

REGULARIZATION METHOD AND A POSTERIORI ERROR ESTIMATES FOR THE BILATERAL OBSTACLE PROBLEM

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ABSTRACT. In this paper, we present a relatively complete analysis of the regularization method of the bilateral obstacle problem with a non-homogeneous condition on the boundary. We provide both the convergence result and a priori estimates. Then by the duality theory from the convex analysis, we determine the dual problem associated with the bilateral obstacle problem. Based on this dual problem, we provide a posteriori error estimates for the continuous and the discrete problems. The a posteriori error estimates are needed for the practical implementation of the regularized problem.

1. INTRODUCTION

Let Ω be an open and bounded subset of $\mathbb{R}^n$, with a smooth boundary denoted by $\partial\Omega$. Let f be an element of $L^2(\Omega)$ and η_1 , η_2 , and η be elements of $H^1(\Omega)$ satisfying

$$\eta_1 \leq \eta_2 \text{ a.e. on } \Omega \quad \text{and} \quad \eta_1 \leq \eta \leq \eta_2 \text{ on } \partial\Omega.$$

Set

$$H_\eta^1(\Omega) := \{\nu \in H^1(\Omega) : \nu = \eta \text{ on } \partial\Omega\},$$

and

$$K := \{v \in H_\eta^1(\Omega) : \eta_1 \leq v \leq \eta_2 \text{ a.e. on } \Omega\}.$$

We consider the following problem:

$$\begin{cases} \text{Find } u \in K \text{ such that} \\ \int_{\Omega} \nabla u \cdot \nabla (v - u) dx + \int_{\Omega} f(v - u) dx \geq 0 \quad \forall v \in K. \end{cases} \quad (1.1)$$

This variational inequality is called a bilateral obstacle problem, where the solution describes the position of one uniform elastic membrane fixed at the boundary of a domain Ω , subjected to external forces, constrained by two rigid obstacles. It is well known that problem (1.1) admits a unique solution u , and, if $\Delta\eta_1, \Delta\eta_2 \in L^2(\Omega)$, then u is an element of $H^2(\Omega)$, see [6, 10, 13]. There are several alternative formulations and resolutions of the bilateral obstacle problem, see [2, 3, 4, 6, 10, 11, 13].

Date: Received: Aug 1, 2024; Accepted: Sep 2, 2024.

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2020 *Mathematics Subject Classification.* Primary 49J40 ; Secondary 65K15.

Key words and phrases. Variational inequalities, bilateral obstacle problem, a posteriori error estimates, regularization method, duality theory.

The bilateral problem (1.1) can be reformulated as an unconstrained variational inequality problem, see [3]. The difficulty in solving this problem is that it has a non-differentiable term Ψ (see (2.4)). Our aim in this paper is to introduce a regularization approach for resolving the non-differentiable minimization problem associated with the bilateral problem (see (2.7)). The proposed regularization method consists of substituting the non-differentiable function Ψ with a sequence of differentiable functions denoted by Ψ_ξ , where ξ goes to zero, see [6, 8, 9]. We give two examples of regularization for which we establish the convergence of the approximate solutions and the a priori error estimates. Subsequently, by employing the duality method from convex analysis, using conjugate functions (as detailed in [5]), we establish a posteriori error estimates for both the continuous and discrete problems. Finally, we present some concluding remarks.

Assume that we evaluated both the approximate solution u_h of the discrete problem and the approximate solution $u_{h,\xi}$ of the discrete regularized problem. Theoretically, achieving more accurate approximations necessitates the use of smaller values for ξ . However, in practical applications, setting ξ too small is not feasible because it worsens the conditioning of the regularized problem, making it difficult to compute an accurate numerical solution. Thus, the practical implementation of the discrete regularized problem requires a posteriori error bounds, or quantitative estimates, once we have solutions of the regularized problems. We may use the a posteriori error estimates as a stopping criterion in actual computation, see [7].

In the papers [1, 8], the authors formulated a regularization approach for a unilateral obstacle problem and deduced a posteriori error estimates through the utilization of duality theory. In this paper, we give an extension study of the regularization method to the bilateral obstacle problem (1.1).

2. REFORMULATION AND REGULARIZATION OF THE BILATERAL OBSTACLE PROBLEM (1.1)

In the sequel of this work, we use the following notations:

- $a(u, v) := \int_{\Omega} \nabla u \cdot \nabla v dx \quad \forall (u, v) \in [H^1(\Omega)]^2,$
- $b(u, v) := \int_{\Omega} uv dx \quad \forall (u, v) \in [L^2(\Omega)]^2,$
- $\nu^+ := \max(\nu, 0) \quad \forall \nu \in L^2(\Omega),$
- $\nu^- := \min(\nu, 0) \quad \forall \nu \in L^2(\Omega).$

Problem (1.1) can be expressed as follows:

$$\begin{cases} \text{Find } u \in K \text{ such that} \\ a(u, v - u) + b(f, v - u) \geq 0 \quad \forall v \in K. \end{cases} \quad (2.1)$$

Set $u_0 := u - \eta$, and $K_0 := \{v \in H_0^1(\Omega) : \eta_1 - \eta \leq v \leq \eta_2 - \eta \text{ a.e. on } \Omega\}$, then problem (1.1) can be written as follows:

$$\begin{cases} \text{Find } u_0 \in K_0 \text{ such that} \\ a(u_0 + \eta, v - u_0) + b(f, v - u_0) \geq 0 \quad \forall v \in K_0. \end{cases} \quad (2.2)$$

K_0 is a closed convex set of $H_0^1(\Omega)$. Moreover, the bilinear form $a(\cdot, \cdot)$ defines an inner product on $H_0^1(\Omega)$; thus, it is continuous and coercive. The function $h : v \mapsto h(v) = b(f, v) + a(\eta, v)$ is a continuous linear form on $H_0^1(\Omega)$. Therefore, the variational inequality of the first kind (2.2) admits a unique solution, see [6].

We assume that $\Delta\eta_1 \in L^2(\Omega)$ and $\Delta\eta_2 \in L^2(\Omega)$. Let

$$f_1 := f - \Delta\eta_1, \quad f_2 := f - \Delta\eta_2 \quad \text{and} \quad \tilde{f} := f - f_1^+ - f_2^-. \quad (2.3)$$

Let Ψ be the function from $L^2(\Omega)$ to $\mathbb{R}$ defined by

$$\Psi(v) := \int_{\Omega} (f_1^+(v - \eta_1)^+ + f_2^-(v - \eta_2)^-) dx, \quad v \in L^2(\Omega).$$

It is clear that the functional Ψ is convex and continuous on $H_0^1(\Omega)$. See [3] (Proposition 3.1 page 48).

According to the work presented in [3] (Theorem 3.1 page 48), we can transform the bilateral obstacle problem (2.2), which is a variational inequality of the first kind, into a variational inequality of the second kind, see [6]. We have

Theorem 2.1. *The bilateral problem (2.2) is equivalent to the following unconstrained variational inequality:*

$$\begin{cases} \text{Find } u_0 \in H_0^1(\Omega) \text{ such that} \\ a(u_0 + \eta, v - u_0) + \Psi(v + \eta) - \Psi(u_0 + \eta) + b(\tilde{f}, v - u_0) \geq 0 \quad \forall v \in H_0^1(\Omega). \end{cases} \quad (2.4)$$

Given the symmetry of the bilinear form $a(\cdot, \cdot)$, problem (2.4) is equivalent to the following minimization problem:

$$\text{Find } u_0 \in H_0^1(\Omega) : \quad J_0(u_0) = \min_{v \in H_0^1(\Omega)} J_0(v), \quad (2.5)$$

where J_0 is the function defined on $H_0^1(\Omega)$ by

$$J_0(v) := \frac{1}{2}a(v, v) + \Psi(v + \eta) + b(\tilde{f}, v) + a(\eta, v).$$

If we set $u := u_0 + \eta$, then problem (2.4) can be expressed as follows:

$$\begin{cases} \text{Find } u \in H_{\eta}^1(\Omega) \text{ such that} \\ a(u, v - u) + \Psi(v) - \Psi(u) + b(\tilde{f}, v - u) \geq 0 \quad \forall v \in H_{\eta}^1(\Omega), \end{cases} \quad (2.6)$$

which is equivalent to the following minimization problem:

$$\text{Find } u \in H_{\eta}^1(\Omega) : \quad J(u) = \min_{v \in H_{\eta}^1(\Omega)} J(v), \quad (2.7)$$

where J is the function defined on $H_\eta^1(\Omega)$ by

$$J(v) := \frac{1}{2}a(v, v) + \Psi(v) + b(\tilde{f}, v).$$

The functional J is not differentiable since Ψ is not. We therefore approximate the function Ψ by a sequence of differentiable ones Ψ_ξ as follows:

$$\Psi_\xi(v) := \int_\Omega (f_1^+ \psi_\xi(v - \eta_1) + f_2^- \phi_\xi(v - \eta_2)) dx, \quad v \in L^2(\Omega),$$

where ψ_ξ and ϕ_ξ are two functions defined from $\mathbb{R}$ into $\mathbb{R}$, differentiable and convex, satisfying

$$\lim_{\xi \rightarrow 0} \psi_\xi(t) = t^+ \quad \text{and} \quad \lim_{\xi \rightarrow 0} \phi_\xi(t) = t^- \quad \forall t \in \mathbb{R}.$$

We define the regularized problem of (2.4) by

$$\begin{cases} \text{Find } u_{0,\xi} \in H_0^1(\Omega) \text{ such that} \\ a(u_{0,\xi} + \eta, v - u_{0,\xi}) + \Psi_\xi(v + \eta) - \Psi_\xi(u_{0,\xi} + \eta) + b(\tilde{f}, v - u_{0,\xi}) \geq 0 \quad \forall v \in H_0^1(\Omega). \end{cases} \quad (2.8)$$

If we set $u_\xi := u_{0,\xi} + \eta$, then the regularized problem of (2.6) is defined by

$$\begin{cases} \text{Find } u_\xi \in H_\eta^1(\Omega) \text{ such that} \\ a(u_\xi, v - u_\xi) + \Psi_\xi(v) - \Psi_\xi(u_\xi) + b(\tilde{f}, v - u_\xi) \geq 0 \quad \forall v \in H_\eta^1(\Omega). \end{cases} \quad (2.9)$$

Since Ψ_ξ is a convex continuous proper functional, then variational inequality of second kind (2.8) (resp. (2.9)) admits a unique solution $u_{0,\xi}$ (resp. u_ξ), see [6].

Here we may propose the following examples of ψ_ξ and ϕ_ξ :

$$\psi_\xi^1(t) = \begin{cases} t - \frac{\xi}{2} & \text{if } t \geq \xi, \\ \frac{t^2}{2\xi} & \text{if } 0 \leq t \leq \xi, \\ 0 & \text{if } t \leq 0. \end{cases} \quad \text{and} \quad \phi_\xi^1(t) = \begin{cases} 0 & \text{if } t \geq 0 \\ -\frac{t^2}{2\xi} & \text{if } -\xi \leq t \leq 0, \\ t + \frac{\xi}{2} & \text{if } t \leq -\xi. \end{cases} \quad (2.10)$$

$$\psi_\xi^2(t) = \begin{cases} \sqrt{t^2 + \xi^2} & \text{if } t \geq 0, \\ \xi & \text{if } t \leq 0. \end{cases} \quad \text{and} \quad \phi_\xi^2(t) = \begin{cases} -\xi & \text{if } t \geq 0, \\ -\sqrt{t^2 + \xi^2} & \text{if } t \leq 0. \end{cases} \quad (2.11)$$

Let u_0 and $u_{0,\xi}$ be solutions of problems (2.4) and (2.8), respectively. To show the a priori error, we take $v = u_{0,\xi}$ in the inequality of problem (2.4) and $v = u_0$ in the inequality of problem (2.8). Adding these two inequalities yields

$$a(u_0 - u_{0,\xi}, u_0 - u_{0,\xi}) \leq \Psi(u_{0,\xi} + \eta) - \Psi_\xi(u_{0,\xi} + \eta) + \Psi_\xi(u_0 + \eta) - \Psi(u_0 + \eta). \quad (2.12)$$

We note that the functions ψ_ξ^j and ϕ_ξ^j , $j = 1, 2$, verify

$$|\psi_\xi^j(t) - t^+| \leq \xi \quad \text{and} \quad |\phi_\xi^j(t) - t^-| \leq \xi \quad \forall t \in \mathbb{R}.$$

We denote by $\|\cdot\|_{H_0^1(\Omega)}$ the norm of $H_0^1(\Omega)$ associated with the inner product $a(\cdot, \cdot)$. Consequently, we deduce from relation (2.12) the following a priori error estimate:

$$\|u_0 - u_{0,\xi}\|_{H_0^1(\Omega)}^2 \leq 2\xi (\|f_1^+\|_{L^1(\Omega)} + \|f_2^-\|_{L^1(\Omega)}). \quad (2.13)$$

Hence, we have the convergence $u_{0,\xi} \rightarrow u_0$ in $H_0^1(\Omega)$.

Theoretically, the a priori estimation of the error (2.13) ensures the convergence of the solution of the regularized problem to the one of the initial problem. Practically, we cannot use this estimate as a stopping criterion in the discrete case. Indeed, a stopping criterion equal to 10^{-6} corresponds to a choice of ξ of the order 10^{-12} which is challenging to handle at the computational level. Thus, the practical implementation of the regularized problem (2.8) requires a posteriori error estimate.

3. DUAL PROBLEM ASSOCIATED WITH THE MINIMIZATION PROBLEM (2.7)

According to Section 2, the bilateral problem (1.1) is equivalent to the following minimization problem (see (2.7)):

$$\text{Find } u \in H_\eta^1(\Omega) : \quad J(u) = \min_{v \in H_\eta^1(\Omega)} J(v), \quad (3.1)$$

where J is the function defined on $H_\eta^1(\Omega)$ by

$$J(v) := \frac{1}{2}a(v, v) + \Psi(v) + b(f, v).$$

In this section, we shall determine the dual problem associated with the primal problem (3.1). We need the following abstract frame, see [5, 8]:

- Let X be a topological vector space and X^* be its dual. We denote by $\langle \cdot, \cdot \rangle_X$ the duality pairing between X and X^* .
- Let $h : X \rightarrow \overline{\mathbb{R}}$ be a function and $h^* : X^* \rightarrow \overline{\mathbb{R}}$ be its conjugate function defined by

$$h^*(v^*) := \sup_{v \in X} \{ \langle v^*, v \rangle_X - h(v) \}, \quad v^* \in X^*.$$

- Let Q be a separable topological vector space and Q^* be its dual. We denote by $\langle \cdot, \cdot \rangle_Q$ the duality pairing between Q and Q^* .
- We shall assume the existence of a continuous linear operator $\Lambda : X \rightarrow Q$, $\Lambda \in \mathcal{L}(X, Q)$, with transpose $\Lambda^* \in \mathcal{L}(Q^*, X^*)$.
- Let $\mathcal{R} : X \times Q \rightarrow \overline{\mathbb{R}}$ be a function and $\mathcal{R}^* : X^* \times Q^* \rightarrow \overline{\mathbb{R}}$ be its conjugate function defined by

$$\mathcal{R}^*(v^*, q^*) := \sup_{(v, q) \in X \times Q} \{ \langle v^*, v \rangle_X + \langle q^*, q \rangle_Q - \mathcal{R}(v, q) \}, \quad (v^*, q^*) \in X^* \times Q^*.$$

We consider the following primal problem:

$$\begin{cases} \text{Find } u \in X \text{ such that} \\ \mathcal{R}(u, \Lambda u) = \inf_{v \in X} \mathcal{R}(v, \Lambda v). \end{cases} \quad (3.2)$$

Theorem 3.1 (see [5]). *Assume that X is a reflexive Banach space and Q is a normed vector space. Let $\mathcal{R} : X \times Q \rightarrow \overline{\mathbb{R}}$ be a proper, lower semi-continuous, and strictly convex function satisfying:*

- (1) $\exists \nu \in X$, such that $\mathcal{R}(\nu, \Lambda \nu) < +\infty$ and the function $q \mapsto \mathcal{R}(\nu, q)$ is continuous at $\Lambda \nu$.

- (2) *The following coercivity condition:* $\mathcal{R}(v, \Lambda v) \rightarrow +\infty$, as $\|v\|_X \rightarrow +\infty$, $v \in X$.

Then primal problem (3.2) admits a unique solution u , and we have

$$\mathcal{R}(u, \Lambda u) = \sup_{q^* \in Q^*} -\mathcal{R}^*(\Lambda^* q^*, -q^*). \quad (3.3)$$

Definition 3.2 (see [5]). Let $l : \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}$ be a function, l is called the Carathéodory function if:

- (1) for any $s \in \mathbb{R}^m$, $x \mapsto l(x, s)$ is a measurable function,
- (2) and for almost all $x \in \Omega$, the function $s \mapsto l(x, s)$ is continuous.

Proposition 3.3 (see [5]). Let $l : \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}$ be a Carathéodory function. Let L be the function defined by

$$L(\nu) = \int_{\Omega} l(x, \nu(x)) \, dx, \quad \nu \in [L^2(\Omega)]^m.$$

The conjugate function of L is given by

$$L^*(\nu^*) = \int_{\Omega} l^*(x, \nu^*(x)) \, dx, \quad \forall \nu^* \in [L^2(\Omega)]^m, \quad (3.4)$$

where

$$l^*(x, z) = \sup_{s \in \mathbb{R}^m} \{z \cdot s - l(x, s)\},$$

and the expression $z \cdot s$ denotes the inner product of z and s in $\mathbb{R}^m$.

To apply Theorem 3.1 to the minimization problem (2.7), we take

- $X = H^1(\Omega)$
- $Q = Q^* = [L^2(\Omega)]^n \times L^2(\Omega)$
- $\Lambda : X \rightarrow Q$, $\Lambda v = (\nabla v, v)$, $v \in X$
- $\mathcal{R} : X \times Q \rightarrow \mathbb{R}$, $\mathcal{R}(v, q) = G(v) + L(q)$, where:
 - $G(v) = \begin{cases} b(\tilde{f}, v) & \text{if } v \in H_{\eta}^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$
 - $L(q) = \int_{\Omega} \left(\frac{1}{2} |q_1|^2 + f_1^+(q_2 - \eta_1)^+ + f_2^-(q_2 - \eta_2)^- \right) dx.$

where $q = (q_1, q_2)$, with $q_1 \in [L^2(\Omega)]^n$ and $q_2 \in L^2(\Omega)$. The same notation is applied to represent $q^* \in Q^*$. Then the formulation of problem (2.7) can be restated as:

$$\text{Find } u \in X, \quad \mathcal{R}(u, \Lambda u) = \inf_{v \in X} \mathcal{R}(v, \Lambda v).$$

The fulfillment of the conditions outlined in Theorem 3.1 is evidently apparent, hence the solution u of the bilateral problem (1.1) satisfies:

$$\mathcal{R}(u, \Lambda u) = \inf_{v \in X} \mathcal{R}(v, \Lambda v) = \sup_{q^* \in Q^*} -\mathcal{R}^*(\Lambda^* q^*, -q^*). \quad (3.5)$$

Let $v^* \in X^*$ and $q^* \in Q^*$, we have

$$\begin{aligned} \mathcal{R}^*(v^*, q^*) &= \sup_{(v, q) \in X \times Q} \{ \langle v^*, v \rangle_X + \langle q^*, q \rangle_Q - G(v) - L(q) \} \\ &= \sup_{v \in X} \{ \langle v^*, v \rangle_X - G(v) \} + \sup_{q \in Q} \{ \langle q^*, q \rangle_Q - L(q) \} \\ &= G^*(v^*) + L^*(q^*). \end{aligned}$$

Consequently, (3.5) implies that

$$\mathcal{R}(u, \Lambda u) = \sup_{q^* \in Q^*} \{ -G^*(\Lambda^* q^*) - L^*(-q^*) \}. \quad (3.6)$$

In order to obtain $\mathcal{R}(u, \Lambda u)$, we initially determine the conjugate functions of both G and L . We have

Proposition 3.4. *Let $q^* \in Q^*$, we have*

$$G^*(\Lambda^* q^*) = \begin{cases} \int_{\Omega} (q_1^* \cdot \nabla \eta + (q_2^* - \tilde{f}) \eta) dx & \text{if } -\operatorname{div}(q_1^*) + (q_2^* - \tilde{f}) = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. Let $q^* \in Q^*$, we have

$$\begin{aligned} G^*(\Lambda^* q^*) &= \sup_{v \in X} \{ \langle \Lambda^* q^*, v \rangle_{X^*} - G(v) \} \\ &= \sup_{v \in H_{\eta}^1(\Omega)} \{ \langle q^*, \Lambda v \rangle_{Q^*} - b(\tilde{f}, v) \} \\ &= \sup_{v \in H_{\eta}^1(\Omega)} \left\{ \int_{\Omega} (q_1^* \cdot \nabla v + (q_2^* - \tilde{f}) v) dx \right\} \\ &= \int_{\Omega} (q_1^* \cdot \nabla \eta + (q_2^* - \tilde{f}) \eta) dx + \sup_{\nu \in H_0^1(\Omega)} \left\{ \int_{\Omega} (q_1^* \cdot \nabla \nu + (q_2^* - \tilde{f}) \nu) dx \right\}. \end{aligned}$$

Since, in the distributional sense, we have

$$\sup_{\nu \in H_0^1(\Omega)} \left\{ \int_{\Omega} (q_1^* \cdot \nabla \nu + (q_2^* - \tilde{f}) \nu) dx \right\} = \begin{cases} 0 & \text{if } -\operatorname{div}(q_1^*) + (q_2^* - \tilde{f}) = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

So

$$G^*(\Lambda^* q^*) = \begin{cases} \int_{\Omega} (q_1^* \cdot \nabla \eta + (q_2^* - \tilde{f}) \eta) dx & \text{if } -\operatorname{div}(q_1^*) + (q_2^* - \tilde{f}) = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

□

For $v \in L^2(\Omega)$, we denote

$$\begin{aligned} I(v) &:= \{x \in \Omega : f_2^-(x) \leq v(x) < f_1^+(x) + f_2^-(x)\} \\ J(v) &:= \{x \in \Omega : f_1^+(x) + f_2^-(x) \leq v(x) \leq f_1^+(x)\}. \end{aligned}$$

Proposition 3.5. *Let $q^* \in Q^*$, we have*

$$L^*(-q^*) = \begin{cases} \int_{\Omega} \frac{1}{2} |q_1^*|^2 dx + \int_{I(-q_2^*)} (-\eta_1 q_2^* + f_2^-(\eta_2 - \eta_1)) dx + \dots \\ \int_{J(-q_2^*)} (-\eta_2 q_2^* + f_1^+(\eta_1 - \eta_2)) dx & \text{if } f_2^- \leq -q_2^* \leq f_1^+ \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. To compute the conjugate function L we first have to compute the two following conjugate functions:

$$L_1 : [L^2(\Omega)]^n \rightarrow \mathbb{R}, \quad L_1(v) = \int_{\Omega} \frac{1}{2} |v(x)|^2 dx, \quad v \in [L^2(\Omega)]^n$$

and

$$L_2 : L^2(\Omega) \rightarrow \mathbb{R}, \quad L_2(v) = \int_{\Omega} (f_1^+(v - \eta_1)^+ + f_1^-(v - \eta_2)^-) dx, \quad v \in L^2(\Omega).$$

Let $v^* \in [L^2(\Omega)]^n$, according to Proposition 3.3, we have

$$L_1^*(v^*) = \int_{\Omega} l_1^*(v^*(x)) dx,$$

where

$$l_1^*(z) = \sup_{s \in \mathbb{R}^n} \left\{ z \cdot s - \frac{1}{2} |s|^2 \right\}.$$

Now the quadratic function: $s \mapsto z \cdot s - \frac{1}{2} |s|^2$ admits a maximum on $s = z$, hence

$$l_1^*(z) = \frac{1}{2} |z|^2,$$

consequently,

$$L_1^*(v^*) = \int_{\Omega} \frac{1}{2} |v^*(x)|^2 dx. \quad (3.7)$$

Let $v^* \in L^2(\Omega)$, according to Proposition 3.3, we have

$$L_2^*(v^*) = \int_{\Omega} l_2^*(x, v^*(x)) dx$$

where

$$l_2 : \Omega \times \mathbb{R} \rightarrow \mathbb{R}, \quad l_2(x, s) = f_1^+(x) (s - \eta_1(x))^+ + f_2^-(x) (s - \eta_2(x))^- , \quad x \in \Omega ; s \in \mathbb{R}.$$

Let $x \in \Omega$ and $z \in \mathbb{R}$, we have

$$\begin{aligned} l_2^*(x, z) &= \sup_{s \in \mathbb{R}} \{sz - l_2(x, s)\} \\ &= \sup_{s \in \mathbb{R}} \{sz - f_1^+(x) (s - \eta_1(x))^+ - f_2^-(x) (s - \eta_2(x))^- \}, \\ &= \max \left\{ \sup_{\eta_1(x) \leq s \leq \eta_2(x)} \{s(z - f_1^+(x) - f_2^-(x)) + f_1^+(x)\eta_1(x) + f_2^-(x)\eta_2(x)\}, \dots \right. \\ &\quad \left. \sup_{s \geq \eta_2(x)} \{s(z - f_1^+(x)) + f_1^+(x)\eta_1(x)\}, \sup_{s \leq \eta_1(x)} \{s(z - f_2^-(x)) + f_2^-(x)\eta_2(x)\} \right\}. \end{aligned}$$

We have

$$\begin{aligned} \sup_{\eta_1(x) \leq s \leq \eta_2(x)} \{s(z - f_1^+(x) - f_2^-(x)) + f_1^+(x)\eta_1(x) + f_2^-(x)\eta_2(x)\} &= \dots \\ &\begin{cases} z\eta_2(x) + f_1^+(x)(\eta_1(x) - \eta_2(x)) & \text{if } z \geq f_1^+(x) + f_2^-(x) \\ z\eta_1(x) + f_2^-(x)(\eta_2(x) - \eta_1(x)) & \text{otherwise,} \end{cases} \\ \sup_{s \geq \eta_2(x)} \{s(z - f_1^+(x)) + f_1^+(x)\eta_1(x)\} &= \begin{cases} z\eta_2(x) + f_1^+(x)(\eta_1(x) - \eta_2(x)) & \text{if } z \leq f_1^+(x) \\ +\infty & \text{otherwise,} \end{cases} \\ \sup_{s \leq \eta_1(x)} \{s(z - f_2^-(x)) + f_2^-(x)\eta_2(x)\} &= \begin{cases} z\eta_1(x) + f_2^-(x)(\eta_2(x) - \eta_1(x)) & \text{if } z \geq f_2^-(x) \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

Since

$$\begin{aligned} \max \{z\eta_1(x) + f_2^-(x)(\eta_2(x) - \eta_1(x)), z\eta_2(x) + f_1^+(x)(\eta_1(x) - \eta_2(x))\} &= \dots \\ &\begin{cases} z\eta_2(x) + f_1^+(x)(\eta_1(x) - \eta_2(x)) & \text{if } z \geq f_1^+(x) + f_2^-(x) \\ z\eta_1(x) + f_2^-(x)(\eta_2(x) - \eta_1(x)) & \text{otherwise,} \end{cases} \end{aligned}$$

so

$$l_2^*(x, z) = \begin{cases} z\eta_1(x) + f_2^-(x)(\eta_2(x) - \eta_1(x)) & \text{if } f_2^-(x) \leq z \leq f_1^+(x) + f_2^-(x), \\ z\eta_2(x) + f_1^+(x)(\eta_1(x) - \eta_2(x)) & \text{if } f_1^+(x) + f_2^-(x) \leq z \leq f_1^+(x), \\ +\infty & \text{otherwise.} \end{cases}$$

Consequently,

$$L_2^*(v^*) = \begin{cases} \int_{I(v^*)} (\eta_1 v^* + f_2^-(\eta_2 - \eta_1)) dx + \int_{J(v^*)} (\eta_2 v^* + f_1^+(\eta_1 - \eta_2)) dx & \text{if } f_2^- \leq v^* \leq f_1^+ \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise.} \end{cases} \quad (3.8)$$

Finally, relations (3.7) and (3.8) imply that for any $q^* \in Q^*$ we have

$$L^*(q^*) = \begin{cases} \int_{\Omega} \frac{1}{2} |q_1^*|^2 dx + \int_{I(q_2^*)} (\eta_1 q_2^* + f_2^-(\eta_2 - \eta_1)) dx + \int_{J(q_2^*)} (\eta_2 q_2^* + f_1^+(\eta_1 - \eta_2)) dx & \text{if } f_2^- \leq q_2^* \leq f_1^+ \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise,} \end{cases}$$

then

$$L^*(-q^*) = \begin{cases} \int_{\Omega} \frac{1}{2} |q_1^*|^2 dx + \int_{I(-q_2^*)} (-\eta_1 q_2^* + f_2^-(\eta_2 - \eta_1)) dx + \dots & \\ \int_{J(-q_2^*)} (-\eta_2 q_2^* + f_1^+(\eta_1 - \eta_2)) dx & \text{if } f_2^- \leq -q_2^* \leq f_1^+ \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

Which completes the proof. $\square$

Let $\mathcal{D}$ be the subset of Q^* defined by

$$\mathcal{D} = \left\{ q \in Q^* : -\operatorname{div}(q_1) + (q_2 - \tilde{f}) = 0 \text{ and } f_2^- \leq -q_2 \leq f_1^+ \text{ a.e. in } \Omega \right\}.$$

Let $A : \mathcal{D} \rightarrow \mathbb{R}$ be the operator defined by

$$\begin{aligned} A(q) = & \int_{\Omega} \left(\frac{1}{2} |q_1|^2 + q_1 \cdot \nabla \eta - \tilde{f} \eta \right) dx + \int_{I(-q_2)} (q_2(\eta - \eta_1) + f_2^-(\eta_2 - \eta_1)) dx + \dots \\ & \int_{J(-q_2)} (q_2(\eta - \eta_2) + f_1^+(\eta_1 - \eta_2)) dx, \quad \forall q \in \mathcal{D}. \end{aligned}$$

Let u be the solution of problem (1.1). From (3.6) and Propositions 3.4 and 3.5, we deduce that

$$\mathcal{R}(u, \Lambda u) = \sup_{q \in \mathcal{D}} \{-A(q)\}, \quad (3.9)$$

which is the dual problem associated with the minimization problem (2.7).

4. A POSTERIORI ERROR ESTIMATION

In this section, we establish an a posteriori error estimation between the solution u of the bilateral problem (2.6) and the solution u_{ξ} of the regularized problem (2.9).

Proposition 4.1. *Let u and u_{ξ} be solutions of problems (2.6) and (2.9), respectively, we have*

$$\|u - u_{\xi}\|_{H_0^1(\Omega)}^2 \leq 2\mathcal{R}(u_{\xi}, \Lambda u_{\xi}) + 2A(q), \quad \forall q \in \mathcal{D}. \quad (4.1)$$

Proof. Let u and u_{ξ} be solutions of problems (2.6) and (2.9), respectively. If we take $v = u_{\xi}$ in problem (2.6), we obtain

$$a(u, u_{\xi} - u) + \Psi(u_{\xi}) - \Psi(u) + b(\tilde{f}, u_{\xi} - u) \geq 0.$$

Since

$$\mathcal{R}(u_{\xi}, \Lambda u_{\xi}) - \mathcal{R}(u, \Lambda u) = J(u_{\xi}) - J(u) = \frac{1}{2}a(u_{\xi}, u_{\xi}) - \frac{1}{2}a(u, u) + b(f, u_{\xi} - u) + \Psi(u_{\xi}) - \Psi(u),$$

we have

$$\mathcal{R}(u_{\xi}, \Lambda u_{\xi}) - \mathcal{R}(u, \Lambda u) \geq \frac{1}{2}a(u_{\xi}, u_{\xi}) - \frac{1}{2}a(u, u) - a(u, u_{\xi} - u),$$

hence

$$\frac{1}{2} \|u - u_{\xi}\|_{H_0^1(\Omega)}^2 \leq \mathcal{R}(u_{\xi}, \Lambda u_{\xi}) - \mathcal{R}(u, \Lambda u).$$

Thus, from the equality (3.9), we deduce that

$$\|u - u_{\xi}\|_{H_0^1(\Omega)}^2 \leq 2\mathcal{R}(u_{\xi}, \Lambda u_{\xi}) + 2A(q), \quad \forall q \in \mathcal{D}.$$

□

Proposition 4.2. *Let u and u_ξ be solutions of problems (2.6) and (2.9), respectively, and $\psi_\xi = \psi_\xi^i$, $\phi_\xi = \phi_\xi^i$ for $i = 1, 2$ (choices (2.10) and (2.11)). The a posteriori error estimate is given by*

$$\begin{aligned} \|u - u_\xi\|_{H_0^1(\Omega)}^2 \leq & 2 \int_{\Omega} f_1^+ ((u_\xi - \eta_1)^+ - (u_\xi - \eta_1)\psi'_\xi(u_\xi - \eta_1)) dx + \dots \\ & 2 \int_{\Omega} f_2^- ((u_\xi - \eta_2)^- - (u_\xi - \eta_2)\phi'_\xi(u_\xi - \eta_2)) dx. \end{aligned} \quad (4.2)$$

Proof. Let u_ξ the solution of the regularized problem:

$$\begin{cases} \text{Find } u_\xi \in H_\eta^1(\Omega) \text{ such that} \\ a(u_\xi, v - u_\xi) + \Psi_\xi(v) - \Psi_\xi(u_\xi) + b(f, v - u_\xi) \geq 0 \quad \forall v \in H_\eta^1(\Omega). \end{cases}$$

Then u_ξ is a solution of the following minimization problem:

$$\text{Find } u_\xi \in H_\eta^1(\Omega) : \quad J_\xi(u_\xi) = \min_{v \in H_\eta^1(\Omega)} J_\xi(v),$$

where J_ξ is the function defined on $H_\eta^1(\Omega)$ by

$$J_\xi(v) := \frac{1}{2}a(v, v) + b(\tilde{f}, v) + \int_{\Omega} (f_1^+ \psi_\xi(v - \eta_1) + f_2^- \phi_\xi(v - \eta_2)) dx.$$

The function J_ξ is differentiable, thus the solution u_ξ satisfies the following variational equation:

$$a(u_\xi, v) + b(\tilde{f}, v) + \int_{\Omega} (f_1^+ \psi'_\xi(u_\xi - \eta_1) + f_2^- \phi'_\xi(u_\xi - \eta_2)) v dx = 0, \quad \forall v \in H_0^1(\Omega). \quad (4.3)$$

Then

$$\int_{\Omega} \nabla u_\xi \cdot \nabla v dx + \int_{\Omega} \left(\tilde{f} + f_1^+ \psi'_\xi(u_\xi - \eta_1) + f_2^- \phi'_\xi(u_\xi - \eta_2) \right) v dx = 0, \quad \forall v \in H_0^1(\Omega). \quad (4.4)$$

This implies that, in the distributional sense, we have

$$- \operatorname{div}(\nabla u_\xi) + \left(\tilde{f} + f_1^+ \psi'_\xi(u_\xi - \eta_1) + f_2^- \phi'_\xi(u_\xi - \eta_2) \right) = 0. \quad (4.5)$$

We remark that for $\psi_\xi = \psi_\xi^i$ and $\phi_\xi = \phi_\xi^i$, for $i = 1, 2$, we have

$$0 \leq \psi'_\xi(t) \leq 1 \quad \text{and} \quad 0 \leq \phi'_\xi(t) \leq 1 \quad \forall t \in \mathbb{R}, \quad (4.6)$$

then we have

$$f_2^- \leq f_1^+ \psi'_\xi(u_\xi - \eta_1) + f_2^- \phi'_\xi(u_\xi - \eta_2) \leq f_1^+ \quad \text{a.e. in } \Omega. \quad (4.7)$$

We set

$$q_1 = -\nabla u_\xi \quad \text{and} \quad q_2 = -\left(f_1^+ \psi'_\xi(u_\xi - \eta_1) + f_2^- \phi'_\xi(u_\xi - \eta_2) \right).$$

From relations (4.5), (4.6), and (4.7), we deduce

$$- \operatorname{div}(q_1) + (q_2 - \tilde{f}) = 0 \quad \text{and} \quad f_2^- \leq -q_2 \leq f_1^+ \quad \text{a.e. in } \Omega.$$

Consequently, we have $q \in \mathcal{D}$. Then, according to Proposition 4.1, we have

$$\|u - u_\xi\|_{H_0^1(\Omega)}^2 \leq 2\mathcal{R}(u_\xi, \Lambda u_\xi) + 2A(q),$$

with

$$A(q) = \int_{\Omega} \left(\frac{1}{2} |q_1|^2 + q_1 \cdot \nabla \eta - \tilde{f} \eta \right) dx + \int_{I(-q_2)} (q_2(\eta - \eta_1) + f_2^-(\eta_2 - \eta_1)) dx + \dots$$

$$\int_{J(-q_2)} (q_2(\eta - \eta_2) + f_1^+(\eta_1 - \eta_2)) dx.$$

Since $q_1 = -\nabla u_{\xi}$ and $q_2 = -(f_1^+ \psi'_{\xi}(u_{\xi} - \eta_1) + f_2^- \phi'_{\xi}(u_{\xi} - \eta_2))$, then

$$A(q) = \int_{\Omega} \left(\frac{1}{2} |\nabla u_{\xi}|^2 - \nabla u_{\xi} \cdot \nabla \eta - \tilde{f} \eta \right) dx + \dots$$

$$\int_{I(-q_2)} ((f_1^+ \psi'_{\xi}(u_{\xi} - \eta_1) + f_2^- \phi'_{\xi}(u_{\xi} - \eta_2)) (\eta_1 - \eta) + f_2^-(\eta_2 - \eta_1)) dx + \dots$$

$$\int_{J(-q_2)} ((f_1^+ \psi'_{\xi}(u_{\xi} - \eta_1) + f_2^- \phi'_{\xi}(u_{\xi} - \eta_2)) (\eta_2 - \eta) + f_1^+(\eta_1 - \eta_2)) dx. \quad (4.8)$$

We have

$$\mathcal{R}(u_{\xi}, \Lambda u_{\xi}) = \int_{\Omega} \left(\frac{1}{2} |\nabla u_{\xi}|^2 + \tilde{f} u_{\xi} + f_1^+ (u_{\xi} - \eta_1)^+ + f_2^- (u_{\xi} - \eta_2)^- \right) dx. \quad (4.9)$$

We have $u_{\xi} - \eta \in H_0^1(\Omega)$; so from (4.4), we deduce that

$$\int_{\Omega} (|\nabla u_{\xi}|^2 - \nabla u_{\xi} \cdot \nabla \eta) dx = \int_{\Omega} \left(\tilde{f} + f_1^+ \psi'_{\xi}(u_{\xi} - \eta_1) + f_2^- \phi'_{\xi}(u_{\xi} - \eta_2) \right) (\eta - u_{\xi}) dx. \quad (4.10)$$

Using the equalities (4.8), (4.9), and (4.10), and after simplifications, we obtain

$$\mathcal{R}(u_{\xi}, \Lambda u_{\xi}) + A(q) = \int_{\Omega} f_1^+ ((u_{\xi} - \eta_1)^+ - (u_{\xi} - \eta_1) \psi'_{\xi}(u_{\xi} - \eta_1)) dx + \dots$$

$$\int_{\Omega} f_2^- ((u_{\xi} - \eta_2)^- - (u_{\xi} - \eta_2) \phi'_{\xi}(u_{\xi} - \eta_2)) dx + \dots$$

$$\int_{J(-q_2)} f_1^+ (\eta_2 - \eta_1) (\psi'_{\xi}(u_{\xi} - \eta_1) - 1) dx + \dots$$

$$\int_{I(-q_2)} f_2^- (\eta_2 - \eta_1) (1 - \phi'_{\xi}(u_{\xi} - \eta_1)) dx. \quad (4.11)$$

We notice that

$$f_1^+ (\eta_2 - \eta_1) (\psi'_{\xi}(u_{\xi} - \eta_1) - 1) \leq 0 \quad \text{and} \quad f_2^- (\eta_2 - \eta_1) (1 - \phi'_{\xi}(u_{\xi} - \eta_1)) \leq 0,$$

therefore

$$\mathcal{R}(u_{\xi}, \Lambda u_{\xi}) + A(q) \leq \int_{\Omega} f_1^+ ((u_{\xi} - \eta_1)^+ - (u_{\xi} - \eta_1) \psi'_{\xi}(u_{\xi} - \eta_1)) dx + \dots$$

$$\int_{\Omega} f_2^- ((u_{\xi} - \eta_2)^- - (u_{\xi} - \eta_2) \phi'_{\xi}(u_{\xi} - \eta_2)) dx$$

Since, by Proposition 4.1, we have

$$\|u - u_{\xi}\|_{H_0^1(\Omega)}^2 \leq 2\mathcal{R}(u_{\xi}, \Lambda u_{\xi}) + 2A(q).$$

Hence

$$\begin{aligned} \|u - u_\xi\|_{H_0^1(\Omega)}^2 &\leq 2 \int_{\Omega} f_1^+ ((u_\xi - \eta_1)^+ - (u_\xi - \eta_1)\psi'_\xi(u_\xi - \eta_1)) dx + \dots \\ &\quad 2 \int_{\Omega} f_2^- ((u_\xi - \eta_2)^- - (u_\xi - \eta_2)\phi'_\xi(u_\xi - \eta_2)) dx. \end{aligned}$$

Which completes the proof. $\square$

- For $\psi_\xi = \psi_\xi^1$ and $\phi_\xi = \phi_\xi^1$ (see (2.10)), the a posteriori error estimates are given by

$$\begin{aligned} \|u - u_\xi\|_{H_0^1(\Omega)}^2 &\leq 2 \int_{[0 < u_\xi - \eta_1 < \xi]} f_1^+(u_\xi - \eta_1) \left(1 - \frac{u_\xi - \eta_1}{\xi}\right) dx + \dots \\ &\quad 2 \int_{[0 < \eta_2 - u_\xi < \xi]} f_2^-(u_\xi - \eta_2) \left(1 - \frac{\eta_2 - u_\xi}{\xi}\right) dx. \end{aligned}$$

- For $\psi_\xi = \psi_\xi^2$ and $\phi_\xi = \phi_\xi^2$ (see (2.11)), we have

$$\begin{aligned} \|u - u_\xi\|_{H_0^1(\Omega)}^2 &\leq 2 \int_{[u_\xi > \eta_1]} f_1^+(u_\xi - \eta_1) \left(1 - \frac{u_\xi - \eta_1}{\sqrt{(u_\xi - \eta_1)^2 + \xi^2}}\right) dx + \dots \\ &\quad 2 \int_{[u_\xi < \eta_2]} f_2^-(u_\xi - \eta_2) \left(1 + \frac{u_\xi - \eta_2}{\sqrt{(u_\xi - \eta_2)^2 + \xi^2}}\right) dx. \end{aligned}$$

5. DISCRETE PROBLEM

Let Ω be a bounded open domain of $\mathbb{R}^n$. Let $(\mathcal{T}_h)_h$ be a regular triangulation of the domain Ω .

Let $(V_h)_h$ (resp. $(V_{0,h})_h$) be the sequence of finite dimension subspaces of $H^1(\Omega)$ (resp. $H_0^1(\Omega)$) defined by

$$\begin{aligned} V_h &:= \{v_h \in C(\bar{\Omega}) \cap H^1(\Omega) : \forall T \in \mathcal{T}_h, v_h|_T \in \mathbb{P}_1(T)\}, \\ V_{0,h} &:= \{v_h \in C(\bar{\Omega}) \cap H_0^1(\Omega) : \forall T \in \mathcal{T}_h, v_h|_T \in \mathbb{P}_1(T)\}, \end{aligned}$$

where $\mathbb{P}_1(T)$ denotes the space of polynomial functions on T of degree less or equal to one. For the sake of simplicity, we assume that the boundary condition function η can be represented exactly by a function from V_h .

The discrete problem associated with the bilateral problem (2.6) is given by:

$$\begin{cases} \text{Find } u_h \in V_h, u_h = \eta \text{ on } \partial\Omega, \text{ such that} \\ a(u_h, v_h - u_h) + \Psi(v_h) - \Psi(u_h) + b(\tilde{f}, v_h - u_h) \geq 0 \quad \forall v_h \in V_h, v_h = \eta \text{ on } \partial\Omega. \end{cases} \quad (5.1)$$

If we set $u_{0,h} := u_h - \eta$, then $u_{0,h}$ is the solution of the following problem:

$$\begin{cases} \text{Find } u_{0,h} \in V_{0,h}, \text{ such that} \\ a(u_{0,h} + \eta, v_h - u_{0,h}) + \Psi(v_h + \eta) - \Psi(u_{0,h} + \eta) + b(\tilde{f}, v_h - u_{0,h}) \geq 0 \quad \forall v_h \in V_{0,h}. \end{cases} \quad (5.2)$$

Proposition 5.1. *Let u and u_h be solutions of problems (2.6) and (5.1), respectively. Then we have the following a priori error estimate:*

$$\|u - u_h\|_{H_0^1(\Omega)} \leq Mh,$$

where M is a constant independent of h .

Proof. We take $v = u_h$ in the inequality of problem (2.6), we have

$$a(u, u_h - u) + \Psi(u_h) - \Psi(u) + b(\tilde{f}, u_h - u) \leq 0. \quad (5.3)$$

Let $v_h \in V_h$, $v_h = \eta$ on $\partial\Omega$. Adding the inequalities in (5.1) and (5.3), we obtain

$$a(u_h, v_h - u_h) + a(u, u_h - u) + \Psi(v_h) - \Psi(u) + b(\tilde{f}, v_h - u) \geq 0,$$

that is

$$-a(u_h, v_h - u_h) - a(u, u_h - u) \leq \Psi(v_h) - \Psi(u) + b(\tilde{f}, v_h - u).$$

Which implies

$$\|u - u_h\|_{H_0^1(\Omega)}^2 \leq a(u_h, v_h - u) + \Psi(v_h) - \Psi(u) + b(\tilde{f}, v_h - u).$$

Then we have

$$\|u - u_h\|_{H_0^1(\Omega)}^2 \leq a(u_h - u, v_h - u) + a(u, v_h - u) + \Psi(v_h) - \Psi(u) + b(\tilde{f}, v_h - u). \quad (5.4)$$

Note that

$$a(u_h - u, v_h - u) \leq \frac{1}{2}\|u - u_h\|_{H_0^1(\Omega)}^2 + \frac{1}{2}\|v_h - u\|_{H_0^1(\Omega)}^2$$

we deduce

$$\frac{1}{2}\|u - u_h\|_{H_0^1(\Omega)}^2 \leq \frac{1}{2}\|v_h - u\|_{H_0^1(\Omega)}^2 + a(u, v_h - u) + \Psi(v_h) - \Psi(u) + b(\tilde{f}, v_h - u). \quad (5.5)$$

We have

$$\begin{aligned} \Psi(v_h) - \Psi(u) &= \int_{\Omega} f_1^+ [(v_h - \eta_1)^+ - (u - \eta_1)^+] dx + \int_{\Omega} f_2^- [(v_h - \eta_2)^- - (u - \eta_2)^-] dx \\ &\leq \|f_1^+\|_{L^2(\Omega)} \|(v_h - \eta_1)^+ - (u - \eta_1)^+\|_{L^2(\Omega)} + \dots \\ &\quad \|f_2^-\|_{L^2(\Omega)} \|(v_h - \eta_2)^- - (u - \eta_2)^-\|_{L^2(\Omega)} \\ &\leq \|f_1^+\|_{L^2(\Omega)} \|v_h - u\|_{L^2(\Omega)} + \|f_2^-\|_{L^2(\Omega)} \|v_h - u\|_{L^2(\Omega)} \\ &\leq (\|f_1^+\|_{L^2(\Omega)} + \|f_2^-\|_{L^2(\Omega)}) \|v_h - u\|_{L^2(\Omega)}. \end{aligned} \quad (5.6)$$

According to the works presented in [3, 4], the bilateral obstacle problem (2.6) is equivalent to the following mixed problem:

$$\left\{ \begin{array}{l} \text{Find } (u, \mu_1, \mu_2) \in H_{\eta}^1(\Omega) \times C_1 \times C_2 \text{ such that} \\ a(u, v) + b(\mu_1 + \mu_2 + \tilde{f}, v) = 0 \quad \forall v \in H_0^1(\Omega), \\ b((\lambda_1 + \lambda_2) - (\mu_1 + \mu_2), u) \leq b(\lambda_1 - \mu_1, \eta_1 - \eta) + b(\lambda_2 - \mu_2, \eta_2 - \eta) \quad \forall (\lambda_1, \lambda_2) \in C_1 \times C_2, \end{array} \right. \quad (5.7)$$

where u is the solution of problem (2.6), and

$$C_1 := \{v \in L^2(\Omega) : 0 \leq v \leq f_1^+ \text{ a.e. in } \Omega\},$$

$$C_2 := \{v \in L^2(\Omega) : f_2^- \leq v \leq 0 \text{ a.e. in } \Omega\}.$$

Then, for all $v_h \in V_h$, $v_h = \eta$ on $\partial\Omega$, we have

$$a(u, v_h - u) = -b(\mu_1 + \mu_2 + \tilde{f}, v_h - u),$$

hence, from (5.5) and (5.6) we obtain

$$\begin{aligned} \frac{1}{2} \|u - u_h\|_{H_0^1(\Omega)}^2 &\leq \frac{1}{2} \|v_h - u\|_{H_0^1(\Omega)}^2 + (\|f_1^+\|_{L^2(\Omega)} + \|f_2^-\|_{L^2(\Omega)}) \|v_h - u\|_{L^2(\Omega)} - b(\mu_1 + \mu_2, v_h - u) \\ &\leq \frac{1}{2} \|v_h - u\|_{H_0^1(\Omega)}^2 + (\|f_1^+\|_{L^2(\Omega)} + \|f_2^-\|_{L^2(\Omega)} + \|\mu_1 + \mu_2\|_{L^2(\Omega)}) \|v_h - u\|_{L^2(\Omega)}. \end{aligned} \quad (5.8)$$

Let $u_0 = u - \eta$, we note $\Pi_h(u_0)$ the interpolant of u_0 in $V_{0,h}$. We have the following approximation errors

$$\|\Pi_h(u_0) - u_0\|_{L^2(\Omega)} \leq c_1 \|u_0\|_{H^2(\Omega)} h^2, \quad (5.9)$$

$$\|\Pi_h(u_0) - u_0\|_{H_0^1(\Omega)} \leq c_2 \|u_0\|_{H^2(\Omega)} h, \quad (5.10)$$

where c_1 and c_2 are two positive constants independent of u_0 (see [12]).

If we take $v_h = \Pi_h(u_0) + \eta$ in the inequality (5.8), and we use relations (5.9) and (5.10), then we have

$$\begin{aligned} \frac{1}{2} \|u - u_h\|_{H_0^1(\Omega)}^2 &\leq \frac{1}{2} c_2^2 \|u_0\|_{H^2(\Omega)}^2 h^2 + c_1 (\|f_1^+\|_{L^2(\Omega)} + \|f_2^-\|_{L^2(\Omega)} + \|\mu_1 + \mu_2\|_{L^2(\Omega)}) \|u_0\|_{H^2(\Omega)} h^2 \\ &\leq N h^2, \end{aligned}$$

with $N = \frac{1}{2} c_2^2 \|u_0\|_{H^2(\Omega)}^2 + c_1 (\|f_1^+\|_{L^2(\Omega)} + \|f_2^-\|_{L^2(\Omega)} + \|\mu_1 + \mu_2\|_{L^2(\Omega)}) \|u_0\|_{H^2(\Omega)}$. $\square$

The discrete regularized problem associated with problem (5.1) is

$$\begin{cases} \text{Find } u_{h,\xi} \in V_h, u_{h,\xi} = \eta \text{ on } \partial\Omega, \text{ such that} \\ a(u_{h,\xi}, v_h - u_{h,\xi}) + \Psi_\xi(v_h) - \Psi_\xi(u_{h,\xi}) + b(f, v_h - u_{h,\xi}) \geq 0 \quad \forall v_h \in V_h, v_h = \eta \text{ on } \partial\Omega. \end{cases} \quad (5.11)$$

Similarly, like the continuous problem, we establish the following a posteriori error estimates for the discrete problem:

Proposition 5.2. *Let u_h and $u_{h,\xi}$ be solutions of (5.1) and (5.11), respectively. Then we have the following a posteriori error estimate:*

$$\begin{aligned} \|u_h - u_{h,\xi}\|_{H_0^1(\Omega)}^2 &\leq 2 \int_{\Omega} f_1^+ ((u_{h,\xi} - \eta_1)^+ - (u_{h,\xi} - \eta_1) \psi'_\xi(u_{h,\xi} - \eta_1)) dx + \dots \\ &\quad 2 \int_{\Omega} f_2^- ((u_{h,\xi} - \eta_2)^- - (u_{h,\xi} - \eta_2) \phi'_\xi(u_{h,\xi} - \eta_2)) dx. \end{aligned} \quad (5.12)$$

- For $\psi_\xi = \psi_\xi^1$ and $\phi_\xi = \phi_\xi^1$ (see (2.10)), the a posteriori error estimates are given by

$$\begin{aligned} \|u_h - u_{h,\xi}\|_{H_0^1(\Omega)}^2 &\leq 2 \int_{[0 < u_{h,\xi} - \eta_1 < \xi]} f_1^+(u_{h,\xi} - \eta_1) \left(1 - \frac{u_{h,\xi} - \eta_1}{\xi}\right) dx + \dots \\ &\quad 2 \int_{[0 < \eta_2 - u_{h,\xi} < \xi]} f_2^-(u_{h,\xi} - \eta_2) \left(1 - \frac{\eta_2 - u_{h,\xi}}{\xi}\right) dx. \end{aligned} \quad (5.13)$$

- For $\psi_\xi = \psi_\xi^2$ and $\phi_\xi = \phi_\xi^2$ (see (2.11)), we have

$$\begin{aligned} \|u_h - u_{h,\xi}\|_{H_0^1(\Omega)}^2 &\leq 2 \int_{[u_{h,\xi} > \eta_1]} f_1^+(u_{h,\xi} - \eta_1) \left(1 - \frac{u_{h,\xi} - \eta_1}{\sqrt{(u_{h,\xi} - \eta_1)^2 + \xi^2}}\right) dx + \dots \\ &\quad 2 \int_{[u_{h,\xi} < \eta_2]} f_2^-(u_{h,\xi} - \eta_2) \left(1 + \frac{u_{h,\xi} - \eta_2}{\sqrt{(u_{h,\xi} - \eta_2)^2 + \xi^2}}\right) dx. \end{aligned} \quad (5.14)$$

6. CONCLUSION AND REMARKS

In this paper, we have considered the bilateral obstacle problem, which is a variational inequality of the first kind, we have reformulated this problem into an unconstrained variational inequality problem. Then, we have presented a relatively complete analysis of the regularization approach for the bilateral obstacle problem, we have provided the convergence of the approximate solutions and the a priori error estimates. Practically, we cannot use this estimate as a stopping criterion in the discrete case. Thus, by employing the duality approach, we establish a posteriori error estimates for both the continuous and discrete problems. We note that for $\psi_\xi = \psi_\xi^1$ and $\phi_\xi = \phi_\xi^1$ (see (2.10)), we have

$$\int_{[0 < u_{h,\xi} - \eta_1 < \xi]} f_1^+(u_{h,\xi} - \eta_1) \left(1 - \frac{u_{h,\xi} - \eta_1}{\xi}\right) dx \leq \xi \|f_1^+\|_{L^1(\Omega)}$$

and

$$\int_{[0 < \eta_2 - u_{h,\xi} < \xi]} f_2^-(u_{h,\xi} - \eta_2) \left(1 - \frac{\eta_2 - u_{h,\xi}}{\xi}\right) dx \leq \xi \|f_2^-\|_{L^1(\Omega)}.$$

Hence, for the choice $\psi_\xi = \psi_\xi^1$ and $\phi_\xi = \phi_\xi^1$, the a posteriori error estimates (see (5.13)) provide a more refined evaluation compared to the a priori error estimates (see (2.13)). Theoretically, we cannot confirm this remark for the choice $\psi_\xi = \psi_\xi^2$ and $\phi_\xi = \phi_\xi^2$.

Furthermore, we have

- For the case where $\psi_\xi = \psi_\xi^1$ and $\phi_\xi = \phi_\xi^1$, the a posteriori error bounds will be calculated over sets $[0 < u_{h,\xi} - \eta_1 < \xi]$ and $[0 < \eta_2 - u_{h,\xi} < \xi]$, which is the admissible domain of the approximate solution with an amplitude less than ξ . Due to the limited extent of this domain, the a posteriori error bounds are expected to be small.

- For the case where $\psi_\xi = \psi_\xi^2$ and $\phi_\xi = \phi_\xi^2$, the a posteriori error bounds will be calculated over sets $[\eta_1 < u_{h,\xi}]$ and $[u_{h,\xi} < \eta_2]$, which is the admissible domain of the approximate solution, this domain is not managed by any amplitude.

Based on the information provided, it seems that choosing $\psi_\xi = \psi_\xi^1$ and $\phi_\xi = \phi_\xi^1$ may lead to better a posteriori error estimates for the bilateral obstacle problem.

Declarations of interest: none.

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