$. Hence $d \ge 72$. We have $g(C_1) \le 4$, $g(C_2) \le 17$ and $g \le 2(d/6)\cdot 16 + (d/6-1)^2 \le (d-2)^2/2$.

Now assume $m=3$ and $e_1=e_2=2$. We get $g(C_1) \le 1$, $g(C_2) \le 5$ and $g \le 10e_3+1 = 5d/2+1$.

(c3) Now assume $e_m e_{m-1} \ge 24$, $e_m \le d/9$ and $4e_m e_{m-1}^2 \le 25d^2/162$. From (3.3) we get $g \le (d-24)^2/3 + (d/9-1)^2 + (d-4)^2(1/6-1/81) \le (d-2)^2/2$.

(c4) Now assume $e_m e_{m-1} \ge 24$, $e_m \le d/9$ and $4e_m e_{m-1}^2 > 25d^2/162$. Set $\alpha := \prod_{i=1}^{m-2} e_i$. We have $d^2 = (e_m e_{m-1})^2 \alpha^2 e_m$. By (3.3) to get $g \le 5d^2/8$ it is sufficient to have $4e_m e_{m-1}^2/d^2 \le 5/8 - 1/3 - 1/81 = 181/648$, i.e. $\alpha^2 e_m \ge 4\cdot 648/181 \ge 14.33$. Hence it is sufficient to have $\alpha^2 e_m \ge 15$. Now assume $\alpha^2 e_m \le 14$. We get $\alpha = 2$ and $e_m \in \{2,3\}$. Since $\alpha = 2$, we have $m=3$. We have $g(C_1) \le 1$, $g(C_2) \le 2e_2 + (e_2-1)^2$ and $g \le 4e_3e_2 + 4e_3(e_2-1)^2 + 1 = 4e_3e_2^2 - 4e_3e_2 + 4e_3 + 1 \le 4e_3e_2^2 = d^2/e_3 \le d^2/2$.

(d) Now assume $m=2$. Since m is defined as a maximum, we have $\overline{K}(f_i,f_j) = \overline{K}(C)$ for all $i \ne j$, i.e. for all $i \ne j$ the pencils f_i and f_j are not composed with the same involution. Lemma 2.3 gives $g \le R'(d,k)/2$ and hence $g \le d^2k/(2k-2)+1$. If $k \ge 5$ we also get $g \le 5d^2/8+1$. □

No more cases are needed to complete the proof of Theorem 1.1. However, in case (c) of Theorem 1.1 we didn't get a good upper bound for the integer $\gamma_K(C)$ if $k \gg d/2$.

Take the set-up of part (a) of Theorem 1.1. The condition $C(L) \ne \emptyset$ for some $t \in \{2,\dots,k\}$ and some degree t extension L is quite often satisfied if K is a finite field by Hasse-Weil bound.

Remark 3.1. Take f evincing $\gamma(C)$ and assume the existence of C', m_1 and m_2 such that $f = m_2 \circ m_1$, $\deg(m_1) = e \in \{2,\dots,d-1\}$ and C' and m_1 are defined over K. Since C' and m_1 are defined over K, we have $\gamma_K(C) \le e\gamma_K(C')$. Assume $g(C') > 1$. In this case we have $\gamma_K(C') \le 2g(C')-2$ and hence $\gamma_K(C) \le 2e(d/e-1)^2 - 2e = 2(d-e)^2/e - 2e$. If $g(C')=1$ and $C'(K)=\emptyset$, then $\gamma_K(C')$ may be arbitrarily large ([4, Remark 2.6]). Over a finite field K we may use that $\gamma_K(C') \le g(C')+1$ ([3, Corollary 2.3]) and hence $\gamma_K(C') \le e(d/e-1)^2 + e$.

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