

ON THE FREE BOUNDARY PROBLEM FOR A PREDATOR-PREY MODEL WITH FEAR EFFECT

JOZIL TAKHIROV^{1*} and BUNYODBEK ANVARJONOV¹

ABSTRACT. This manuscript proposes a three-species model with a free food chain boundary to study the effects of fear. The well-posedness and uniform estimates for the solution are proved, as well as the behavior of the components of the solution and the unknown boundary over large time intervals. The findings suggest that the effect of fear of predators can move a system from a chaotic to a stable state. The results may provide useful biological information about ecosystems in which predator-prey interactions occur.

1. INTRODUCTION

It is a common observation that species hunt one species and at the same time are hunted by other species. These phenomena create complex networks within ecosystems, and studying the influence of fear on the population of species experiencing it is of great importance.

Fear of predators is a well-recognized phenomenon that occurs as a result of predation in a natural food chain or food web ecosystem. Wang et al. [25] published the first food chain model that considered the effect of fear of predators on the growth rate of prey. The indirect effects of fear have been studied by incorporating it into several food web models in different ecological contexts.

Due to the increasingly in-depth discussion of prey-predator models, more and more elements were introduced into the model. By introducing the fear effect [26, 20], it is concluded that the prey's fear of a predator changes the rules of its growth and development. The presence of a predator can significantly change the behavior of the prey to the point that it can influence the prey more influentially than direct predation. Recently, a large number of scientific articles have been published on the influence of fear in predator-prey systems [3, 4, 6, 12, 17, 18]. More generally, some recent notable works analyzing nonlinear differential equations can be found in [10, 7, 2].

For example, in [16] the authors develop a mathematical model of the food chain dynamics of an ecological community, and in [15] the authors consider a model of the predator-prey fear effect with mutual interference or group defense.

Date: Received: Aug 3, 2024; Accepted: Nov 8, 2024.

* Corresponding author.

2020 *Mathematics Subject Classification.* Primary: 35K51, 35R35; Secondary 37N25.

Key words and phrases. Prey-predator model, Fear effect, Existence and Stability, free boundary.

For the mutual interference model, it is shown that the internal equilibrium is globally stable, and mutual interference can stabilize the predator-prey system. In [21], a prey-predator model with a linear functional response was considered and it was shown that fear does not affect the stability of equilibrium, but the time difference between the two populations has a positive effect.

The work [1] considers "a predator-prey model with density-dependent prey taxis and the structure of predator stages". The existence of classical solutions with a time-uniform boundary in the one-dimensional case is established. Zhang et. al. [27] studied the influence of fear and prey refuge on the stability of the predator-prey system and showed that the effect of fear can stabilize the system. Tiwari et al. [23] examined a predator-prey model with a functional Beddington-DeAngelis response and the influence of fear on prey, and discussed the stability and bifurcations of the system.

2. PRELIMINARIES

2.1. About ODE mathematical models. Here we present the main assumptions of [16] to substantiate the model and use them when constructing a spatial model. In [16], the authors constructed an ODE model of the fear effect in "a three-species food chain model of an ecological community with a generalist predator". Three species are considered: prey, intermediate predator, and main predator, and their densities are denoted by $x(t)$, $y(t)$, and $z(t)$ under the following assumptions:

1. "In the absence of predators, prey species grow logistically at their growth rate r , and carrying capacity r/d ; d is the density-dependent mortality rate."
2. "Abduction of prey species by intermediate predators is quantified using the Holling type II functional response, which is defined as $\frac{\alpha_1 xy}{m + x}$, where α_1 is the capture rate and m is the half-saturation constant."
3. Intermediate predators belong to a specialized type, i.e., their food source is only the prey in question.
4. "The density of intermediate predators decreases due to natural mortality, as well as intraspecific competition. Let us denote natural mortality and the coefficient of intraspecific competition of intermediate predators by δ_1 and δ_2 , respectively. Their densities are also reduced by predation by top predators, which is modeled by the bilinear functional response $\alpha_2 yz$."
5. "The growth rate is proportional to the functional response with a constant of proportionality θ_2 . The density of top predators decreases due to natural mortality at a rate of δ_3 ."
6. "Unlike intermediate predators, apex predators are universal. Growth is determined by the relation $\frac{\eta z}{1 + \eta_0 z}$, where η is the per capita reproduction rate and η_0 is the population size, depending on population density."

Then the model of [16] takes the following form:

$$\begin{cases} \frac{dx}{dt} = rx - dx^2 - \frac{\alpha_1 xy}{m+x}, \\ \frac{dy}{dt} = \frac{\theta_1 \alpha_1 xy}{m+x} - \delta_1 y - \delta_2 y^2 - \alpha_2 yz, \\ \frac{dz}{dt} = \frac{\eta z}{1+\eta_0 z} + \theta_2 \alpha_2 yz - \delta_3 z, \end{cases} \quad (2.1)$$

where $x(0) > 0$, $y(0) > 0$ and $z(0) > 0$. Note that above, the value of η must be greater than δ_3 - the natural mortality of top predators.

The aggregation process increases intraspecific competition between prey due to limited resources. To model this mathematically, the cost of fear is taken into account by adjusting parameters that control the prey population growth and intraspecific competition. Specifically, the growth rate r of the prey species is multiplied by a decreasing function $\frac{1}{1+\omega_1 y}$. The effect of fear on intraspecific competition is taken into account by applying the increasing function $1+\omega_2 y$ to d . Here, ω_1 and ω_2 are the fear parameters.

The predation rate of intermediate predators varies by a factor of $\frac{1}{1+\omega_3 z}$, where ω_3 is the fear parameter and z is the density of top predators. The period of growth in a mesopredator population is proportional to the modified functional response with a constant of proportionality θ_1 , including the same coefficient $\frac{1}{1+\omega_3 z}$. Consequently, the coefficient of intraspecific competition δ_2 is adjusted by an increasing function $1+\omega_4 z$, where ω_4 is the fear parameter. This modification of the model formulation reflects the complex interaction of participants and has the form

$$\begin{cases} \frac{dx}{dt} = \frac{rx}{1+\omega_1 y} - d(1+\omega_2 y)x^2 - \frac{1}{1+\omega_3 z} \frac{\alpha_1 xy}{m+x}, \\ \frac{dy}{dt} = \frac{1}{1+\omega_3 z} \frac{\theta_1 \alpha_1 xy}{m+x} - \delta_1 y - \delta_2(1+\omega_4 z)y^2 - \alpha_2 yz, \\ \frac{dz}{dt} = \frac{\eta z}{1+\eta_0 z} + \theta_2 \alpha_2 yz - \delta_3 z, \end{cases} \quad (2.2)$$

with initial condition $x(0) > 0$, $y(0) > 0$ and $z(0) > 0$.

The positivity and condition of boundedness of solutions of system (2.2) are established, ensuring the proposed system's meaningfulness from an environmental point of view. Next, equilibrium points are determined to understand the system's long-term behavior. Equilibrium points represent stable states at which changes in population size cease. It is easy to see that the last model has three axial equilibria: $E_0(0, 0, 0)$, $E_1(\hat{x}, 0, 0)$ and $E_2(0, 0, \hat{z})$; flat equilibrium $E_3(\hat{x}, 0, \hat{z})$, where $\hat{x} = \frac{r}{d}$ and $\hat{z} = (\eta - \delta_3)/\eta_0 \delta_3 > 0$, since $\eta > \delta_3$.

The authors investigated the potential existence of a Hopf bifurcation by examining the direction and stability of the resulting periodic solution resulting from this bifurcation phenomenon.

The results demonstrate the stability of a predator-free intermediate equilibrium when the maximum growth rate falls below a critical threshold. In such cases, the intermediate predator's survival in the system is at risk. This is due to the significant role of alternative food sources in accelerating the growth of apex predators, which increases their population density and, in turn, reduces the population of intermediate predators. While prey species initially benefit from this, their densities eventually reach saturation due to resource limitations. Consequently, the ever-increasing density of apex predators, coupled with limited prey availability, eventually leads to the extinction of intermediate predators. However, when fear of predators is minimal and natural mortality rates among apex predators are higher, the role of additional food sources in supporting apex predator growth becomes crucial for maintaining the ecosystem's dynamic stability.

It is noteworthy that all parameters of fear have a stabilizing effect, making a significant contribution to the stability of the system.

2.2. Comparison principles. Here we recall some comparison principles that were proven in [24, 9] and will be used in our work to evaluate the solution and determine the criteria that determine propagation and attenuation. The comparison principle for similar problems was first proven in [9], and in [24] the authors adapted them to their problems.

[24] considers the DFB problem

$$\begin{cases} u_t - u_{xx} = u(1 - u - av), t > 0, 0 < x < h(t) \\ v_t - dv_{xx} = v(c - v + bu), t > 0, 0 < x < h(t) \\ u = v = 0, t > 0, x = 0, \\ u = v = 0, h'(t) = -\mu(u_x + \rho v_x), t > 0, x = h(t), \\ u(0, x) = u_0(x), v(0, x) = v_0(x), x \in [0, h_0], \\ h(0) = h_0. \end{cases} \quad (DFB)$$

If we accept the condition on the left boundary in the form $u_x = v_x = 0, t > 0, x = 0$, then we obtain the NFB problem.

Lemma 2.1. [24] (*Comparison principle*). *Let $\bar{h} \in C^1([0, \infty))$ and $\bar{h}(t) > 0$ in $[0, \infty)$ and $\bar{u}, \bar{v} \in C(\Omega) \cap C^{1,2}(\Omega)$, with $\Omega(t) = \{(t, x) : t > 0, 0 < x < \bar{h}(t)\}$. Assume that $(\bar{u}, \bar{v}, \bar{h})$ satisfies*

$$\begin{cases} \bar{u}_t - \bar{u}_{xx} \geq \bar{u}(1 - \bar{u}), t > 0, 0 < x < \bar{h}(t), \\ \bar{v}_t - d\bar{v}_{xx} \geq \bar{v}(c - \bar{v} + b\bar{u}), t > 0, 0 < x < \bar{h}(t), \\ \bar{u}(t, 0) \geq 0, \bar{v}(t, 0) \geq 0, \\ \bar{u}(t, \bar{h}(t)) = \bar{v}(t, \bar{h}(t)) = 0, t > 0, \\ \bar{h}'(t) \geq -\mu(\bar{u}_x(t, \bar{h}(t)) + \rho \bar{v}_x(t, \bar{h}(t))), t > 0, \end{cases} \quad (2.3)$$

If $\bar{h}(0) \geq h_0$, $\bar{u}(0, x), \bar{v}(0, x) \geq 0$ on $[0, \bar{h}(0)]$, and $\bar{u}(0, x) \geq u_0(x)$, $\bar{v}(0, x) \geq v_0(x)$ on $[0, h_0]$. Then the solution (u, v, h) of (DFB) satisfies $h(t) \leq \bar{h}(t)$ on $[0, \infty)$, and $u \leq \bar{u}$, $v \leq \bar{v}$ on D , where

$$D(t) = \{(t, x) : t > 0, 0 \leq x \leq h(t)\}.$$

In (2.3), if the conditions $\bar{u}(t, 0) \geq 0$ and $\bar{v}(t, 0) \geq 0$ are replaced by $\bar{u}_x(t, 0) \leq 0$ and $\bar{v}_x(t, 0) \leq 0$, then the conclusion still holds for the solution of (NFB).

Lemma 2.2. [24] (*Comparison principle*). Let $\underline{h} \in C^1([0, \infty))$ with $\underline{h}(t) > 0$ for all $t \in [0, \infty)$, and $\underline{v} \in C(\Omega_1) \cap C^{1,2}(\Omega_1)$ with

$$\Omega_1(t) = \{(t, x) : t > 0, 0 < x < \underline{h}(t)\}.$$

Assume that $(\underline{v}, \underline{h})$ satisfies

$$\begin{cases} \underline{v}_t - d\underline{v}_{xx} \leq \underline{v}(c - \underline{v}), t > 0, 0 < x < \underline{h}(t), \\ \underline{v}(t, 0) = \underline{v}(t, \underline{h}(t)) = 0, t > 0, \\ \underline{h}'(t) \leq -\mu\rho\underline{v}_x(t, \underline{v}(t)), t > 0, \end{cases} \quad (2.4)$$

and $0 < \underline{h}(0) \leq h_0$, $0 \leq \underline{v}(0, x) \leq v_0(x)$ on $[0, \underline{h}(0)]$. Then the solution (u, v, h) of (DFB) satisfies $h(t) \geq \underline{h}$ on $[0, \infty)$, and $v(t, x) \geq \underline{v}(t, x)$ on $\Omega_1(t)$. If, in (2.4), the condition $\underline{v}(t, 0) = 0$ is replaced by $\underline{v}_x(t, 0) \geq 0$, then the conclusion still holds for the solution of (NFB).

3. MATHEMATICAL MODEL WITH A FREE BOUNDARY

Next, we shall build a model to control the spatial and temporal evolution of prey, intermediate predators (specialists), and top predators (generalists), as well as free boundaries represented by the predator distribution front. Moreover, we assume that top predators can survive in the ecosystem even when the intermediate predators are absent. It should be noted that the theoretical studies of predator-prey dynamics and food web structure have largely focused on specialist predators.

Keeping in mind that due to mobility the distribution of individuals across an area is uneven, modeling this phenomenon often leads to reaction-diffusion equations. To compensate for the gradual spread of predators, the best modeling method is to introduce a free boundary.

Currently, the Stefan problem has become widely used to describe biomedical and environmental processes [28, 5, 14, 22]. To describe the process of gradual expansion and change of the region under consideration, a free boundary was introduced, which represents the propagation front of the main component of the model.

From the authors' perspective, the introduction of free boundary problems into the theory of nonlinear dissipative structures is completely reasonable, as they more adequately describe natural processes.

Therefore, based on (2.4), we propose a model with a free boundary to control the spatial and temporal evolution of predators and prey in limited areas

$$Q = \{(t, x) : t > 0, 0 < x < l\}, D = \{(t, x) : t > 0, 0 < x < h(t)\},$$

$$\begin{aligned}
u_t - (d_1(u)u_x)_x &= \frac{ru}{1 + \omega_1 v} - d(1 + \omega_2 v)u^2 - \frac{\alpha_1 uv}{(m + u)(1 + \omega_3 w)}, \quad (t, x) \in Q \\
v_t - (d_2(v)v_x)_x &= \frac{\theta_1 \alpha_1 uv}{(m + u)(1 + \omega_3 w)} - \delta_1 v \\
&\quad - \delta_2(1 + \omega_4 w)v^2 - \alpha_2 vw, \quad (t, x) \in D, \\
w_t - (d_3(w)w_x)_x &= \frac{\eta w}{(1 + \eta_0)} + \theta_2 \alpha_2 vw - \delta_3 w, \quad (t, x) \in D, \\
u_x(t, 0) &= u_x(t, l) = 0, v(t, h(t)) = w(t, h(t)) = 0, t \geq 0, \\
v_x(t, 0) &= w_x(t, 0) = 0, h'(t) = -\mu(w_x(t, h(t)) + \rho v_x(t, h(t))), t > 0, \\
u(0, x) &= u_0(x), 0 \leq x \leq l, \\
v(0, x) &= v_0(x), w(0, x) = w_0(x), 0 \leq x \leq h_0 = h(0),
\end{aligned} \tag{3.1}$$

where $u(t, x)$, $v(t, x)$, and $w(t, x)$ represent the populations of prey, intermediate predators (specialists), and top predators (generalists), respectively. Here we use the variation region $(0, h(t))$ to denote the predatory habitat of the predators, where the free boundary $x = h(t)$ represents the expansion front of the predators. Above, $x = h(t)$ is a moving boundary that is defined together with $u(t, x)$, $v(t, x)$ and $w(t, x)$. Moreover, $\omega_i, \delta_i (i = 1, 2, 3)$, $d_i, \alpha_i, C_i, (i = 1, 2)$, $\omega_4, r, d, m, \eta, \eta_0$ and h_0 are positive constants. We emphasize that the boundary conditions for $u(t, x)$ are set based on biological considerations.

We assume that the initial functions d_i, u_0, v_0 and w_0 satisfy

- A) $d_i(\xi) > d_i(0) > 0, i = 1, 2, 3; d_i(\xi) \in C^{1+\alpha}(\Omega)$ for all x belonging to a closed set;
- B) $u_0(x) \in C_2[0, l], v_0(x), w_0(x) \in C_2([0, h_0]);$
- C) $u_0 > 0$ in $(0, l), v_0, w_0 > 0$ in $(0, h_0);$
- D) The corresponding alignment conditions at the corner points are met.

In [19], the fear effect and stage structure are introduced into a space-time problem with a free boundary predator-prey model.

The goal of this work is to study the properties of the solution and prove the global solvability of problem (3.1). Moreover, we develop a new method for obtaining Schauder-type a priori estimates for a new class of free boundary problems for mixed two-phase equations. We also study the properties of free boundaries in the region under consideration and the asymptotic regime of $t \rightarrow \infty$.

4. A PRIORI ESTIMATES

It is known that the results of local existence and uniqueness are valid for any quasi-linear parabolic equation if the given functions are sufficiently smooth, without any restrictions on the type of growth of these functions from the unknown and its derivative with respect to x (see, for example, [11, 13]). They are necessary when considering the global solvability of boundary value problems.

4.1. Positivity and boundedness of solutions. Herein we obtain a positivity result and establish a boundedness condition of the solutions of (3.1), which

ensures that the proposed system (3.1) makes sense from a mathematical and environmental point of view. It is important to ensure that the formulated model reflects interactions between species and real ecological principles, such as the stability of populations, their limitation, and the absence of endless growth or extinction.

Lemma 4.1. *Assume that $(u, v, w, h(t))$ is a solution of (3.1). Then $\exists M_i > 0$, $i = \overline{1, 4}$, independent of T , such that*

$$\begin{aligned} 0 < u(t, x) &\leq M_1 = \max \left\{ \|u_0\|, \frac{r}{d} \right\} \text{ in } Q_T = [(t, x) : 0 < t \leq T, 0 < x < l], \\ 0 < v(t, x) &\leq M_2 \text{ in } D_T = [(t, x) : 0 < t \leq T, 0 < x < h(t)], \\ 0 < w(t, x) &\leq M_3 \text{ in } D_T, \\ 0 < h'(t) &\leq M_4, 0 < t \leq T. \end{aligned}$$

Proof. The positivity of the original functions and the homogeneity of the boundary functions imply that the sought functions inside the domain are positive. Indeed, let $P_0(t_0, x_0)$ be the first interior point from below, where $u(P_0) = 0$, $0 < t_0 < T$. This will be the minimum for $u(t, x)$ in the region $\{(t, x) : 0 < t < t_0; 0 < x < l\}$. Then the equation for $u(t, x)$ at point P_0 leads to a contradiction. Similar reasoning is valid for $v(t, x)$, $w(t, x)$. Thus, $v(P_0) > 0$ and $w(P_0) > 0$. Then, since P_0 is arbitrary, we obtain the positivity of $v(p)$, $w(p)$ for any $P \in D_T$. Therefore, $v(t, x) > 0$, $w(t, x) > 0$. By Hopf's boundary lemma we obtain that $v_x(t; h(t)) < 0$, $w_x(t; h(t)) < 0$. Then the Stefan condition gives $h'(t) > 0$.

Next, we set upper bounds for u, v, w . From the first equation of (3.1) with $y(t) = u(t, \text{const})$ we obtain

$$y' = \frac{ry}{1 + \omega_1 v} - d(1 + \omega_2 v)y^2 - \frac{\alpha_1 y v}{(m + y)(1 + \omega_3 w)} \leq ry - dy^2.$$

Therefore,

$$\limsup_{t \rightarrow \infty} y(t) \leq M_1 = \max \left\{ \|u_0\|, \frac{r}{d} \right\}.$$

Apply the comparison theorem, we find that $u(t, x) \leq y(t) \leq M_1$.

The estimate for $v(t, x)$ can be established as follows. Let $v(t, x)$ reach a positive maximum at some interior point $P \in D_T$. Consider the second equation for $v(t, x)$ at point P . We have

$$\begin{aligned} 0 \leq v_t - (d_2 v_x)_x &= \frac{\theta_1 \alpha_1 u v}{(m + u)(1 + \omega_3 w)} - \delta_1 v - \delta_2 (1 + \omega_4 w) v^2 - \alpha_2 v w \\ &< \theta_1 \alpha_1 v - \delta_1 v - \delta_2 v^2, \end{aligned}$$

giving

$$V(P) \leq \frac{\theta_1 \alpha_1 - \delta_1}{\delta_2}, \quad \theta_1 \alpha_1 - \delta_1 > 0.$$

Then

$$0 < v(t, x) \leq M_2 = \max \left\{ \|v_0\|, \frac{\theta_1 \alpha_1 - \delta_1}{\delta_2} \right\}.$$

For the function w , using similar reasoning, we find

$$0 \leq w_t - (d_3 w_x)_x = \frac{\eta w}{1 + \eta_0 w} + \theta_2 \alpha_2 v w - \delta_3 w < \frac{\eta}{\eta_0} + \left(\theta_2 \alpha_2 \frac{(\theta_1 \alpha_1 - \delta_1)}{\delta_2} - \delta_3 \right) w.$$

As a result, we get that

$$W(P) < \frac{1}{\delta_3 - \frac{\theta_1 \alpha_1 - \delta_1}{\delta_2}} \frac{\eta}{\eta_0} = M_3.$$

Then $0 < w(t, x) \leq M_3 = \max\{\|w_0\|, M_3\}$.

To establish estimates for $h'(t)$ from above, we look at following the problem for $w(t, x)v(t, x)$:

$$\begin{aligned} w_t - (d_3 w_x)_x + a(v, w)w &= 0, \\ w(0, x) &= w_0(x), 0 \leq x \leq h_0, \\ w_x(t, 0) &= w(t, h(t)) = 0, t \geq 0, \\ h'(t) &= -\mu(w_x(t, h(t)) + \rho v_x(t, h(t))), t \geq 0, \end{aligned}$$

where $a(v, w) = -\theta_2 \alpha_2 v w + \delta_3 - \frac{\eta}{1 + \eta_0 v} < 0$.

Introducing a new function $U(t, x) = w(t, x) + N(x - h(t))$, gives the following problem for $U(x, t)$:

$$\begin{aligned} U_t - (d_3 U_x)_x - N d'_3(U) U_x + a(v, w)U &= -N h'(t) + a(v, w)N(x - h(t)) < 0, \\ U(0, x) &= v_0(x) + N(x - h_0) \leq 0, \\ U_x(t, 0) &= N > 0, U(t, h(t)) = 0, t \geq 0. \end{aligned}$$

Therefore, according to the maximum principle, $U(t, x) \leq 0$. Then $U_x(t, h(t)) \geq 0$ or $w_x(t, h(t)) \geq -N$, $v_x(t, h(t)) \geq -N$. Stefan's condition then gives

$$h'(t) = -\mu(w_x(t, h(t)) + \rho v_x(t, h(t))) \leq 2\mu N = M_4.$$

Lemma 4.1 is proved. $\square$

4.2. High-order prior estimates. We will derive Hölder norm bounds $|\cdot|_{1+\alpha}$ and $|\cdot|_{2+\alpha}$ in Q_T and D_T . For each equation of the system, we formulate the corresponding problems:

$$\begin{aligned} (d_1(u)u_x)_x + u a_1(u, v, w) - u_t &= 0 \text{ in } Q_T = \{(t, x) : 0 < t \leq T; 0 < x < l\}, \\ u(0, x) &= u_0(x), 0 \leq x \leq l, \\ u(t, 0) &= u(t, l) = 0, \end{aligned} \tag{4.1}$$

where $a_1(u, v, w) = \frac{ru}{1 + \omega_1 u} - d(1 + \omega_2 v)u^2 - \frac{\alpha_1 uv}{(m + u)(1 + \omega_3 w)}$,

$$\begin{aligned} (d_2(v)v_x)_x + a_2(u, v, w) - v_t &= 0, \text{ in } D_T = \{(t, x) : 0 < t \leq T; 0 < x < h(t)\}, \\ v(0, x) &= v_0(x), 0 \leq x \leq h_0, \\ v_x(t, 0) &= 0, v(t, h(t)) = 0, t \geq 0, \\ h'(t) &= -\mu(w_x(t, h(t)) + \rho v_x(t, h(t))), t \geq 0 \end{aligned} \tag{4.2}$$

where $a_2(u, v, w) = \frac{\theta_1 \alpha_1 uv}{(m+u)(1+\omega_3 w)} - \delta_1 v - \delta_2(1+\omega_4 w)v^2 - \alpha_2 vw$, and

$$\begin{aligned} w_t - (d_3(w)w_x)_x + a_3(v, w)w &= 0, \\ w(0, x) &= w_0(x), 0 \leq x \leq h_0, \\ w_x(t, 0) &= w(t, h(t)) = 0, t \geq 0, \\ h'(t) &= -\mu(w_x(t, h(t)) + \rho v_x(t, h(t))), t \geq 0, \end{aligned} \quad (4.3)$$

where $a_3(u, v, w) = \frac{\eta w}{1 + \eta_0 w} + \theta_2 \alpha_2 vw - \delta_3 w$.

Theorem 4.2. *Assume that $u(t, x), u_x(t, x)$ are continuous in Q_T and suppose that $u(t, x)$ is a solution of (4.1). Then*

$$|u_x(t, x)| \leq C_1(M_1, M_2) \text{ in } Q_T. \quad (4.4)$$

Moreover, if $U_{xx}, U_{tx} \in L_2(Q_T)$, then there exists $\alpha = \alpha(M_1)$, such that

$$|u|_{1+\alpha, Q_T} \leq C_2(M_1, C_1). \quad (4.5)$$

Additionally, assume that, $u(t, x)$ satisfying (4.1) in Q_T , is continuous with its derivatives u_t, u_x, u_{xx} and $|u|_{2+\alpha, Q_T} < \infty$. Then

$$|u|_{2+\alpha, Q_T} \leq C_3(M_1, C_1, C_2). \quad (4.6)$$

Proof. Estimates (4.4)-(4.6) for $(t, x) \in Q_T$ are a direct consequence of the results of [11].

A priori estimates for (4.2) and (4.3) are constructed as follows. Internal estimates in $\{(t, x) : 0 \leq t \leq T; 0 \leq x \leq h_0\}$ are shown in the same way as in [11]. To obtain bounds near unknown curves, we carry out the transformation $\tau = t; y = \frac{x}{h(t)}$, to straighten the boundary out. Then D_t is mapped to $\Omega = \{(\tau, y) : 0 < \tau \leq T, 0 < y < 1\}$, while for the $V(y; \tau) = v(y\tau(\tau), \tau)$ we obtain an equation with bounded coefficients and a right-hand side.

Using the results of [11, 13], we establish the estimates $|V_y|, |V|_{1+\gamma}$ in $\bar{\Omega}$. For higher order derivatives, estimates were obtained using linear theory [13, Chapter IV, Thms 5.2 and 5.3]. Estimates for $w(t, x)$ are established similarly. $\square$

5. THE EXISTENCE AND UNIQUENESS OF THE SOLUTION

5.1. The uniqueness of a solution. First, we derive an integral expression for the free boundary. For this equations for the functions $v(t, x), w(t, x)$ we integrate over the domain D_t and obtain

$$\begin{aligned} \int_{\partial D_t} (d_2 v_x dy + v d\xi) &= - \iint_{D_t} f(u, v, w) dy d\xi, \\ \int_{\partial D_t} (d_3 w_x dy + w d\xi) &= - \iint_{D_t} g(v, w) dy d\xi \end{aligned}$$

where

$$\begin{aligned} f(u, v, w) &= \frac{\theta_1 \alpha_1 uv}{(m+u)(1+\omega_3 w)} - \delta_1 v - \delta_2(1+\omega_4 w)v^2 - \alpha_2 vw, \\ g(v, w) &= \frac{\eta w}{1+\eta_0 w} + \theta_2 \alpha_2 vw - \delta_3 w. \end{aligned}$$

Realizing the boundary conditions, we find that

$$\begin{aligned} h(t) &= h_0 + \frac{\mu}{d_0} \int_0^{h_0} (v_0(\xi) + \rho w_0(\xi)) d\xi - \frac{\mu}{d_0} \int_0^{s(t)} (v(\xi, \xi) + \rho w(t, \xi)) d\xi \\ &\quad + \iint_{D_t} (f + \rho g) dy d\xi. \end{aligned} \quad (5.1)$$

Theorem 5.1. *If the conditions of Lemma 4.1 and Theorem 4.2 are satisfied, then there is at most one solution to problem (3.1).*

Proof. Let the functions $u_i, v_i, w_i, h_i(t), i = 1, 2$ are a solutions to system (3.1) and let $s_1(t) = \min(h_1(t), h_2(t))$, $s_2(t) = \max\{h_1(t), h_2(t)\}$.

Considering the problem for the difference between these two classes of solutions and, using (5.1), we have that

$$\begin{aligned} |h_1(t) - h_2(t)| &\leq C_1 \left(\int_{s_1(t)}^{s_2(t)} |z_j(t, \xi)| d\xi + \int_{s_1(y)}^{s_2(y)} |F(u_j, v_j, w_j)| d\xi \right. \\ &\quad \left. + \int_0^t dy \int_0^{s_1(y)} |F(u_1, v_1, w_1) - F(u_2, v_2, w_2)| d\xi \right), \end{aligned}$$

where

$$\begin{aligned} z_j(t, \xi) &= v_j(t, \xi) + \rho w_j(t, \xi), \\ F(u, v, w) &= f(u, v, w) + \rho g(u, v, w), \\ (u_j, v_j, w_j) &= \begin{cases} u_1, v_1, w_1, & \text{if } h_1(t) > h_2(t) \\ u_2, v_2, w_2, & \text{if } h_1(t) < h_2(t) \end{cases}, j = 1, 2. \end{aligned}$$

Using the ideas and results of [22, Thm 2], we finish the proof of Theorem 5.1. $\square$

5.2. On the existence of a solution.

Theorem 5.2. *For any given (u_0, v_0, w_0) satisfying conditions A) – C) and for $0 < t \leq T < \infty$ problem (3.1) admits a unique global solution*

$$(u, v, w, h(t)) \in C^{1+\alpha, 2+\alpha}(Q_T) \times [C^{1+\alpha, 2+\alpha}(D_T)]^2 \times [C^{1+\alpha}([0, T])],$$

and there holds

$$|u|_{C^{1+\alpha, \alpha}(Q_T)} + |v|_{C^{1+\alpha, 2+\alpha}(D_T)} + |w|_{C^{1+\alpha, 2+\alpha}(D_T)} + |h|_{C^{1+\alpha}([0, T])} \leq C,$$

where $\alpha \in (0, 1)$, C -depends on $h_0, \alpha, |u_0, v_0, w_0|_{C^2}$

Proof. To prove the solvability of nonlinear problems, you can use various theorems from the theory of nonlinear equations, taking into account that the uniqueness theorems of the classical solution are valid for them. Usually, the Leray-Schoeder principle is used [11], which uses established a priori estimates $|\cdot|_{1+\alpha}$ for all possible solutions of nonlinear problems and the solvability theorem in Hölder classes for linear problems. Here the existence theorems for systems are the same as for the case of one equation since each of the equations can be considered as a linear equation with continuous Hölder coefficients. More complete information on the application of the technique can be found, for example, in the works (section VI, [13]; section VII, [8]). $\square$

6. ASYMPTOTIC BEHAVIOR OF THE FREE BOUNDARY

Now we will show that the free boundary $x = h(t)$ does not intersect the lateral boundary $x = l$, which means that predators do not completely occupy the territory under consideration. This is also necessary for the coexistence of all species and the correctness of the mathematical problem on any time interval. In light of Lemma 4.1, we get that $x = h(t)$ is monotonically increasing and, therefore, $\exists h_\infty \in (0, \infty]$ such that $\lim_{t \rightarrow \infty} h(t) = h_\infty$.

Using the established a priori estimates from problem (3.1), we can isolate the problem in the form (2.3) for $v(t, x)$ and, using Lemma 2.2 [24], obtain a lower bound for $v(t, x) \geq v^* > 0$.

Lemma 6.1. *Let (u, v, w, h) be the unique solution of (3.1) and*

$$d_1(0) = d_2(0) = d_0, \frac{1}{\theta_2} = \rho, \eta \leq \delta_3 + \delta_2 \theta_2 v^*, \alpha_1 < \frac{\delta_1}{\theta_1},$$

$$h_0 + \frac{1}{\theta_1} \int_0^{h_0} (v_0(x) + \rho w_0(x)) dx < l.$$

Then $0 < h(t) < l$ and $h_\infty < l$.

Proof. Let's consider

$$\begin{aligned}
I &= \frac{d}{dt} \int_0^{h(t)} \left(\frac{1}{\theta_1} v + \frac{1}{\theta_1 \theta_2} w \right) dx = \int_0^{h(t)} \left(\frac{1}{\theta_1} v_t + \frac{1}{\theta_1 \theta_2} w_t \right) dx \\
&= \int_0^{h(t)} \frac{1}{\theta_1} \left[(d_2(v) v_x)_x + \frac{\theta_1 \alpha_1 u v}{(m+u)(1+\omega_3 w)} - \delta_1 v - \delta_2 (1 + \omega_4 w) v^2 - \alpha_2 v w \right] dx \\
&\quad + \int_0^{h(t)} \frac{1}{\theta_1 \theta_2} \left[(d_3(w) w_x)_x + \frac{\eta w}{1 + \eta_0 w} + \theta_2 \alpha_2 v w - \delta_3 w \right] dx \\
&= \frac{1}{\theta_1} d_2(0) v_x(t, h(t)) + \frac{1}{\theta_1 \theta_2} d_3(0) w_x(t, h(t)) \\
&\quad + \int_0^{h(t)} \left[\frac{\alpha_1 u v}{(m+u)(1+\omega_3 w)} - \frac{\delta_1}{\theta_1} v - \frac{\delta_2}{\theta_1} (1 + \omega_4 w) v^2 - \frac{\alpha_2}{\theta_1} v w \right. \\
&\quad \left. + \frac{\eta w}{\theta_1 \theta_2 (1 + \eta_0 w)} + \frac{\alpha_2}{\theta_1} v w - \frac{\delta_3}{\theta_1 \theta_2} w \right] dx.
\end{aligned}$$

Considering that

$$\frac{u}{m+u} < 1, \quad \frac{w}{1+\omega_4 w} < 1,$$

we find that

$$\begin{aligned}
I &\leq -\frac{d_2(0)}{\mu \theta_1} h'(t) \\
&\quad + \int_0^{h(t)} \left[\alpha_1 v - \frac{\delta_1}{\theta_1} v - \frac{\delta_2}{\theta_1} (1 + \omega_4 w) v^2 - \frac{\alpha_2}{\theta_1} v w + \frac{\eta w}{\theta_1 \theta_2 (1 + \eta_0 w)} + \frac{\alpha_2}{\theta_1} v w - \frac{\delta_3}{\theta_1 \theta_2} w \right] \\
&\leq -Ah'(t) + \int_0^{h(t)} \left[w \left(\frac{\eta}{\theta_1 \theta_2} - \frac{\delta_3}{\theta_1 \theta_2} - \frac{\delta_2}{\theta_1} v^* \right) + v \left(\alpha_1 - \frac{\delta_1}{\theta_1} \right) \right] d\eta,
\end{aligned}$$

where we set

$$A = \frac{d_0}{\mu \theta_1}.$$

Thus

$$\frac{d}{dt} \int_0^{h(t)} \left(\frac{1}{\theta_1} v(t, x) + \frac{1}{\theta_1 \theta_2} w(t, x) \right) dx \leq -Ah'(t).$$

Integrating the last inequality from 0 to t gives

$$\int_0^{h(t)} \left(\frac{v(t, x)}{\theta_1} + \frac{w(t, x)}{\theta_1 \theta_2} \right) dx - \int_0^{h_0} \left(\frac{v_0(x)}{\theta_1} + \frac{w_0(x)}{\theta_1 \theta_2} \right) dx \leq -Ah(t) + Ah_0,$$

or

$$h(t) \leq h_0 + \frac{1}{\theta_1} \int_0^{h_0} (v_0(x) + \rho w_0(x)) dx.$$

Letting $t \rightarrow \infty$, we have $h_\infty < l$. □

7. CONCLUSIONS

A free-boundary model consisting of systems of parabolic equations for a three-species food web ecosystem is proposed, in which various parameters capture the effects of fear.

Qualitative studies were conducted to obtain results about the long-term dynamics of the model. The solution's properties have been analyzed and the global solvability of the problem has been established. A method has been developed to obtain Schauder-type priori estimates for a new class of problems with a free boundary. The properties of the free boundary in the region under consideration, as well as in the asymptotic regime of $t \rightarrow \infty$ are studied.

It has been proven that the unknown boundary $x = h(t)$ is always to the left of the lateral boundary $x = l$ for $\forall t > 0$, meaning that predators do not completely occupy the territory under consideration. On the other hand, this is necessary for the coexistence of all types of populations and the correctness of the mathematical problem on any time interval.

The results are described in terms of model parameters and enable us to understand how fear effects could impact the outcomes of interactions within a food web system.

The dynamics detected by the system may have some biological and environmental implications for biological control. Depending on the particular species in the food chain, these species may have different objectives regarding their extinction or persistence.

REFERENCES

1. Ailing, C. X., & Liangchen, W. (2022). Boundedness of a predator-prey model with density-dependent motilities and stage structure for the predator. *Electronic Research Archive (ERA)*, 30(5), 1954–1972. <https://doi.org/10.3934/era.2022099>.
2. Ait Dada Alla, S., Azraibi, O., & El Haji, B. (2024). On some nonlinear elliptic equations with measurable boundary conditions in anisotropic weighted Sobolev spaces. *Gulf Journal of Mathematics*, 17(1), 130–141. <https://doi.org/10.56947/gjom.v17i1.1712>
3. Clinchy, M., Sheriff, M. J., & Zanette, L. Y. (2013). Predator-induced stress and the ecology of fear. *Functional Ecology*, 27(1), 56–65. <https://doi.org/10.1111/1365-2435.12033>.
4. Cong, P., Fan, M., & Zou, X. (2021). Dynamics of a three-species food chain model with fear effect. *Communications in Nonlinear Science and Numerical Simulation*, 99, 105809. <https://doi.org/10.1016/j.cnsns.2021.105809>.
5. Du, Y., & Lin, Z. (2010). Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary. *SIAM Journal on Mathematical Analysis*, 42(1), 377–405. <https://doi.org/10.1137/090770297>.
6. Dubey, B., & Kumar, A. (2021). Stability switching and chaos in a multiple delayed prey-predator model with fear effect and anti-predator behavior. *Mathematics and Computation Simulation*, 188, 164–192. <https://doi.org/10.1007/s00285-020-01660-5>.

7. El-Aqqad, B. (2024). The equations coupled by Von Karman system with thermoelasticity. *Gulf Journal of Mathematics*, 17(2), 190–207. <https://doi.org/10.56947/gjom.v17i2.2171>
8. Friedman, A. (2008). *Partial differential equations of parabolic type*. Courier Dover Publications.
9. Guo, J. S., & Wu, C. H. (2012). On a free boundary problem for a two-species weak competition system. *Journal of Dynamical and Differential Equations*, 24, 873–895. <https://doi.org/10.1007/s10884-012-9267-0>.
10. Harfash, A., & Al-Musawi, G. A. (2024). Global existence of a pair of coupled Cahn-Hilliard equations with nondegenerate mobility and logarithmic potential. *Gulf Journal of Mathematics*, 16(1), 9–35. <https://doi.org/10.56947/gjom.v16i1.1368>
11. Kruzhkov, S. N. (1967). Nonlinear parabolic equations with two variables. *Proceedings of the Moscow Mathematical Society*, 16, 329–346.
12. Kumar, V., & Kumari, N. (2020). Controlling chaos in three species food chain model with fear effect. *AIMS Mathematics*, 5(2), 828–842. <https://doi.org/10.3934/math.2020.2.828>.
13. Ladyzhenskaya, O. A., Solonnikov, V. A., & Ural'ceva, N. N. (1968). *Linear and quasilinear equations of parabolic type*. American Mathematical Society.
14. Liu, Y., Guo, Z., El Smaili, M., & Wang, L. (2019). Biological invasion in a predator-prey model with a free boundary. *Boundary Value Problems*, 33(22). <https://doi.org/10.1186/s13661-019-1147-7>.
15. He, M. A., & Zhong, L. (2022). Stability of a fear effect predator-prey model with mutual interference or group defense. *Journal of Biological Dynamics*, 16(1), 480–498. <https://doi.org/10.1080/17513758.2022.2091800>.
16. Pal, S., Tiwari, P. K., Misra, A. K., & Wang, H. (2023). Fear effect in a three-species food chain model with generalist predator. *Mathematical Biosciences and Engineering*, 21(1), 1–33. <https://doi.org/10.3934/mbe.2024001>.
17. Panday, P., Pal, N., Samanta, S. & Chattopadhyay, C. (2019). A three species food chain model with fear-induced trophic cascade. *International Journal of Applied Computational Mathematics*, 5(4), 1–26. <https://doi.org/10.1007/s40819-019-0688-x>.
18. Sasmal, S. K., & Takeuchi, Y. (2022). Modeling the Allee effects induced by cost of predation fear and its carry-over effects. *Journal of Mathematical Analysis and Applications*, 505, 125485. <https://doi.org/10.1016/j.jmaa.2021.125485>.
19. Shao, C. (2023). The free boundary problem of a predator-prey model with fear effect and stage structure. *Mathematics and Computer Science*, 8(2), 62–67. <https://doi.org/10.11648/j.mcs.20230802.15>.
20. Sk, N., Tiwari, P. K., Pal, S., & Martcheva, M. (2021). A delay non-autonomous model for the combined effects of fear, prey refuge and additional food for predator. *Journal of Biological Dynamics*, 15(1), 580–622. <https://doi.org/10.1080/17513758.2021.2001583>.
21. Sourav, B., & Sasmal, K. (2018). Population dynamics with multiple Allee effects induced by fear factors - a mathematical study on prey-predator interactions. *Applied Mathematical Modelling*, 61, 102–112. <https://doi.org/10.1016/j.apm.2018.05.020>.
22. Takhirov, J. O. (2019). A free boundary problem for a reaction-diffusion equation appearing in biology. *Indian Journal of Pure and Applied Mathematics*, 50(1), 95–112. <https://doi.org/10.1007/s13226-019-0309-8>.
23. Tiwari, V., Tripathi, J., Mishra, S., & Upadhyay, R. (2020). Modeling the fear effect and stability of nonequilibrium patterns in mutually interfering predator-prey systems. *Applied Mathematics and Computation*, 371, 124948. <https://doi.org/10.1016/j.amc.2019.124948>.
24. Wang, M. (2014). On some free boundary problems of the prey-predator model. *Journal of Differential Equations*, 256, 3365–3394. <https://doi.org/10.1016/j.jde.2014.02.013>.

25. Wang, X., Zanette, L., & Zou, X. (2016). Modelling the fear effect in predator-prey interactions. *Journal of Mathematical Biology*, 73(5), 1179–1204. <https://doi.org/10.1007/s00285-016-0989-1>.
26. Yadav, D. K., Gupta, R. P., & Kumar, A. (2022). Nonlinear dynamics of a three species prey-predator system incorporating fear effect and harvesting. *Research Square*. <https://doi.org/10.21203/rs.3.rs-1938961/v1>.
27. Zhang, H., Cai, Y., Fu, S., & Wang, W. (2019). Impact of the fear effect in a prey-predator model incorporating a prey refuge. *Applied Mathematics and Computation*, 356, 328–337. <https://doi.org/10.1016/j.amc.2019.03.034>.
28. Zhao, M., Li, W. T., & Zhang, Y. (2019). Dynamics of an epidemic model with advection and free boundaries. *Mathematical Biosciences and Engineering*, 16(5), 5991–6014. <https://doi.org/10.3934/mbe.2019300>.

¹ INSTITUTE OF MATHEMATICS, UZBEKISTAN ACADEMY OF SCIENCES, 9, UNIVERSITY STR., TASHKENT 100174, UZBEKISTAN

Email address: prof.takhirov@yahoo.com

Email address: bunyodbek.anvarjonov@bk.ru