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SAMPLED-DATA CONTROLLERS AND OBSERVER-DESIGN FOR UNCERTAIN SEMILINEAR SYSTEMS WITH POSITIVITY CONSTRAINTS

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Abstract. This paper deals with the problem of positive stabilization of a class of uncertain semilinear diffusion systems using sampled-data controllers. The conditions for the existence of such controllers are obtained through certain inequalities. By solving these inequalities, we determine upper bounds for the sampling intervals that guarantee exponential stability, as well as the corresponding decay rates. Furthermore, the design of a positive observer for systems with discrete measurements is studied. The stability of the estimation error is ensured through the same LMIs. Numerical simulations are presented to illustrate the effectiveness of the proposed approach.

1. Introduction

Positive systems occupy a central place in systems and control theory, attracting a keen interest among many researchers in the fields of modeling, analysis, and control (see, for example, [3, 11, 29, 30]). These systems are characterized by models where state trajectories remain nonnegative for any nonnegative input and initial condition. This property is motivated by several real phenomena observed in physics, biology, and economics (see, for example, [4, 20]).

In the literature, various problems in control theory related to positivity have been addressed. Several approaches have been introduced to stabilize specific classes of positive linear systems through state feedback, ensuring that the closed-loop system's trajectories remain positive and stable. (for example, [1, 11, 31, 40, 30]). Moreover, the design of positive observers has attracted growing interest (see [28, 13, 33, 6, 7]), with researchers introducing new positive observers based on standard Luenberger-like models to guarantee non-negative state estimates at all times.

In the last decades, research on sampled data control systems has grown significantly, largely facilitated by technological advances in numerical control, leading to notable advances in analysis and synthesis. Several strategies have been developed to study these systems. Concerning the stability analysis of sampled

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systems in finite dimension, three main methods can be identified: The first models the sampled system as a discrete-time system incorporating uncertainties, as indicated by the work of [26, 35, 16], where robust linear matrix inequalities are applied to establish sufficient stability conditions. The second method approaches the sampled system as a continuous system with an input delay, explored in [15, 32, 36], by obtaining stability conditions via the Lyapunov–Krasovskii method. The third approach treats the sampled system as an impulsive system, as discussed in [25], which allows for the incorporation of additional dynamical features. This latter method focuses on creating Lyapunov functionals that depend on the sampling time, in order to decrease the conservatism typically associated with classical Lyapunov functionals.

Although there have been significant advancements in the analysis and synthesis of finite-dimensional sampled-data systems, research on the control of distributed parameter systems with sampled-data remains limited. However, important contributions in this area include studies that address discrete-time methods for sampled-data boundary control [10] and sampled-data state feedback control [24, 39], as well as model reduction approaches that use approximate finite-dimensional models to represent the dominant dynamics of infinite-dimensional systems [17]. Unfortunately, the findings from these studies do not support performance analysis for closed-loop systems. Recently, a novel strategy for sampled-data state-feedback control was introduced in [14], specifically for one-dimensional uncertain semilinear diffusion systems. This strategy involves partitioning the spatial domain into N intervals, with sensors placed centrally in each interval. By leveraging the spatial state measurements provided by these sensors, along with zero-order-hold (ZOH) devices, the resulting state feedback control law is formulated as a piecewise constant signal in both time and space. Since this sampled-data approach requires state measurements to be available at a finite number of fixed spatial points at each sampling instant, it lends itself to straightforward implementation in real-world applications. (for further insightful reading on this topic, you may refer to [39, 38]). The literature review the area of sampled-data controllers and observers, especially for positive semilinear systems, is notably unexplored. To our knowledge, there is currently no research addressing the problem of sampled data controllers or observers for positive semilinear systems.

The problem of stabilizing an uncertain semilinear diffusion equation with positivity constraints under sampled-data controllers remains unresolved. The first objective of this article is to address this issue for such systems under point measurements. (LMIs) are employed to find sufficient conditions that guarantee the positivity of the closed-loop system and exponential stability under sampled-data controllers. The main challenge lies in ensuring the positivity and stability of the closed-loop system simultaneously. Another objective is to design a positive observer for the uncertain semilinear diffusion equation in cases where only sampled-output measurements are available. We derive sufficient LMI conditions for the positivity of the observers and the exponential stability of the estimation error system using an appropriate Lyapunov functional. Although control methods for uncertain systems are well studied, our approach presents a novelty in its application to sampled data of positive semilinear systems. By sampling data spatially while maintaining temporal continuity, we use states available only at specific spatial points to ensure stability and positivity despite present uncertainties.

The organization of this paper is presented below:In the next section, Section 2, we describe the class of systems being considered and outline the two problems at hand. Section 3 explores the positive stabilization issue using sampled-data controllers. The design of a positive observer, taking into account sampled-output measurements, is detailed in Section 4. Section 5 presents several numerical simulations. Finally, Section 6 offers additional conclusions and discusses open questions.

Notation. We denote by $\mathcal{C}^1$ and $x_s(s)$ (resp. $x_{ss}(s)$) the space of continuous differentiable functions and the first (resp. second) derivative of the function $x_s(s)$. $L_2\left[0,l\right]$ denotes the space of square Lebesgue integrable functions with corresponding norm

$$\|x(s,t)\|_{L_2[0,l]} = \left(\int_0^l x^2(s,t)ds\right)^{\frac{1}{2}}$$

and $\mathcal{H}^1\left[0,l\right] = W^{1,2}\left(\left[0,l\right],\mathbb{R}\right)$ is the Sobolev space of absolutely continuous functions x defined from the interval $[0,l]$ to $\mathbb{R}$ with square Lebesgue integrable derivative $x_s(s)$. $\mathcal{H}^2\left[0,l\right]$ is the Sobolev space of absolutely continuous functions x defined from the interval $[0,l]$ to $\mathbb{R}$ with square Lebesgue integrable derivatives $x_s(s)$ and $x_{ss}(s)$.

2. System Description and Problem Statement

Let the following semilinear system:

$$\begin{cases} x_t(s,t) = \dfrac{\partial}{\partial s}\left[a(s)x_s(s,t)\right] + f\left(x(s,t),s,t\right)x(s,t) + u(s,t), \ \ s \in \left[0,l\right], \\ x(s,0) = x_0(s), \end{cases} \tag{2.1}$$

Here, $l > 0$, x denotes the state, and u represents the control input. The functions a and f are of class $\mathcal{C}^1$ and may be unknown. We assume that $0 < a_0 \leq a$ and $f_m \leq f \leq f_M$, where a_0, f_m, and f_M are known bounds.

We equip the system with one of the following boundary conditions for all $r > 0$:

Dirichlet boundary conditions:

$$x(0,r) = x(l,r) = 0 \tag{2.2}$$

Mixed boundary conditions:

$$x_s(0,r) = \gamma x(l,r) = 0, \quad x(l,r) = 0 \tag{2.3}$$

Remark 2.1. Note the system (2.1) models many physical phenomena. A typical example is the convection–diffusion equation described by (see [34])

$$\begin{cases} x_t(s,r) = a_1 x_{ss}(s,r) - a_2 x_s(s,r) + g\left(x(s,r),s,r\right)x(s,r) + u(s,r), \\ x_s(0,r) = \beta_0 x(0,r),\ x(l,r) = 0 \\ x(s,0) = x_0(s), \end{cases} \tag{2.4}$$

where $a_1 > 0$, $a_2 \in \mathbb{R}$ and the unknown function g belong to class $\mathcal{C}^1$ is such that $g_m \leq g \leq g_M$. Here, using the change variables $z(s,r) = e^{-\frac{a_2}{a_1}s}x(s,r)$ in (2.4) leads to

$$\begin{cases} z_r(s,r) = a_0 z_{ss}(s,r) + g_0\left(z(s,r),s,r\right)z(s,r) + e^{-\frac{a_2}{a_1}s}u(s,r), \\ z_s(0,r) = (\beta_0 - \dfrac{a_2}{a_1})z(0,r),\ x(l,r) = 0 \\ z(s,0) = z_0(s), \end{cases} \tag{2.5}$$

where $g_0\left(z(s,r),s,t\right) = g\left(e^{\frac{a_2}{a_1}s}z(s,r),s,r\right) - \frac{a_2^2}{4a_1}$.

Let the system (2.1) with the boundary condition (2.2) or (2.3), and divide the interval $[0,l]$ into N sampling sub-intervals as follows

$$0 = s_0 < s_1 < s_2 < \cdots < s_N = l\ ,\gamma \geq 0 \text{ and } s_{j+1} - s_j \leq \Delta.$$

Additionally, We presume that N sensors are positioned in $\bar{s}_j = \frac{s_{j+1}+s_j}{2}$, $j = 0,1,\ldots,N-1$ of intervals $[s_j, s_{j+1})$. In this case, the sensors provide a discrete measurements of the state, and the system output is given by

$$y_j(t) = x(\bar{s}_j,t),\quad j = 0,1,\ldots,N-1,\ t \geq 0 \tag{2.6}$$

At some points in time, the state $x(s,t)$ can be measured for all $\bar{s}_j$ in the interval $\jmath = 0,1,\ldots,N-1$.

The primary contribution of the study is:

A. Demonstrate the well-posedness of the closed-loop system with the Dirichlet boundary conditions (2.2) when using the controller (sampled in space and continuous in time):

$$u(s,t) = -Kx(\bar{s}_j,t),\ s \in \left[s_j, s_{j+1}\right),\ j = 0,1,\ldots,N-1, \tag{2.7}$$

with a gain $K > 0$. The well-posedness under the mixed conditions (2.3) can be established in a similar manner.

B. Positive stabilization (see Definition 2.6) of the system (2.1) via sampled data state-feedback controller in the context of an infinite dimensional setting.

C. Design a positive semilinear observer (see Definition 2.8) under the sampled-data measurements given by (2.6) for the system (2.1)-(2.2).

For the example given by (2.4), the control law (2.8) should be modified as follows:

$$u(s,t) = -Ke^{-\frac{a_2}{a_1}(\bar{s}_j - s)}x(\bar{s}_j,t) = -Ke^{\frac{a_2}{a_1}s}z(\bar{s}_j,t),\ s \in \left[s_j, s_{j+1}\right),\ j = 0,1,\ldots,N-1. \tag{2.8}$$

Let's review certain definitions and results concerning semilinear systems. To do so, consider the following semilinear system on the Hilbert space X with a positive cone X^+.

$$\begin{cases} \dot{x}(r) = \mathbf{A}x(r) + \mathbf{F}\left(r, x(r)\right) + \mathbf{B}u(r), \\ x(0) = x_0, \end{cases} \tag{2.9}$$

where $\mathbf{A}$ generates a C_0-semigroup $\left(T_{\mathbf{A}}(r)\right)_{r\geq 0}$ on X (see [27]), $\mathbf{F}$ is a function of class $\mathcal{C}^1$, and $\mathbf{B}$ is a bounded linear operator from U to X, where U denotes an ordered Banach space with a positive cone U^+. The input is denoted by $u(\cdot)$. Positive semilinear systems are defined as follows.

Definition 2.2. Let $x_0 \in X^+$ and any input $u \in U^+$, the system (2.9) is said to be positive if $x(r)$ remains in X^+ for all $r \geq 0$.

Definition 2.3. [3]
The C_0-semigroup $\left(T_{\mathbf{A}}(r)\right)_{r\geq 0}$ is positive if all operators $T_{\mathbf{A}}(r)$, $r \geq 0$, are positive i.e. $T_{\mathbf{A}}(r)X^+ \subset X^+$, for all $r \geq 0$.

Theorem 2.4. *Let $\mathbf{A}$ be the generator of the C_0-semigroup $\left(T_{\mathbf{A}}(r)\right)_{r\geq 0}$ on X. Then, the system (2.9) is positive if $\left(T_{\mathbf{A}}(r)\right)_{r\geq 0}$, $\mathbf{F}$, and $\mathbf{B}$ are all positive.*

Proof. The mild solution $x(r)$ of the system (2.9) can be expressed as:

$$x(r) = T_{\mathbf{A}}(r)x_0 + \int_0^r T_{\mathbf{A}}(r-s)\mathbf{F}(x(s))ds + \int_0^r T_{\mathbf{A}}(r-s)\mathbf{B}u(s)ds.$$

The C_0-semigroup $\left(T_{\mathbf{A}}(r)\right)_{r\geq 0}$ is positive on X, which means that $T_{\mathbf{A}}(r)x_0 \in X^+$ for every $x_0 \in X^+$. The positivity of the nonlinear operator ensures that $\int_0^r T_{\mathbf{A}}(r-s)\mathbf{F}(x(s))ds$ is positive. Additionally, since both $\mathbf{B}$ and $\left(T_{\mathbf{A}}(r)\right)_{r\geq 0}$ are positive, it follows that $\int_0^r T_{\mathbf{A}}(r-s)\mathbf{B}u(s)ds$ is also positive. Hence, the system (2.9) is positive. □

Definition 2.5. The system (2.9) is defined to be exponentially stable if there exist two positive constants M and ω such that

$$\|x(r)\| \leq Me^{-\omega r}\|x_0\| \quad \forall r > 0. \tag{2.10}$$

Definition 2.6. The (2.9) is positively exponentially stabilizable if there exists a state feedback $\mathbf{K}$ such that the corresponding closed-loop system is exponentially stable and positive.

The semilinear system (2.9) is augmented by the following output:

$$y(r) = \mathbf{C}x(r), \tag{2.11}$$

where $\mathbf{C}$ is a positive linear operator defined from X into Y, where Y is a finite dimensional Hilbert space.

The Luenberuger like observer is given by

$$\begin{cases} \dot{\hat{x}}(r) = \left(\mathbf{A} + \mathbf{LC}\right)\hat{x}(r) + \mathbf{F}\left(\hat{x}(r)\right) - \mathbf{L}y(r), \\ \hat{x}(0) = \hat{x}_0, \end{cases} \tag{2.12}$$

where $\mathbf{L}$ (observer gain) is a bounded linear operator from Y to H.

Remark 2.7. Ensuring the positivity of the observer is essential for systems where state variables, such as concentrations or populations, must remain non-negative. This constraint is incorporated using (LMIs), which guarantee that the state estimates produced by the observer are always positive, thereby maintaining physical realism and reliability.

Definition 2.8. Consider the positive system (2.9) and (2.11). We refer to the system (2.12) as a positive observer for the system (2.9) and (2.11) if the following two conditions are met.

i. the system (2.12) is positive, that is for any positive initial condition $\hat{x}_0 \in X^+$ and the corresponding positive output $y(r)$, the trajectory $\hat{x}(r)$ remain in x^+ for all $r \geq 0$.
ii. the estimation error $e(r) = \hat{x}(r) - x(r)$ tends to zero as $r \to \infty$.

Lemma 2.9. *Let $x \in \mathcal{H}^1[0,l]$ be a scalar function for which either $x(0) = 0$ or $x(l) = 0$. Then, Wirtinger's inequality holds*

$$\int_0^l x^2(s)ds \leq \frac{4l^2}{\pi^2}\int_0^l x_s^2(s)ds.$$

In addition, if $x(0) = x(l) = 0$, then

$$\int_0^l x^2(s)ds \leq \frac{4}{\pi^2}\int_0^l x_s^2(s)ds.$$

3. Positive stabilization of a semilinear system

In this section, we establish the well-posedness of the closed-loop system with Dirichlet boundary conditions (2.2) under the controller defined by (2.8). Additionally, we examine the stability of the resulting closed-loop system using the Lyapunov method.

3.1. Well-posedness of the closed-loop system.

The closed-loop system with the Dirichlet boundary conditions (2.2) subject to the control input specified in equation (2.8), is expressed in the following manner:

$$\begin{cases} x_t(s,t) = \dfrac{\partial}{\partial s}\left[a(s)x_s(s,t)\right] + f\left(x(s,t),s,t\right)x(s,t) - Kx(s,t) + K\displaystyle\int_{\bar{s}_j}^{s} x_\theta(\theta,t)d\theta, \\ \qquad\qquad s_j \leq s < s_{j+1},\ \bar{s}_j = \dfrac{s_{j+1}+s_j}{2},\ 0 \leq j \leq N-1, \\ x(0,t) = x(l,t) = 0, \\ x(s,0) = x_0(s), \quad s \in [0,l]\,,\ l > 0. \end{cases} \tag{3.1}$$

where, we have used the relation $x(\bar{s}_j,t) = x(s,t) - \int_{\bar{s}_j}^{s} x_\theta(\theta,t)d\theta,\ s_j \leq s < s_{j+1},\ \bar{s}_j = \frac{s_{j+1}+s_j}{2},\ 0 \leq j \leq N-1.$

The problem (3.1) can be rewritten on $H = L_2[0,l]$ as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + F(t,x),\ t \geq 0, \\ x(0) = x_0, \end{cases} \tag{3.2}$$

where, A is given by

$$A\cdot = \frac{\partial}{\partial s}\left[a(s)\frac{\partial\cdot}{\partial s}\right] \tag{3.3}$$

on its domain

$$\mathcal{D}(A) = \left\{s \in \mathcal{H}^2[0,l] :\ x(0) = x(l) = 0\right\}.$$

$F:\ \mathbb{R}\times\mathcal{H}\,[0,l] \longrightarrow L_2\,[0,l]$ is defined by

$$F\,(t, x(\cdot,t)) = f\,(x(s,t), s, t)\,x(s,t) - Kx(s,t) + K\int_{\bar{s}_j}^{s} x_\theta(\theta,t)d\theta \tag{3.4}$$

We know that A generates a strong continuous exponentially stable semigroup $T_A(t)$ on H. Moreover, there exist $M \geq 1$ and $\omega > 0$ such that:

$$\|T_A(t)\| \leq Me^{-\omega t}, \forall t \geq 0.$$

Note that $\mathcal{D}(A) = A^{-1}H$ is a Hilbert space with the graph inner product $\langle v, w\rangle_{\mathcal{D}(A)} = \langle Av, Aw\rangle$ for all v and w in $\mathcal{D}(A)$. The operator $-A$ is positive, so the square root $(-A)^{\frac{1}{2}}$ of $-A$ is well defined on its domain given by

$$\mathcal{D}(A) = (-A)^{\frac{1}{2}} = H_{\frac{1}{2}} = \left\{s \in \mathcal{H}^1[0,l] :\ x(0) = x(l) = 0\right\}.$$

The space $H_{\frac{1}{2}}$ is a Hilbert space with the scalar product $\langle v, w\rangle_{\frac{1}{2}} = \left\langle (-A)^{\frac{1}{2}}v, (-A)^{\frac{1}{2}}w\right\rangle$ for all v and w in $H_{\frac{1}{2}}$ (see [37]). Its know also that the operator A has an extension to a bounded operator denoted by A defined from $H_{\frac{1}{2}}$ to $H_{-\frac{1}{2}}$, where $H_{-\frac{1}{2}}$ is the dual space with respect to the pivot space H. In addition, the injection $\mathcal{D}(A) \subset H_{\frac{1}{2}} \subset H$ is continuous.

Next, we recall the definition of strong solution of the system (3.2).

Definition 3.1. The function $x:\ [0,T] \longrightarrow \mathcal{H}_{\frac{1}{2}}$ is said to be strong solution of (3.2) if

$$x(t) = x(0) + \int_0^t \left[Ax(t) + F(\theta, x(\theta))\right] d\theta \tag{3.5}$$

is valid for all $t \in [0,T]$. The integral is calculated in $\mathcal{H}_{-\frac{1}{2}}$.

The function f is of a class $\mathcal{C}^1$ then

$$\|F(t_1,x_1) - F(t_2,x_2)\|_2 \leq L\left[\mid t_1 - t_2 \mid + \left\|(-A)^{\frac{1}{2}}(x_1 - x_2)\right\|_2\right]$$

holds locally in $(t_i, x_i) \in \mathbb{R}\times H_{\frac{1}{2}}$, $i = 1,2$, for some constant $L > 0$. All assumptions required by Theorem 3.3.3 of [21] are satisfied. Thus, the closed-loop system (3.2) has a locally unique strong solution $x(t) \in H_{\frac{1}{2}}$. Moreover, the function as f is bounded, then there exists $L_1 > 0$ such that

$$\|F(t,x)\|_{L_2[0,l]} \leq L_1\left(\left\|(-A)^{\frac{1}{2}}x\right\|_{L_2[0,l]}\right),\quad \forall x \in H_{\frac{1}{2}}.$$

Hence, the strong solution initialized with $x_0 \in H_{\frac{1}{2}}$ exists for all $t \geq 0$.

3.2. **Positive stabilization via sampled data controllers.** The positive stabilization problem involves returning to finding the conditions of existence of state feedback satisfying the conditions of Definition 2.6. Let us consider the following Lyapunov function.

$$V(t) = \int_0^l x^2(s,t)ds \tag{3.6}$$

We will give conditions that guarantee $\dot{V}(t) + 2\omega V(t) \leq 0$ along the closed-loop (3.1) under the boundary conditions.

$$\begin{aligned} \dot{V}(t) + 2\omega V(t) \leq 0 &\Leftrightarrow V(t) \leq e^{-2\omega t} V(t) \\ &\Leftrightarrow \int_0^l x^2(s,t)ds \leq e^{-2\omega t} \int_0^l x^2(s,0)ds \end{aligned} \tag{3.7}$$

For the strong solution of the closed-loop (3.2) under the boundary conditions is exponentially stable with the decay rate ω.

Theorem 3.2. *Assume that there exist $K > 0, \omega > 0$ and $\Delta > 0$ such that the following inequalities holds*

$$\begin{cases} R^{-1}K\frac{\Delta}{\pi} - 2a_0 \leq 0 \\ RK\frac{\Delta}{\pi} + 2\omega + 2(f_M - K) + \frac{\pi^2}{bl^2}\left(R^{-1}K\frac{\Delta}{\pi} - 2a_0\right) \leq 0 \\ K \leq f_m \end{cases} \tag{3.8}$$

where b and R are given positive values.

i. *The closed-loop system (3.2) under mixed boundary conditions (2.3) is positively exponentially stable if the LMIs (3.8) are feasible with $b = 4$.*
ii. *The closed-loop system (3.2) under the Dirichlet boundary conditions (2.2) is positively exponentially stable if the LMIs (3.8) are feasible with $b = 1$.*

Proof. To show the exponential stability of the closed-loop, we consider the following Lyapunov function

$$V(t) = \int_0^l x^2(s,t)ds \tag{3.9}$$

Differentiating V along the closed-loop, we obtain:

$$\begin{aligned} \dot{V}(t) &= 2\int_0^l x(s,t)x_t(s,t)ds \\ &= 2\int_0^l x(s,t)\left[\frac{\partial}{\partial s}\left[a(s)x_s(s,t)\right] + f\left(x(s,t),s,t\right)x(s,t) - Kx(s,t)\right]ds \\ &\quad + 2\sum_{j=0}^{N-1}\int_{s_j}^{s_{j+1}} Kx(s,t)\left[x(s,t) - x(\bar{s}_j,t)\right]ds \end{aligned} \tag{3.10}$$

Integrating by parts and using the boundary conditions, we have

$$2\int_0^l x(s,t)\left[\frac{\partial}{\partial s}\left[a(s)x_s(s,t)\right]\right]ds = \left[2a(s)x(s,t)(s)x_s(s,t)\right]_0^l - 2\int_0^l a(s){x_s}^2(s,t)ds$$
$$\le -2a_0\int_0^l {x_s}^2(s,t)ds. \tag{3.11}$$

As $f \le f_M$, then

$$2\int_0^l x(s,t)\left[f\left(x(s,t),s,t\right)x(s,t) - Kx(s,t)\right]ds \le 2\int_0^l \left(f_M - K\right)x^2(s,t)ds.$$

The substitution of these inequalities into inequality (3.10) yields:

$$\dot V(t) \le -2a_0\int_0^l {x_s}^2(s,t)ds + 2\int_0^l \left(f_M - K\right)x^2(s,t)ds$$
$$+ 2\sum_{j=0}^{N-1}\int_{s_j}^{s_{j+1}} Kx(s,t)\left[x(s,t) - x(\bar s_j,t)\right]ds.$$

By virtue Young's inequality, for any scalar $\tilde R > 0$, we have that

$$2\sum_{j=0}^{N-1}\int_{s_j}^{s_{j+1}} Kx(s,t)\left[x(s,t) - x(\bar s_j,t)\right]ds$$
$$\le K\left[\tilde R\int_0^l x^2(s,t)ds + \frac{1}{\tilde R}\sum_{j=0}^{N-1}\int_{\bar s_j}^{s_{j+1}} \left(x(s,t) - x(\bar s_j,t)\right)^2 ds\right].$$

Using Wirtinger's inequality, we obtain:

$$\int_{s_j}^{s_{j+1}} \left(x(s,t) - x(\bar s_j,t)\right)^2 ds = \int_{s_j}^{\bar s_j} \left(x(s,t) - x(\bar s_j,t)\right)^2 ds + \int_{\bar s_j}^{s_{j+1}} \left(x(s,t) - x(\bar s_j,t)\right)^2 ds$$
$$\le \frac{\Delta^2}{\pi^2}\int_{s_j}^{s_{j+1}} x_s^2(s,t)ds.$$

Denoting $\tilde R = \frac{\Delta}{\pi}R$, we conclude that:

$$\dot V(t) + 2\omega V(t) \le \left(\frac{K\Delta}{\pi R} - 2a_0\right)\int_0^l x_s^2(s,t)ds + \left(RK\frac{\Delta}{\pi} + 2\omega + 2(f_M - K)\right)\int_0^l x^2(s,t)ds.$$

By applying again Wirtinger's inequality

$$\dot V(t) + 2\omega V(t) \le \left(\frac{K\Delta}{\pi R} - 2a_0\right)\int_0^l x_s^2(s,t)ds + \left(RK\frac{\Delta}{\pi} + 2\omega + 2(f_M - K)\frac{bl^2}{\pi^2}\right)\int_0^l x_s^2(s,t)ds.$$

$\dot V(t) + 2\omega V(t) \le 0$ if

$$\begin{cases} \frac{K\Delta}{\pi R} - 2a_0 \le 0, \\ RK\frac{\Delta}{\pi} + 2\omega + 2(f_M - K) + \frac{\pi^2}{bl^2}\left(R^{-1}K\frac{\Delta}{\pi} - 2a_0\right) \le 0, \end{cases} \tag{3.12}$$

where $b = 4$ under mixed boundary conditions and $b = 1$ under Dirichlet boundary conditions.

Therefore, the closed-loop system (3.2) is exponentially stable with the decay rate ω if inequalities (3.12) are feasible. Moreover, it is a positive system if $0 \leq f - K$, i.e., $0 \leq f_m - K$, which completes the proof.

□

Positive stabilization of the system (2.1) under continuous feedback in space and time, with the controller $u(s,t) = Kx(s,t)$, is provided by the following corollary.

Corollary 3.3. *Consider the system* (2.1)*. The following results hold.*

i. *The closed-loop system, subject to the continuous feedback controller $u(s,t) = Kx(s,t)$ and the boundary condition* (2.3)*, is positively exponentially stable with a decay rate $\omega > 0$ if*

$$\begin{cases} K \geq f_M + \omega - \frac{a_0\pi^2}{4l^2}, \\ K \leq f_m. \end{cases}$$

ii. *The closed-loop system, governed by the continuous feedback controller $u(s,t) = Kx(s,t)$ with the boundary condition* (2.2)*, is positively exponentially stable with a decay rate $\omega > 0$ if*

$$\begin{cases} K \geq f_M + \omega - \frac{a_0\pi^2}{l^2}, \\ K \leq f_m. \end{cases}$$

This result can be proved using similar arguments as in the proof of Theorem 3.2.

4. Observer design

The objective of this section is to develop a positive observer for the system described by (2.1), designed in the following form:

$$\begin{cases} \hat{x}_t(s,t) = \dfrac{\partial}{\partial s}\left[a(s)\hat{x}_s(s,t)\right] + f\left(\hat{x}(s,t),s,t\right)\hat{x}(s,t) + u(s,t) + L\left(y_j(t) - \hat{x}(\bar{s}_j,t)\right), \\ \qquad\qquad s_j \leq s < s_{j+1}, j = 1,2,3,\cdots,N, \\ \hat{x}(0,t) = \hat{x}(l,t) = 0, \quad s \in [0,l]\,, \; l > 0, \\ \hat{x}(s,0) = \hat{x}_0(s), \end{cases} \tag{4.1}$$

where L is injection gain. By using similar techniques as in the previous section, we can reformulate the system (4.1) as an abstract differential equation of the form (3.2). The estimation error satisfies the following equation:

$$\begin{cases} e_t(s,t) = \dfrac{\partial}{\partial s}\left[a(s)e_s(s,t)\right] + f\left(\hat{x}(s,t),s,t\right)\hat{x}(s,t) - f\left(x(s,t),s,t\right)x(s,t) + L\left(y_j(t) - \hat{x}(\bar{s}_j,t)\right) \\ \qquad\quad = \dfrac{\partial}{\partial s}\left[a(s)e_s(s,t)\right] - Le(\bar{s}_j,t) + f\left(\hat{x}(s,t),s,t\right)\hat{x}(s,t) - f\left(x(s,t),s,t\right)x(s,t) \end{cases} \tag{4.2}$$

The estimation error (4.2) can be reformulated as an abstract differential equation on the state space H as follows

$$\begin{cases} \dot{e}(t) = Ae(t) + F(t, e), \\ e(0) = e_0, \end{cases} \tag{4.3}$$

where the operator A is given in (3.3) and the nonlinear term F is defined by

$$F(t, e(\cdot, t)) = f(\hat{x}(s,t), s, t)\, \hat{x}(s,t) - f(x(s,t), s, t)\, x(s,t) - Le(s,t) + L\int_{\bar{s}_j}^{s} e_\theta(\theta, t)d\theta, \quad s_j \leq s < s_{j+1}, j = 1, 2, 3, \cdots, N. \tag{4.4}$$

We observe that the estimation error system has the same form as the closed-loop system (3.1). In this case, we can use Theorem 3.2 to determine the observer gain L, guaranteeing the exponential stability of the estimation error system. For the positivity of the observer system, it is sufficient to replace the condition $K \leq f_m$ in the inequality (3.8) with $L \leq f_m$. This condition ensures the positivity of the observer (4.1). The main result of this section is given in the following corollary, and the proof can be carried out using similar techniques as in the proof of Theorem 3.2.

Corollary 4.1. *Assume that there exist $L > 0, \omega > 0$ and $\Delta > 0$ such that the following inequalities holds*

$$\begin{cases} R^{-1}L\frac{\Delta}{\pi} - 2a_0 \leq 0 \\ RL\frac{\Delta}{\pi} + 2\omega + 2(F_M - L) + \frac{\pi^2}{l^2}\left(R^{-1}L\frac{\Delta}{\pi} - 2a_0\right) \leq 0 \\ L \leq f_m \end{cases} \tag{4.5}$$

where $F_M = f_M - f_m$, b and R are given positive values. Then, the system given (4.1) is a positive observer for the system (2.1) under the Dirichlet boundary conditions (2.2) if the LMIs (4.5) are feasible.

Remark 4.2. The observer proposed given by (4.1) is designed to meet the specific requirements of positive systems. Unlike traditional observers, our design takes into account positivity constraints by integrating additional assumptions on the system, especially on the nonlinear parts. These assumptions are essential to ensure that state estimates remain positive and convergence of the estimation error to zero. This approach makes it possible to maintain the precision of the estimates while respecting the positivity constraints of the system. Note that, for infinite positive linear systems, a new (positive) observer design based on a classical Luenberger-like observer is proposed in [6]. Here, our approach does not require modifications to the Luenberger-like observer to maintain the positivity of the observer and to ensure the convergence of the estimation error to zero.

5. Applications

Consider the following diffusion equation

$$\begin{cases} x_t(s,t) = \dfrac{\partial}{\partial s}\left[a(s)x_s(s,t)\right] + f\left(x(s,t),s,t\right)x(s,t) + u(s,t), \ s \in [0,l], \ l > 0, \\ x(s,0) = x_0(s), \end{cases} \tag{5.1}$$

under the mixed boundary conditions (2.3). The function f belongs to the class $\mathcal{C}^1$ and satisfies $f_m \leq f \leq f_M$, where $f_m = 2.75$ and $f_M = 4.5$. We consider the case where $a \equiv 1$. To apply Corollary 3.3, we choose the following values: $l = \pi$ and $a_0 = 1$. According to Corollary 3.3, the state feedback $u(s,t) = -2.5x(s,t)$ positively and exponentially stabilizes the system (5.1) with a decay rate $\omega = 0.47$.

We use the finite difference method and the numerical simulation results are programmed in Matlab.

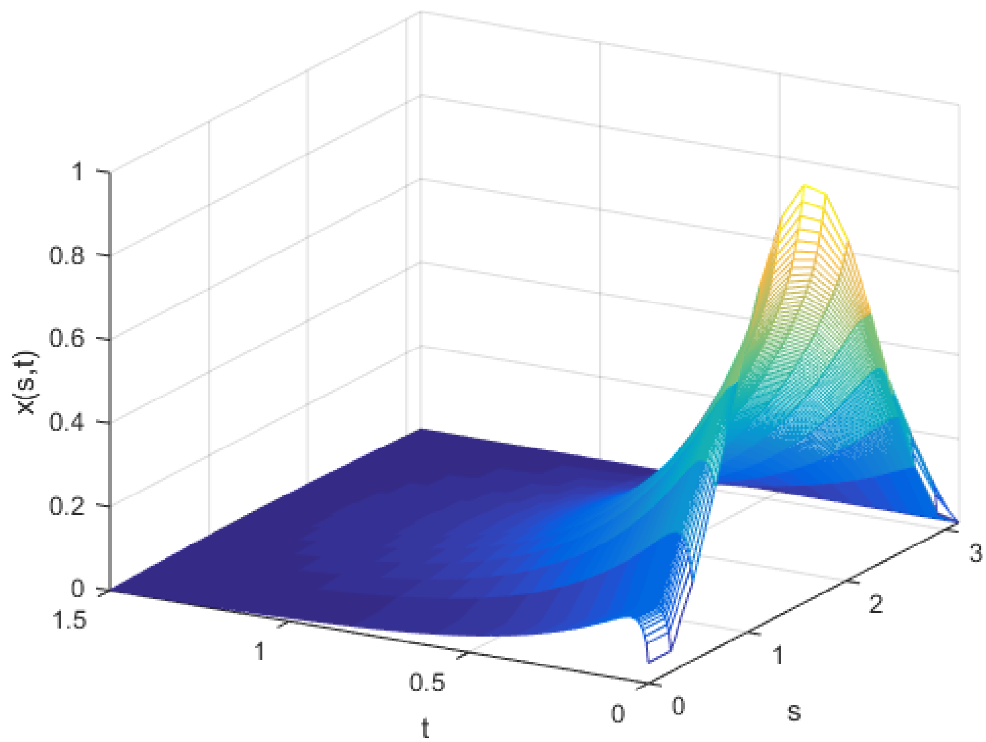

Figure 1. Evolution of closed-loop under feedback control $u(s,t) = -2.5x(s,t)$

Fig. 1 shows the behavior of the closed-loop system under feedback control $u(s,t) = -2.5x(s,t)$ over the time and the length, we see that the closed-loop system is exponentially stable and positive.

For the continuous in time and sampled in space controller $u(s,t) = -2.5x(\bar{s}_j,t)$, we apply Theorem 3.2. We set $R = 1$, in this case and from the conditions (3.8), the closed-loop system is exponentially stable with $\Delta \leq 2.5$. We set $s_1 = \frac{\pi}{2}$ in the middle of $[0,\pi]$, which corresponds to two sensors placed in $\bar{s}_1 = \frac{\pi}{4}$ and $\bar{s}_2 = \frac{3\pi}{2}$. Moreover, the controller $u(s,t) = -2.5x(\bar{s}_j,t)$, $j = 1,2$ positively and exponentially stabilizes the system (5.1) with decay rate $\omega = 0.47$.

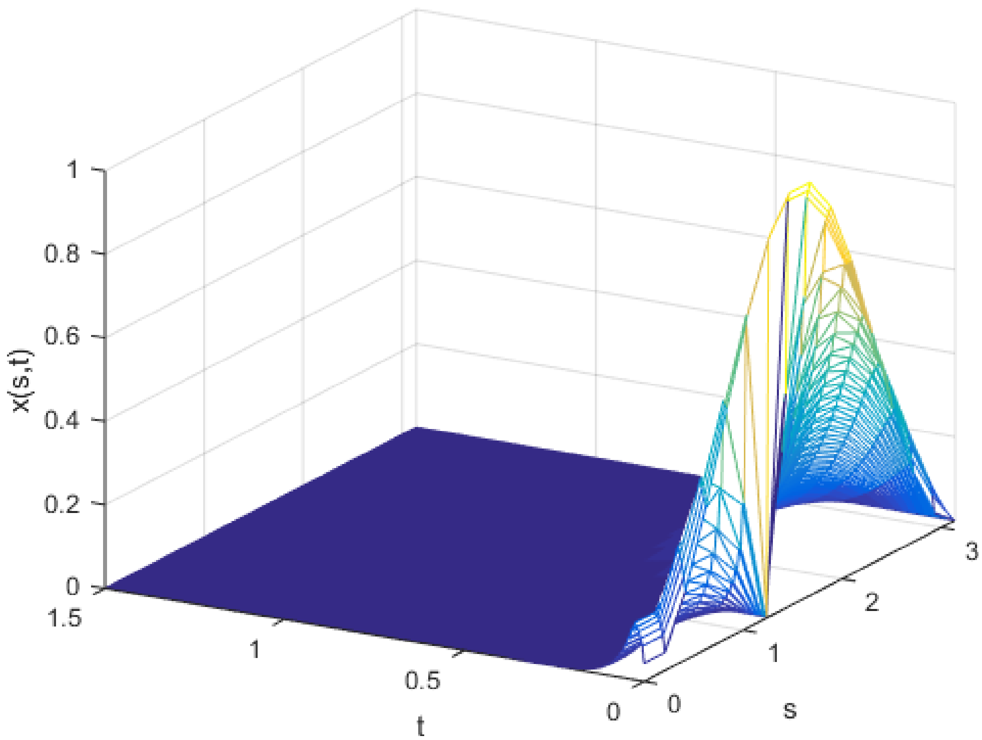

FIGURE 2. Evolution of closed-loop under the sampled-data controller (2.8)

The profile of evolution of closed-loop system under the sampled-data controller $u(s,t) = -2.5x(\bar{s}_j,t)$, $j = 1,2$ is shown in Fig. 2, we see that it is exponentially stable and positive.

Next, we consider the system (5.1) under the Dirichlet boundary conditions (2.2). The objective is to construct a positive observer under the sampled-data measurements (2.6). We apply a modified version of Theorem 3.2 to give a positive estimation of considered system. To do that, we set $R = 1$, and from the conditions (3.8), the estimation error $e(s,t)$ of the observer (4.1) under the sampled-data measurements (2.6) is exponentially stable with observer gain $L = 2.35$ and $\Delta \leq 2.67$. Note that with this gain the condition $L \leq f_m$ is satisfied. We choose $s_1 = \frac{\pi}{2}$ in the middle of $[0,\pi]$, which corresponds to two sensors placed in $\bar{s}_1 = \frac{\pi}{4}$ and $\bar{s}_2 = \frac{3\pi}{2}$. Moreover, the system (4.1) with gain $L = 2.35$ under the two sampled-data measurements $y_1(t) = x(\bar{s}_1, t)$ and $y_2(t) = x(\bar{s}_2, t)$ is a positive observer for the system (5.1).

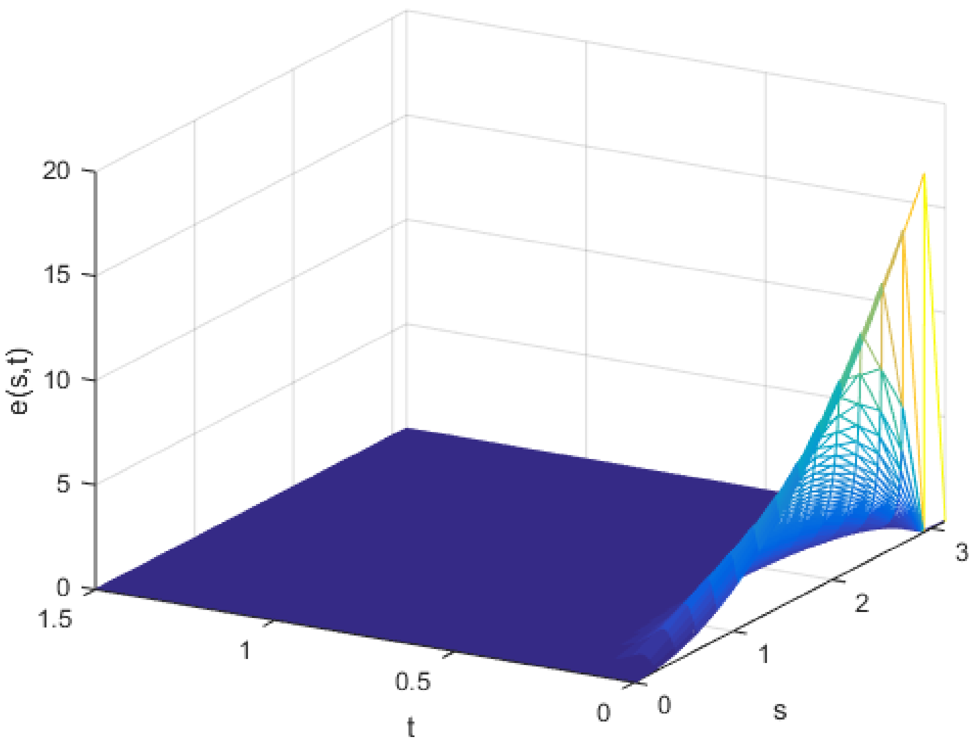

FIGURE 3. Evolution of estimation error $e(s,t)$ of the observer (4.1) under the sampled-data measurements (2.6)

Fig. 3 shows the behavior of estimation error $e(s,t)$ of the observer system under the sampled-data measurements $y_1(t) = x(\bar{s}_1, t)$ and $y_2(t) = x(\bar{s}_2, t)$. We observe that $e(t,s)$ is exponentially stable and the result is satisfactory.

6. Conclusion

In this research, we address the issue of positive stabilization for a class of uncertain diffusion equations with sampled data. We derive sufficient conditions based on (LMIs) to ensure the positivity and stability of the closed-loop system. Additionally, we design a positive observer for these systems with discrete measurements. We demonstrate that the estimation error system exhibits exponential stability using the same LMIs. Results from numerical simulations highlight the effectiveness of the proposed design methods.

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