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THE EQUATIONS COUPLED BY VON KARMAN SYSTEM WITH THERMOELASTICITY

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Abstract. The objective of this paper is to study the model of the von Karman evolution coupled to the thermoelastic equation, with rotational inertia and clamped boundary conditions. We study the existence and uniqueness of a weak solution related to the dynamic model. Finally, we employ the finite difference method to approximate the solution to the problem.

1. Introduction

A dynamic von Karman equation incorporating rotational inertia ($\alpha > 0$),[5], describes the phenomenon of small nonlinear vibration with a vertical displacement to the elastic plates. The model of the wave-structure interaction with clamped boundary conditions, in the note account of rotational terms, can be formulated as follows [5]. Find $(u, \psi, \theta) \in L^2\big([0,T], H^2(\Omega) \times H_0^2(\Omega) \times H^1(\Omega)\big)$ such that

$$(\mathbb{P}_0)\begin{cases} P_\alpha u_{tt} + \Delta^2 u - [\psi + F_0, u] + \Delta\theta = p(x,t) & in \quad \Omega \times [0,T] \\ \theta_t - \Delta\theta - \Delta u_t = 0, & in \quad \Omega \times [0,T], \\ u_{|t=0} = u_0, \quad (u_t)_{|t=0} = u_1, \quad \theta_{|t=0} = \theta_0 & in \quad \Omega, \\ u = \partial_\nu u = \theta = 0 & on \quad \partial\Omega \times [0,T], \end{cases}$$

and

$$(\mathbb{Q})\begin{cases} \Delta^2\psi + [u,u] = 0 & in \quad \Omega \times [0,T], \\ \psi = 0, \ \partial_\nu \psi = 0 & on \quad \partial\Omega \times [0,T]. \end{cases}$$

Where, u is the displacement, θ is the thermal function and ψ denotes the Airy stress function, Ω is the surface plate, Δ is the Laplace operator, $P_\alpha = (1 - \alpha\Delta)$, ϕ_0, ϕ_1 and θ_0 are initials datums, $\partial_\nu u = \frac{\partial u}{\partial n}$ is the normal derivative inside the boundary $\partial\Omega$ and $[.,.]$ is the so-called Monge-Ampère operator defined by [5]

$$[\psi, u] = \partial_{11}\psi\partial_{22}u + \partial_{11}u\partial_{22}\psi - 2\partial_{12}\psi\partial_{12}u. \tag{1.1}$$

F_0 is a internal force and $p(x,t)$ is interacted with an enclosed acoustic field filling a bounded domain Σ in $\mathbb{R}^3$, the aerodynamic pressure on the plate is given by

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the formula (see [5]):

$$p(x,t) = p_0(x_1, x_2) + \mu(\phi_t)_{|x_3=0}, \ \ with \ \ x = (x_1, x_2, x_3), \tag{1.2}$$

where $p_0 \in L^2(\Omega)$ and $\mu > 0$ is proportional to the intensity of interaction between the chamber of wave and the plate. The parameter $\alpha > 0$ accounts for rotational inertia and the acoustic velocity potential $\phi(x,t)$ satisfies the following problem, [5]:

$$(\mathbb{S}) \begin{cases} \phi_{tt} = \Delta\phi & in \quad \Sigma \times [0,T], \\ \phi_{|t=0} = \phi_0, \quad (\phi_t)_{|t=0} = \phi_1 & in \quad \Sigma, \\ \partial_{x_3}\phi = u_t & on \quad \Omega \times [0,T], \\ \partial_\nu \phi = 0 & on \quad \partial\Sigma/\Omega \times [0,T], \end{cases}$$

such that ϕ_0 and ϕ_1 are initials data, $T > 0$ and $x = (x_1, x_2, x_3 = 0)$.

Most authors study this type of problem theoretically, Chueshov and Lasiecka using the theory of nonlinear semi-group and the Galerkin method in [2, 1, 3] to show the existence and the uniqueness of a weak solution of our problem, for the uniqueness weak solution the authors used the weak continuity of nonlinear terms involving Airy stress function. On the other hand, in this paper our approach is based on an iterative problem $(\mathbb{P}_n)_{n\geq 0}$ whose the sequence-solution $(u_n, \psi_n, \theta_n)_{n\geq 0}$ converges to the unique solution of the problem under consideration [11, 10]. For illustrating our theoretical results, we apply the finite difference method developed by Bilbao in [4].

The remainder of this paper is organized as follows. In section 2 we present to some notations and some results needed in the sequel. In section 3 we use the iterative method for established the existence and the uniqueness of a weak solution of the considered problem. Finally in section 4 describe a numerical simulation using the finite difference method.

2. Preliminaries and main results

2.1. **Notations and preliminary results.**

In this paper, Σ denotes a nonempty bounded domain in $\mathbb{R}^3$, with regular boundary $\partial\Sigma$ and Ω is a nonempty bounded open domain in $\mathbb{R}^2$, with $\Omega \subset \partial\Sigma$. Let $p \geq 1$ is a positive real number and $m \geq 1$ be an integer. We denote by $|.|_p$ the classical norm of $L^p(\Omega)$ and by $\|.\|_m$ that of $H^m(\Omega)$. For $u \in H_0^2(\Omega)$ we set $\|u\| := |\Delta u|_2^2 = \displaystyle\int_\Omega (\Delta u)^2$ and for simplicity, we note for

$$\|u\|_\alpha := \|u\| + \alpha\, |\nabla u_t|_2^2 + |u_t|_2^2 \,. \tag{2.1}$$

The following results are interesting.

Theorem 2.1. [6, 9] *Let $f \in L^2(\Omega)$. The following problem*

$$(\mathbb{D}) \begin{cases} \Delta^2 w = f & in \quad \Omega, \\ w = 0 & on \quad \partial\Omega, \\ \partial_\nu w = 0 & on \quad \partial\Omega, \end{cases}$$

has one and only one solution $w \in H_0^2(\Omega) \cap H^4(\Omega)$ thus satisfying

$$\|w\| \leq c_0\, |f|_1 \,,$$

where $c_0 > 0$ is a constant depending only on $mes(\Omega)$.

Remark 2.2. If $f \in L^2\big([0,T]\,, L^2(\Omega)\big)$, then the unique solution of the problem $(\mathbb{D})$ is in $L^2\big([0,T]\,, H^4(\Omega) \cap H_0^2(\Omega)\big)$.

Theorem 2.3. [9] *Let $g \in H^2(\Omega)$ and $w_0 \in L^2(\Omega)$. Then the following problem:*

$$\begin{cases} w_t - \Delta w = \Delta g & in \;\; \Omega \times [0,T]\,, \\ w_{|t=0} = w_0 & in \;\; \Omega, \\ w = 0 & on \;\; \partial\Omega \times [0,T]\,, \end{cases}$$

has one and only one solution $w \in C\big([0,T]\,; H^2(\Omega) \cap H_0^1(\Omega)\big) \cap C^1\big([0,T]\,; L^2(\Omega)\big)$, satisfies the following inequality:

$$\forall\, 0 \leq t \leq T, \quad |w|_2^2 + \int_0^t |\nabla w|_2^2 \leq |w_0|_2^2 + \int_0^t |\nabla g|_2^2\,. \tag{2.2}$$

Theorem 2.4. [5] *For $f \in L^2(\Omega)$, $\alpha > 0$ and $(w_0, w_1) \in H_0^2(\Omega) \times H_0^1(\Omega)$. Then the problem*

$$\begin{cases} P_\alpha w_{tt} + \Delta^2 w = f & in \;\; \Omega \times [0,T]\,, \\ w = \partial_\nu w = 0 & on \;\; \partial\Omega \times [0,T]\,, \\ w_{|t=0} = w_0, \;\; (w_t)_{|t=0} = w_1 & in \;\; \Omega, \end{cases}$$

has a unique solution $w \in C\big([\,0,T\,]\,, H_0^2(\Omega)\big)$, $w_t \in C\big([\,0,T\,]\,, H_0^1(\Omega)\big)$ and the energy equality

$$E_0(w, w_t) = E_0(w_0, w_1) + \int_0^t \int_\Omega f w_t$$

holds, here

$$E_0(w_0, w_1) = \frac{1}{2} \int_\Omega \big(\, \|w_0\|_2^2 + |w_1|_2^2 + \alpha\, |\nabla w_1|_2^2\,\big).$$

Theorem 2.5. [11] *Let $g \in L^2\big([0,T], L^2(\Omega)\big)$, $\phi_0 \in L^2\big([0,T], H^1(\Sigma)\big)$ and $\phi_1 \in L^2\big([0,T], L^2(\Sigma)\big)$. Then the following problem:*

$$(\mathbb{S}) \begin{cases} \phi_{tt} = \Delta\phi & in \;\; \Sigma \times [0,T]\,, \\ \phi_{|t=0} = \phi_0, \;\; (\phi_t)_{|t=0} = \phi_1 & in \;\; \Sigma, \\ \partial_{x_3}\phi = g & on \;\; \Omega \times [0,T]\,, \\ \partial_\nu \phi = 0 & on \;\; \partial\Sigma/\Omega \times [0,T]\,, \end{cases}$$

has one and only one solution $\phi \in L^2([0,T], H^1(\Sigma))$ and $\phi_t \in L^2([0,T], L^2(\Sigma))$. In addition we have the following inequality:

$$\forall\, 0 \leq t \leq T, \qquad |\phi_t|_2^2 + |\nabla\phi|_2^2 \leq e^T \Big(|\nabla\phi_0|_2^2 + |\phi_1|_2^2 + \int_0^T |g|_2^2\Big).$$

2.2. **Main results.** In this subsection, we will demonstrate Some results that we will use in section 3. We are considering the following problem:

$$(\mathbb{S}_1)\begin{cases} P_\alpha u_{tt} + \Delta^2 u + \Delta\theta - \mu(\phi_t)_{|x_3=0} = f & in \quad \Omega\times[0,T], \\ \theta_t - \Delta\theta = \Delta u_t & in \quad \Omega\times[0,T], \\ \phi_{tt} = \Delta\phi & in \quad \Sigma\times[0,T], \\ u = \partial_\nu u = \theta = 0 & on \quad \partial\Omega\times[0,T], \\ (u)_{|t=0} = u_0, (u_t)_{|t=0} = u_1, (\theta)_{|t=0} = \theta_0 & in \quad \Omega, \\ \phi_{|t=0} = \phi_0, (\phi_t)_{|t=0} = \phi_1 & in \quad \Sigma, \\ \partial_{x_3}\phi = u_t & on \quad \Omega\times[0,T], \\ \partial_\nu\phi = 0 & on \quad \partial\Sigma/\Omega\times[0,T], \end{cases}$$

Theorem 2.6. *Let $f \in L^2([0,T], L^2(\Omega))$, $\theta_0 \in H_0^1(\Omega)$, $(u_0,u_1) \in H_0^2(\Omega)\times H_0^1(\Omega)$ and $(\phi_0,\phi_1) \in H^1(\Sigma)\times L^2(\Sigma)$. Then the problem $(\mathbb{S}_1)$ has one and only one weak solution (u,θ,ϕ) such that $(u,\theta\phi) \in C^0\big([\,0,T\,], H_0^2(\Omega)\times H_0^1(\Omega)\times H^1(\Sigma)\big)$ and $(u_t,\phi_t) \in H_0^1(\Omega)\times L^2(\Sigma)$. Moreover we have this inequality*

$$\|u\|_\alpha + |\theta|_2^2 + 2\int_0^t |\nabla\theta|_2^2 + \mu|\phi_t|_2^2 + \mu|\nabla\phi|_2^2 \le e^T\Big(\|u_0\|^2 + \alpha\,|\nabla u_1|_2^2 + |u_1|_2^2 + |\theta_0|_2^2 + \mu|\phi_1|_2^2 + \mu|\nabla\phi_0|_2^2 + \int_0^T |f|_2^2\Big). \tag{2.3}$$

Proof. For Demonstrating this result, We will employ a comparable approach as in [2, 8]. We consider the n-order approximate solution of problem $(\mathbb{S}_1)$ and its associate variational problem. We divide the proof into fourth steps.
Step 1: Let $\{\, e_k^1, e_k^2, e_k^3 \,\}$ be a basis in the space $H_0^2(\Omega)\times H_0^1(\Omega)\times H^1(\Sigma)$. We describe an n-order Galerkin approximate solution to the problem $(\mathbb{S}_1)$ with clamped boundary conditions on the interval $[\,0,T\,]$, as a function $(u^n(t),\theta^n(t),\phi^n(t))$ such that:

$$u^n = \sum_{k=1}^n h_k^1(t)e_k^1\,,\quad \theta^n = \sum_{k=1}^n h_k^2(t)e_k^2 \;\; and \;\; \phi^n = \sum_{k=1}^n h_k^3(t)e_k^3 \quad n = 1,2,3,...,$$

where $(h_k^1(t),h_k^2(t),h_k^3(t)) \in W^{2,+\infty}(0,T,I\!R)\times\big(W^{1,+\infty}(0,T,I\!R)\big)^2$. Let (u^n,θ^n,ϕ^n) be a solution of the iterative problem $(\mathbb{S}_1^n)$ associated with the problem $(\mathbb{S}_1)$ given by:

$$(\mathbb{S}_1^n)\begin{cases} P_\alpha u_{tt}^n + \Delta^2 u^n + \Delta\theta^n - \mu(\phi_t^n)_{|x_3=0} = f & in \quad \Omega\times[0,T], \\ \theta_t^n - \Delta\theta^n = \Delta u_t^n & in \quad \Omega\times[0,T], \\ \phi_{tt}^n = \Delta\phi^n & in \quad \Sigma\times[0,T], \\ u^n = \partial_\nu u^n = \theta^n = 0 & on \quad \partial\Omega\times[0,T], \\ (u^n)_{|t=0} = u_{n0}, (u_t^n)_{|t=0} = u_{n1}, (\theta^n)_{|t=0} = \theta_{n0} & in \quad \Omega, \\ \phi_{|t=0}^n = \phi_{n0}, (\phi_t^n)_{|t=0} = \phi_{n1} & in \quad \Sigma, \\ \partial_{x_3}\phi^n = u_t^n & on \quad \Omega\times[0,T], \\ \partial_\nu\phi^n = 0 & on \quad \partial\Sigma/\Omega\times[0,T], \end{cases}$$

where the initial data $(u_{n0},\theta_{n0},\phi_{n0})$ and (u_{n1},ϕ_{n1}) are chosen such that $(u_{n0},\theta_{n0},\phi_{n0})$ converges to (u_0,θ_0,ϕ_0) in $L^2([0,T\,], H_0^2(\Omega)\times H_0^1(\Omega)\times H^1(\Sigma))$ and (u_{n1},ϕ_{n1}) converges to (u_1,ϕ_1) in $L^2([0,T\,], H_0^1(\Omega)\times L^2(\Sigma))$.

Now, we multiply the first equation of $(\mathbb{S}_1^n)$ by u_t^n, the second equation by θ^n and the third equation by ϕ_t^n and we then integrate both them over Ω, by applying standard integral techniques, we get

$$\int_\Omega u_{tt}^n u_t^n + \alpha \int_\Omega \nabla u_{tt}^n \nabla u_t^n + \int_\Omega \Delta u^n \Delta u_t^n + \int_\Omega \Delta\theta^n u_t^n - \mu \int_\Omega \phi_t^n u_t^n = \int_\Omega f u_t^n,$$

$$\int_\Omega \theta_t^n \theta^n - \int_\Omega (\nabla\theta^n)^2 = \int_\Omega \Delta u_t^n \theta^n \quad \text{and} \quad \int_\Sigma \phi_{tt}^n \phi_t^n + \int_\Sigma \nabla\phi^n \nabla\phi_t^n = -\int_\Omega u_t^n \phi_t^n.$$

Since $(u_t^n, \theta^n) \in H_0^2(\Omega) \times H_0^1(\Omega)$ and $\int_\Omega \Delta\theta^n u_t^n = \int_\Omega \theta^n \Delta u_t^n$, we have

$$\frac{1d}{2dt}(|u_t^n|_2^2 + \|u^n\|^2 + \alpha\, |\nabla u_t^n|_2^2) + \int_\Omega \theta^n \Delta u_t^n - \mu \int_\Omega \phi_t^n u_t^n = \int_\Omega f u_t^n,$$

$$\frac{d}{2dt}\, |\theta^n|_2^2 + |\nabla\theta^n|_2^2 = \int_\Omega \Delta\theta^n u_t^n \quad \text{and} \quad \mu \frac{d}{2dt}\big(\, |\phi_t^n|_2^2 + |\nabla\phi^n|_2^2\,\big) = -\mu \int_\Omega u_t^n \phi_t^n.$$

Hence

$$\frac{d}{2dt}\big(\, |u_t^n|_2^2 + \|u^n\|^2 + \alpha\, |\nabla u_t^n|_2^2\,\big) + \frac{d}{2dt}\, |\theta^n|_2^2 + |\nabla\theta^n|_2^2 + \mu \frac{d}{2dt}\big(\, |\phi_t^n|_2^2 + |\nabla\phi^n|_2^2\,\big) = \int_\Omega f u_t^n.$$

Integrating this latter equality over $[0,t]$, using the fact that $u^n_{|t=0} = u_{n0}$, $(u_t^n)_{|t=0} = u_{n1}$, $\theta^n_{|t=0} = \theta_{n0}$, $\phi^n_{|t=0} = \phi_{n0}$ and $(\phi_t^n)_{|t=0} = \phi_{n1}$, we deduce that

$$\begin{aligned}\|u^n\|_\alpha + |\theta^n|_2^2 + \mu\big(\, |\phi_t^n|_2^2 + |\nabla\phi^n|_2^2\,\big) + 2\int_0^t |\nabla\theta^n|_2^2 &= |u_{n1}|_2^2 + \alpha\, |\nabla u_{n1}|_2^2 + \|u_{n0}\|^2 + |\theta_{n0}|_2^2 \\ &\quad + \mu\, |\phi_{n1}|_2^2 + \mu\, |\nabla\phi_{n0}|_2^2 + 2\int_0^t \int_\Omega f u_t^n.\end{aligned}$$

For $s \in [0,T]$, we have

$$\begin{aligned}&\|u^n\|_\alpha + |\theta^n|_2^2 + \mu\big(\, |\phi_t^n|_2^2 + |\nabla\phi^n|_2^2\,\big) + 2\int_0^t |\nabla\theta^n|_2^2 \le |u_{n1}|_2^2 + \alpha\, |\nabla u_{n1}|_2^2 + \|u_{n0}\|^2 + |\theta_{n0}|_2^2 \\ &+ \mu\, |\phi_{n1}|_2^2 + \mu\, |\nabla\phi_{n0}|_2^2 + \int_0^T |f|_2^2 + \int_0^t \Big(\, \|u^n\|_\alpha + |\theta^n|_2^2 + \mu\big(\, |\phi_t^n|_2^2 + |\nabla\phi^n|_2^2\,\big) + 2\int_0^s |\nabla\theta^n|_2^2\Big).\end{aligned} \tag{2.4}$$

Step 2: For any $0 \le s \le t$, we put

$$J(s) = \|u^n\|_\alpha + |\theta^n|_2^2 + \mu\big(\, |\phi_t^n|_2^2 + |\nabla\phi^n|_2^2\,\big) + 2\int_0^s |\nabla\theta^n|_2^2\,.$$

The inequality (2.4) yields

$$\begin{aligned}e^{-s}\Big(J(s) - \int_0^s J(\sigma)d\sigma\Big) \le e^{-s}\Big(\, &|u_{n1}|_2^2 + \alpha\, |\nabla u_{n1}|_2^2 + \|u_{n0}\|^2 + |\theta_{n0}|_2^2 + \mu\, |\phi_{n1}|_2^2 \\ &+ \mu\, |\nabla\phi_{n0}|_2^2 + \int_0^T |f|_2^2\Big).\end{aligned}$$

It follows that

$$\begin{aligned}\frac{d}{ds}\Big(e^{-s}\int_0^s J(\sigma)d\sigma\Big) &= e^{-s}J(s)-e^{-s}\int_0^s J(\sigma)d\sigma = e^{-s}\Big(J(s)-\int_0^s J(\sigma)d\sigma\Big),\\ &\le e^{-s}\Big(\, |u_{n1}|_2^2+\alpha\,|\nabla u_{n1}|_2^2+\|u_{n0}\|^2+|\theta_{n0}|_2^2+\mu\,|\phi_{n1}|_2^2\\ &\qquad +\mu\,|\nabla\phi_{n0}|_2^2+\int_0^T |f|_2^2\Big),\end{aligned}$$

and

$$|u_{n1}|_2^2+\alpha\,|\nabla u_{n1}|_2^2+\|u_{n0}\|^2+|\theta_{n0}|_2^2+\mu\,|\phi_{n1}|_2^2+\mu\,|\nabla\phi_{n0}|_2^2+\int_0^T |f|_2^2 = J(0)+\int_0^T |f|_2^2$$

does not depend on s, then

$$\begin{aligned}\int_0^t \frac{d}{ds}\Big(e^{-s}\int_0^s J(\sigma)d\sigma\Big)ds &\le \Big(\int_0^t e^{-s}ds\Big)\Big(\, |u_{n1}|_2^2+\alpha\,|\nabla u_{n1}|_2^2+\|u_{n0}\|^2+|\theta_{n0}|_2^2\\ &\qquad +\mu\,|\phi_{n1}|_2^2+\mu\,|\nabla\phi_{n0}|_2^2+\int_0^T |f|_2^2\Big),\end{aligned}$$

from which we deduce

$$\begin{aligned}e^{-t}\int_0^t J(\sigma)d\sigma &\le (1-e^{-t})\Big(\, |u_{n1}|_2^2+\alpha\,|\nabla u_{n1}|_2^2+\|u_{n0}\|^2+|\theta_{n0}|_2^2+\mu\,|\phi_{n1}|_2^2\\ &\qquad +\mu\,|\nabla\phi_{n0}|_2^2+\int_0^T |f|_2^2\Big).\end{aligned}$$

Since

$$\int_0^t \Big(\, \|u^n\|_\alpha+|\theta^n|_2^2+\mu\big(\,|\phi_t^n|_2^2+|\nabla\phi^n|_2^2\,\big)\Big)=\int_0^t J(\sigma)d\sigma,$$

then we have

$$\begin{aligned}\int_0^t J(\sigma)d\sigma &\le \tfrac{(1-e^{-t})}{e^{-t}}\Big(\, |u_{n1}|_2^2+\alpha\,|\nabla u_{n1}|_2^2+\|u_{n0}\|^2+|\theta_{n0}|_2^2+\mu\,|\phi_{n1}|_2^2+\mu\,|\nabla\phi_{n0}|_2^2\\ &\qquad +\int_0^T |f|_2^2\Big),\\ &\le (e^t-1)\Big(\, |u_{n1}|_2^2+\alpha\,|\nabla u_{n1}|_2^2+\|u_{n0}\|^2+|\theta_{n0}|_2^2+\mu\,|\phi_{n1}|_2^2+\mu\,|\nabla\phi_{n0}|_2^2\\ &\qquad +\int_0^T |f|_2^2\Big),\\ &\le (e^T-1)\Big(\, |u_{n1}|_2^2+\alpha\,|\nabla u_{n1}|_2^2+\|u_{n0}\|^2+|\theta_{n0}|_2^2+\mu\,|\phi_{n1}|_2^2+\mu\,|\nabla\phi_{n0}|_2^2\\ &\qquad +\int_0^T |f|_2^2\Big).\end{aligned}$$

This, with (2.4), yields

$$\|u^n\|_\alpha + |\theta^n|_2^2 + \mu\big(|\phi_t^n|_2^2 + |\nabla\phi^n|_2^2\big) + 2\int_0^t |\nabla\theta^n|_2^2 \le \Big(|u_{n1}|_2^2 + \alpha\,|\nabla u_{n1}|_2^2 + \|u_{n0}\|^2$$
$$+ |\theta_{n0}|_2^2 + \mu\,|\phi_{n1}|_2^2 + \mu\,|\nabla\phi_{n0}|_2^2 + \int_0^T |f|_2^2\Big) + (e^T - 1)\Big(|u_{n1}|_2^2 + \alpha\,|\nabla u_{n1}|_2^2 + \|u_{n0}\|^2$$
$$+ |\theta_{n0}|_2^2 + \mu\,|\phi_{n1}|_2^2 + \mu\,|\nabla\phi_{n0}|_2^2 + \int_0^T |f|_2^2\Big) \le e^T\Big(|u_{n1}|_2^2 + \alpha\,|\nabla u_{n1}|_2^2 + \|u_{n0}\|^2 + |\theta_{n0}|_2^2$$
$$+\mu\,|\phi_{n1}|_2^2 + \mu\,|\nabla\phi_{n0}|_2^2 + \int_0^T |f|_2^2\Big).$$

This estimate implies that there exists a subsequence $(u^{n_l}, \theta^{n_l}, \phi^{n_l})$ such that $(u^{n_l}, \theta^{n_l}, \phi^{n_l}) \rightharpoonup (u, \theta, \phi)$ weakly in $H_0^2(\Omega)\times L^2(\Omega)\times H^1(\Sigma)$ and $((u^{n_l})_t, \nabla\theta^{n_l}, (\phi^{n_l})_t) \rightharpoonup (u_t, \nabla\theta, \phi_t)$ weakly in $H_0^1(\Omega)\times L^2(\Omega)\times L^2(\Sigma)$.

Step 3: In this step, we will establish that (u, θ, ϕ) is a weak solution of the problem $(\mathbb{S}_1)$, by following the same way as in [8]. Let $\varphi_j \in C^1(0, T)$, $1 \le j \le j_0$, such that $\varphi_j(T) = 0$ and

$$\psi = \sum_{j=1}^{j_0} \varphi_j \otimes e_j^1, \quad \varphi = \sum_{j=1}^{j_0} \varphi_j \otimes e_j^2, \quad \chi = \sum_{j=1}^{j_0} \varphi_j \otimes e_j^3.$$

After the variational problem, we have

$$-\int_0^T\int_\Omega u_t^{nl}\psi_t + \alpha\int_0^T\int_\Omega \nabla u_t^{nl}\nabla\psi_t + \int_0^T\int_\Omega \nabla\theta^{nl}\nabla\psi + \int_0^T\int_\Omega \Delta u^{nl}\Delta\psi$$
$$-\mu\int_0^T\int_\Omega \phi_t^{nl}\psi = \int_0^T\int_\Omega f\psi - \int_\Omega u_{nl1}\psi(0) - \alpha\int_\Omega \nabla u_{nl1}\nabla\psi(0), \tag{2.5}$$

$$-\int_0^T\Big(\int_\Omega \theta^{nl}\varphi_t + \int_\Omega \nabla\theta^{nl}\nabla\varphi - \int_\Omega \nabla u^{nl}\nabla\varphi_t\Big) = -\int_\Omega \theta_{nl0}\varphi(0) + \int_\Omega \nabla u_{nl1}\nabla\varphi(0) \tag{2.6}$$

and

$$-\int_0^T\int_\Sigma \phi^{nl}\chi_t + \int_\Sigma \nabla\phi^{nl}\nabla\chi = -\int_\Omega u_t^{nl}\chi + \int_\Sigma \phi_{nl1}\chi(0). \tag{2.7}$$

Letting $nl \to +\infty$, in (2.5), (2.6) and (2.7), we deduce that

$$-\int_0^T\int_\Omega u_t\psi_t + \alpha\int_0^T\int_\Omega \nabla u_t\nabla\psi_t + \int_0^T\int_\Omega \nabla\theta\nabla\psi + \int_0^T\int_\Omega \Delta u\Delta\psi - \mu\int_0^T\int_\Omega \phi_t\psi$$
$$= \int_0^T\int_\Omega f\psi - \int_\Omega u_1\psi(0) - \alpha\int_\Omega \nabla u_1\nabla\psi(0),$$

$$-\int_0^T\Big(\int_\Omega \theta\varphi_t + \int_\Omega \nabla\theta\nabla\varphi - \int_\Omega \nabla u\nabla\varphi_t\Big) = -\int_\Omega \theta_0\varphi(0) + \int_\Omega \nabla u_1\nabla\varphi(0)$$

and

$$-\int_0^T\int_\Sigma \phi\chi_t + \int_\Sigma \nabla\phi\nabla\chi = -\int_\Omega u_t\chi + \int_\Sigma \phi_1\chi(0).$$

hold true for all $\psi \in L^2([0,T], H_0^2(\Omega))$, $\psi_t \in L^2([0,T], H^1(\Omega))$, $\varphi \in L^2([0,T], H_0^1(\Omega))$, $\varphi_t \in L^2([0,T], L^2(\Omega))$, $\chi \in L^2([0,T], H^1(\Sigma))$ and $\chi_t \in L^2([0,T], L^2(\Sigma))$ such that

$\psi(T) = \varphi(T) = \chi(0) = 0$. It means that (u, θ, ϕ) is a weak solution of the problem $(\mathbb{S}_1)$, by the some method as in the last proof, we deduce the following inequality:

$$\|u\|_\alpha + |\theta|_2^2 + 2\int_0^t |\nabla\theta|_2^2 + \mu|\phi_t|_2^2 + \mu|\nabla\phi|_2^2 \leq e^T\Big(\|u_0\|^2 + \alpha\,|\nabla u_1|_2^2 + |u_1|_2^2 + |\theta_0|_2^2$$
$$+\mu|\phi_1|_2^2 + \mu|\nabla\phi_0|_2^2 + \int_0^T |f|_2^2\Big).$$

Step 4: Now we show the uniqueness, let (u_1, θ_1, ϕ_1) and (u_2, θ_2, ϕ_2) be two solutions of the problem $(\mathbb{S}_1)$. Then $(u_1 - u_2, \theta_1 - \theta_2, \phi_1 - \phi_2)$ is a solution of the following problem:

$$\begin{cases} P_\alpha(u_1 - u_2)_{tt} + \Delta^2(u_1 - u_2) + \Delta(\theta_1 - \theta_2) & \\ \qquad\qquad -\mu((\phi_1 - \phi_2)_t)_{|x_3} = 0 & in\ \ \Omega \times [0, T]\,, \\ (\theta_1 - \theta_2)_t - \Delta(\theta_1 - \theta_2) = \Delta(u_1 - u_2)_t & in\ \ \Omega \times [0, T]\,, \\ (\phi_1 - \phi_2)_{tt} = \Delta(\phi_1 - \phi_2) & in\ \ \Sigma \times [0, T]\,, \\ u_1 - u_2 = \partial_\nu(u_1 - u_2) = \theta_1 - \theta_2 = 0 & on\ \ \partial\Omega \times [0, T]\,, \\ (u_1 - u_2)_{|t=0} = 0, ((u_1 - u_2)_t)_{|t=0} = 0, (\theta_1 - \theta_2)_{|t=0} = 0 & in\ \ \Omega, \\ (\phi_1 - \phi_2)_{|t=0} = 0, ((\phi_1 - \phi_2)_t)_{|t=0} = 0 & in\ \ \Sigma, \\ \partial_{x_3}(\phi_1 - \phi_2) = (u_1 - u_2)_t & on\ \ \Omega \times [0, T]\,, \\ \partial_\nu(\phi_1 - \phi_2) = 0 & on\ \ \partial\Sigma/\Omega \times [0, T]\,. \end{cases}$$

According to (2.3), we have

$$\|u_1 - u_2\|_\alpha + |\theta_1 - \theta_2|_2^2 + 2\int_0^t |\nabla(\theta_1 - \theta_2)|_2^2 + \mu|(\phi_1 - \phi_2)_t|_2^2 + \mu|\nabla(\phi_1 - \phi_2)|_2^2$$
$$\leq e^T\Big(\|(u_1 - u_2)_0\|^2 + \alpha\,|\nabla(u_1 - u_2)_1|_2^2 + |(u_1 - u_2)_1|_2^2 + |(\theta_1 - \theta_2)_0|_2^2$$
$$+\mu|(\phi_1 - \phi_2)_1|_2^2 + \mu|\nabla(\phi_1 - \phi_2)_0|_2^2\Big).$$

Then we deduce $u_1 = u_2$, $\theta_1 = \theta_2$ and $\phi_1 = \phi_2$. Hence the demonstration is complete. □

For the sake of simplicity, let us put

$$G(u, \psi) = [\psi + F_0, u] + p.$$

we also need this result after.

Proposition 2.7. *Let $(w, z) \in (H_0^2(\Omega))^2$ and $F_0 \in H^4(\Omega)$ be with small norms. Let $\psi, \varphi \in H_0^2(\Omega)$ be the solutions of the following two problems:*

$$\Delta^2\psi = -[w, w] \quad \text{and} \quad \Delta^2\varphi = -[z, z]\,.$$

Then the following estimations:

$$\Big|\,[w, \psi] - [z, \varphi]\,\Big|_2 \leq c_1\|w - z\|$$

and

$$\|G(w, \psi) - G(z, \varphi)\|_{L^2(\Omega)} \leq c_1\,\|w - z\|$$

hold for some $0 < c_2 < 1$.

Proof. Using [5], we have

$$\Big|\,[w,\psi]-[z,\varphi]\,\Big|_2 \le c_0\big(\,\|w\|^2+\|z\|^2\,\big)\,\|w-z\|\,,$$

for some $c_0>0$. Let $c>0$ be small enough such that $\|w\|\le c$ and $\|z\|\le c$. We have

$$\Big|\,[w,\psi]-[z,\varphi]\,\Big|_2 \le 2c_0c^2\,\|w-z\|$$

and so

$$\begin{aligned}\|G(w,\psi)-G(z,\varphi)\|_{L^2(\omega)} &\le \Big|\,[\psi+F_0,w]-[\varphi+F_0,z]\,\Big|_2,\\ &\le \Big|\,[\psi,w]-[\varphi,z]\,\Big|_2+\Big|\,[F_0,w-z]\,\Big|_2,\\ &\le \big(2c_0c^2+4\,\|F_0\|_{4,\Omega}\big)\,\|w-z\|\,.\end{aligned}$$

If we choose

$$\|F_0\|_{4,\Omega}<\frac{1}{4}\quad \textit{and}\quad 0<c<\sqrt{\frac{1-4\,\|F_0\|_{4,\Omega}}{2c_0}},$$

we have

$$0<c_1=2c_0c^2+4\,\|F_0\|_{4,\Omega}<1,$$

we then conclude the proof. □

3. Iterative Approach

To establish the existence and the uniqueness of a weak solution for $(\mathbb{P}_0)$ and $(\mathbb{Q})$, with rotational terms $\alpha>0$, we will use the following iterative approach. Let $n\ge 2$ and let $0\ne u_1\in H_0^2(\Omega)$ be given. First we find $\psi_{n-1}\in H_0^2(\Omega)$ as the unique solution of the equation $\Delta^2\psi_{n-1}=-\left[u_{n-1},u_{n-1}\right]$ and (u_n,θ_n,ϕ_n) as the solution of the following problem:

$$(\mathbb{P}_n)\begin{cases} P_\alpha(u_n)_{tt}+\Delta^2u_n-\mu\big((\phi_n)_t\big)_{/x_3=0}=F(u_{n-1},\phi_{n-1},\theta_n) & in\quad \Omega\times[0,T]\,,\\ (\theta_n)_t-\Delta\theta_n=\Delta(u_n)_t & in\quad \Omega\times[0,T]\,,\\ (\phi_n)_{tt}=\Delta\phi_n & in\quad \Sigma\times[0,T]\,,\\ u_n=\partial_\nu u_n=\theta_n=0 & on\quad \partial\Omega\times[0,T]\,,\\ (u_n)_{|t=0}=u_0,((u_n)_t)_{|t=0}=u_1,(\theta_n)_{|t=0}=\theta_0 & in\quad \Omega,\\ (\phi_n)_{|t=0}=\phi_0,\quad ((\phi_n)_t)_{|t=0}=\phi_1 & in\quad \Sigma,\\ \partial_{x_3}\phi_n=(u_n)_t & on\quad \Omega\times[0,T]\,,\\ \partial_\nu\phi_n=0 & on\quad \partial\Sigma/\Omega\times[0,T]\,.\end{cases}$$

With $F(u,\psi,\theta)=G(u,\psi)-\Delta\theta$ and G is defined above. We are now in a position to present the main result of this section.

Theorem 3.1. *Let $p\in L^2(\Omega)$, $(u_0,\,u_1)\in H_0^2(\Omega)\times H_0^1(\Omega)$, $\theta_0\in H_0^1(\Omega)$ and $(\phi_0,\phi_1)\in H^1(\Sigma)\times L^2(\Sigma)$. We assume that all the following norms*

$$\|F_0\|_{4,\Omega}\,,\ |p|_2\,,\ \|u_0\|^2+|u_1|_2^2\,,\ |\nabla u_1|_2^2\quad \|\theta_0\|_{1,\Omega}^2\quad |\nabla\phi_0|_2^2\quad \textit{and}\quad |\phi_1|_2^2$$

are small enough. Then the problem $(\mathbb{P}_0)$ has one and only one weak solution (u,ψ,θ,ϕ) in $L^2\big([0,T]\,,H_0^2(\Omega)\times H_0^2(\Omega)\times H_0^1(\Omega)\times H^1(\Sigma)\big)$ such that $(u_t,\phi_t)\in L^2\big([0,T]\,,H_0^1(\Omega)\times L^2(\Sigma)\big)$ and $u_{tt}\in L^2\big([0,T]\,,L^2(\omega)\big)$.

Proof. We divide it into four steps.
Step 1: Using the problem $(\mathbb{P}_n)$, where $0 \neq u_1$. We will use the following notation throughout this proof

$$\|(u,\theta)\|_* := \|u\|_\alpha + |\theta|_2^2 + 2\int_0^t |\nabla\theta|_2^2, \text{ and } \|\phi\|_{**} := |\phi_t|_2^2 + |\nabla\phi|_2^2.$$

Where $\|.\|_\alpha$ is defined by (2.1). By using an induction method, We will show that

$$\forall n \in \mathbb{N}^*, \quad \|u_n\|_\alpha = \|u_n\|^2 + \alpha\,|\nabla(u_n)_t|_2^2 + |(u_n)_t|_2^2 \leq \|u_1\|^2 \text{ and } \|\psi_n\| \leq \|u_1\|$$

for all $t \in [0,T]$. For $n = 1$, we have

$$\|u_1\|_\alpha = \|u_1\|^2 + \alpha\,|\nabla(u_1)_t|_2^2 + |(u_1)_t|_2^2 = \|u_1\|^2$$

since u_1 independent of t. Furthermore, let ψ_1 be the solution of $\Delta^2\psi_1 = -[\,u_1,u_1\,]$, Theorem 2.1 ensures that there exists $c_0 > 0$ such that

$$\|\psi_1\| \leq c_0\,|[\,u_1,u_1\,]|_1\,,$$

using the proof of Proposition 2.7 with $\|u_1\| < c$ and $0 < 4c_0c < 1$, we can deduce that

$$\|\psi_1\| \leq 4c_0\,\|u_1\|^2 \leq 4c_0c\,\|u_1\| \leq \|u_1\|\,.$$

This implies that inequalities are true for $n = 1$. Assume that for $k = 2,...,n$ and $0 \leq t \leq T$, we have

$$\|u_k\|_\alpha \leq \|u_1\|^2 \text{ and } \|\psi_k\| \leq \|u_1\|\,.$$

According to Proposition 2.7 and Theorem 2.1, we have

$$\|\psi_n\| \leq c_0\,|[\,u_n,u_n\,]|_1 \leq 4c_0\,\|u_n\|^2 \leq 4c_0c\,\|u_n\| \leq c_1\,\|u_n\|\,.$$

Since $(u_{n+1},\theta_{n+1},\phi_{n+1})$ is a solution of $(\mathbb{P}_{n+1})$, Theorem 2.6, Proposition 2.7 and Theorem 2.1 imply that there exists $0 < c_1 = 2c_0c^2 + 4\,\|F_0\|_{4,\omega} < 1$ such that

$$\begin{aligned}
\|(u_{n+1},\theta_{n+1})\|_* + \mu\|\phi_{n+1}\|_{**} &\leq e^T\Big(\,\|u_0\|^2 + \alpha\,|\nabla u_1|_2^2 + |u_1|_2^2 + |\theta_0|_2^2 + \mu|\phi_1|_2^2 \\
&\qquad + \mu|\nabla\phi_0|_2^2 + \int_0^T \|G(u_n,\psi_n)\|_{L^2(\Omega)}^2\Big), \\
&\leq e^T\Big(\,\|u_0\|^2 + \alpha\,|\nabla u_1|_2^2 + |u_1|_2^2 + |\theta_0|_2^2 + \mu|\phi_1|_2^2 + \mu|\nabla\phi_0|_2^2 + 2\int_0^T (c_1^2\|u_n\|^2 + |p|_2^2)\Big) \\
&\leq e^T\Big(\,\|u_0\|^2 + \alpha\,|\nabla u_1|_2^2 + |u_1|_2^2 + |\theta_0|_2^2 + \mu|\phi_1|_2^2 + \mu|\nabla\phi_0|_2^2 + 2T|p|_2^2\Big) + 2Te^Tc_1^2\|u_1\|^2.
\end{aligned}$$

Because $\|u_n\|^2 \leq \|u_1\|^2$. If we choose $c > 0$ sufficiently small, then $0 < c_1 < 1,\ 0 < c_2 := 2Te^Tc_1^2 < 1$ and we can choose

$$\|u_0\|^2 + \alpha\,|\nabla u_1|_2^2 + |u_1|_2^2 + |\theta_0|_2^2 + \mu|\phi_1|_2^2 + \mu|\nabla\phi_0|_2^2 + 2T|p|_2^2 \leq \frac{(1-c_2)}{e^T}\,\|u_1\|^2\,.$$

We have

$$\|(u_{n+1},\theta_{n+1})\|_* + \mu\|\phi_{n+1}\|_{**} \leq \|u_1\|^2.$$

It follows that

$$\|u_{n+1}\|_\alpha \leq \|u_1\|^2\,.$$

Further, we have

$$\|\psi_{n+1}\| \leq c_0\big|\,[\,u_{n+1},u_{n+1}\,]\,\big|_1$$

with $\|u_1\| < c$ and $0 < 4c_0c < 1$, hence

$$\|\psi_{n+1}\| \leq 4c_0 \|u_{n+1}\|^2 \leq 4c_0 \|u_1\|^2 \leq 4c_0c \|u_1\| \leq \|u_1\|.$$

In summary, we have proved that for all $n \geq 1$ and any $\forall 0 \leq t \leq T$, we have

$$\|u_n\|_\alpha \leq \|u_1\|^2 \quad \text{and} \quad \|\psi_n\| \leq \|u_1\|.$$

Moreover, we have

$$|\theta_n|_2^2 + 2\int_0^t |\nabla\theta_n|_2^2 \leq \|(u_n,\theta_n)\|_* \leq \|u_1\|^2 \quad \text{and} \quad \|\phi_n\|_{**} \leq \frac{1}{\mu}\|u_1\|^2.$$

Step 2: For $n \geq 2$, let (u_n, θ_n, ϕ_n) be a solution of $(\mathbb{P}_n)$. Let $2 \leq m \leq n$, then it is easy to see that $u_{nm} = u_n - u_m$, $\theta_{nm} = \theta_n - \theta_m$ and $\phi_{nm} = \phi_n - \phi_m$ are solutions of the following problem:

$$\begin{cases} P_\alpha(u_{nm})_{tt} + \Delta^2(u_{nm}) + \Delta\theta_{nm} - \mu\big((\phi_{nm})_t\big)_{/x_3=0} & \\ \qquad\qquad = G(u_{n-1},\psi_{n-1},\theta_n) - G(u_{m-1},\psi_{m-1},\theta_m) & in \quad \Omega\times[0,T], \\ (\theta_{nm})_t - \Delta\theta_{nm} = \Delta(u_{nm})_t & in \quad \Omega\times[0,T], \\ (\phi_{nm})_{tt} = \Delta\phi_{nm} & in \quad \Sigma\times[0,T], \\ u_{nm} = \partial_\nu u_{nm} = \theta_{nm} = 0 & on \quad \partial\Omega\times[0,T], \\ (u_{nm})_{|t=0} = 0, ((u_{nm})_t)_{|t=0} = 0, (\theta_{nm})_{|t=0} = \theta_0 & in \quad \Omega, \\ (\phi_{nm})_{|t=0} = 0, \quad ((\phi_{nm})_t)_{|t=0} = 0 & in \quad \Sigma, \\ \partial_{x_3}\phi_{nm} = (u_{nm})_t & on \quad \Omega\times[0,T], \\ \partial_\nu\phi_{nm} = 0 & on \quad \partial\Sigma/\Omega\times[0,T]. \end{cases}$$

And $\psi_{nm} = \psi_n - \psi_m$ is a solution of the following problem:

$$\begin{cases} \Delta^2\psi_{nm} + [u_{nm}, u_{nm}] = 0 & in \quad \Omega\times[0,T], \\ \psi_{nm} = 0, \ \partial_\nu\psi_{nm} = 0 & on \quad \partial\Omega\times[0,T], \end{cases}$$

Using Proposition 2.7 and Theorem 2.1 we deduce, for all $0 \leq t \leq T$,

$$\|\psi_{n-1} - \psi_{m-1}\| \leq 4c_0c \|u_{n-1} - u_{m-1}\|.$$

According to Theorem 2.6 and Proposition 2.7, again we have, with $0 < c_3 = Te^Tc_1^2 < 1$,

$$\begin{aligned} \|(u_{nm},\theta_{nm})\|_* + \mu\|\phi_{nm}\|_{**} &\leq e^T\int_0^T \big\|G(u_{n-1},\psi_{n-1}) - G(u_{m-1},\psi_{m-1})\big\|_{L^2(\Omega)}^2, \\ &\leq e^T\int_0^T c_1^2\|u_{n-1} - u_{m-1}\|^2. \end{aligned}$$

It follows that

$$\begin{aligned} \|(u_{nm},\theta_{nm})\|_* + \mu\|\phi_{nm}\|_{**} &\leq e^Tc_1^2\int_0^T \|u_{n-1} - u_{m-1}\|^2, \\ &\leq (e^Tc_1^2)^{m-2}\int_0^T \cdots \int_0^T \|u_{n-m+2} - u_1\|^2, \\ &\leq (e^Tc_1^2)^{m-2}\int_0^T \cdots \int_0^T \sum_{k=0}^{n-m+1}(e^Tc_1^2)^k\int_0^T \cdots \int_0^T \|u_2 - u_1\|^2, \\ &\leq (c_3)^{m-2}\sum_{k=0}^{n-m+1}(c_3)^k(2\|u_1\|^2). \end{aligned}$$

Consequently

$$\|u_n - u_m\|_\alpha \leq (c_3)^{m-2} \sum_{k=0}^{n-m+1} (c_3)^k (2\,\|u_1\|^2).$$

The sequence $(u_n, \psi_{n-1})_{n\geq 2}$ is a Cauchy sequence in $L^2\big([0,T],(H_0^2(\Omega))^2\big)$, it follows that (u_n, ψ_{n-1}) converges to $L^2\big([0,T],(H_0^2(\Omega))^2\big)$, $(u_n)_t$ converges to u_t in $L^2\big([0,T],L^2(\Omega)\big)$ and $\nabla(u_n)_t$ converges to ∇u_t in $L^2\big([0,T],L^2(\Omega)\big)$.
Step 3: Using the inequality (2.2), we have

$$|\theta_{n-1} - \theta_{m-1}|_2^2 \quad + \quad \int_0^t |\nabla(\theta_{n-1} - \theta_{m-1})|_2^2 \qquad \leq \qquad \int_0^t |\nabla(u_{n-1} - u_{m-1})_t|_2^2\,.$$

We deduce that θ_n is a Cauchy sequence in $L^2([0,T]\,, H_0^1(\Omega))$, then θ_n converges to θ in $L^2([0,T]\,, H_0^1(\Omega))$. By Proposition 2.7 we have $G(u_{n-1},\psi_{n-1})$ converges to $G(u,\psi)$ in $L^2(\Omega)$. In addition, $((\phi_n)_t, \nabla\phi_n)$ is also Cauchy sequence in $L^2\big([0,T],(L^2(\Sigma))^2\big)$ and hence $((\phi_n)_t, \nabla\phi_n)$ converges to $(\phi_t, \nabla\phi)$ in $L^2\big([0,T],(L^2(\Sigma))^2\big)$. By the continuity of the operators "trace" and "∂_ν",

$$(u_n, \psi_{n-1})_{\partial\Omega} = (\partial_\nu u_n, \partial_\nu \psi_{n-1}) = (0,0) \;\text{ and }\; ((u_n)_t)_{|\partial\Omega}$$

imply that

$$(u, \psi)_{\partial\Omega} = (\partial_\nu u, \partial_\nu \psi) = (0,0) \;\text{ and }\; ((u)_t)_{|\partial\Omega}.$$

Thanks to Theorem 2.6, we have $(u_n, (u_n)_t, \phi_n, (\phi_n)_t) \in C^0([0,T]\,, H_0^2(\Omega)\times H_0^1(\Omega)\times H^1(\Sigma)\times L^2(\Sigma))$ with $(u_n)_{|t=0} = u_0$, $((u_n)_t)_{|t=0} = u_1$, $(\phi_n)_{|t=0} = \phi_0$ and $((\phi_n)_t)_{|t=0} = \phi_1$, which implies that $(u)_{|t=0} = u_0$, $((u)_t)_{|t=0} = u_1$, $(\phi)_{|t=0} = \phi_0$ and $((\phi)_t)_{|t=0} = \phi_1$.

For showing that (u,ψ,θ,ϕ) is a weak solution of the problems $(\mathbb{P}_0)$ and $(\mathbb{Q})$, we follow the same way as in [8]. Let $\varphi_j \in C^1(0,T)$, $1 \leq j \leq j_0$, such that $\varphi_j(T) = 0$ and

$$\psi = \sum_{j=1}^{j_0} \varphi_j \otimes e_j^1, \quad \varphi = \sum_{j=1}^{j_0} \varphi_j \otimes e_j^2, \quad \chi = \sum_{j=1}^{j_0} \varphi_j \otimes e_j^3,$$

where (e^1), (e^2) and (e^3) are basis of $H_0^2(\Omega)$, $H_0^1(\Omega)$ and $H^1(\Sigma)$. We have the following variational problem:

$$-\int_0^T\int_\Omega u_t^{nl}\psi_t + \alpha\int_0^T\int_\Omega \nabla u_t^{nl}\nabla\psi_t + \int_0^T\int_\Omega \nabla\theta^{nl}\nabla\psi + \int_0^T\int_\Omega \Delta u^{nl}\Delta\psi$$

$$-\mu\int_0^T\int_\Omega \phi_t^{nl}\psi = \int_0^T\int_\Omega G(u_{n-1},\psi_{n-1})\psi - \int_\Omega u_{nl1}\psi(0) - \alpha\int_\Omega \nabla u_{nl1}\nabla\psi(0), \tag{3.1}$$

$$-\int_0^T\Big(\int_\Omega \theta^{nl}\varphi_t + \int_\Omega \nabla\theta^{nl}\nabla\varphi - \int_\Omega \nabla u^{nl}\nabla\varphi_t\Big) = -\int_\Omega \theta_{nl0}\varphi(0) + \int_\Omega \nabla u_{nl1}\nabla\varphi(0) \tag{3.2}$$

and

$$-\int_0^T\int_\Sigma \phi^{nl}\chi_t + \int_\Sigma \nabla\phi^{nl}\nabla\chi = -\int_\Omega u_t^{nl}\chi + \int_\Sigma \phi_{nl1}\chi(0). \tag{3.3}$$

If $nl \to +\infty$, in (3.1), (3.2) and (3.3), we deduce that for all $\psi \in L^2([0,T], H_0^2(\Omega))$, $\psi_t \in L^2([0,T], H^1(\Omega))$, $\varphi \in L^2([0,T], H_0^1(\Omega))$, $\varphi_t \in L^2([0,T], L^2(\Omega))$, $\chi \in L^2([0,T], H^1(\Sigma))$ and $\chi_t \in L^2([0,T], L^2(\Sigma))$ such that $\psi(T) = \varphi(T) = \chi(0) = 0$, we have

$$-\int_0^T \int_\Omega u_t\psi_t + \alpha \int_0^T \int_\Omega \nabla u_t \nabla \psi_t + \int_0^T \int_\Omega \nabla\theta\nabla\psi + \int_0^T \int_\Omega \Delta u \Delta\psi - \mu \int_0^T \int_\Omega \phi_t \psi$$
$$= \int_0^T \int_\Omega G(u,\psi)\psi - \int_\Omega u_1\psi(0) - \alpha \int_\Omega \nabla u_1 \nabla\psi(0),$$
$$-\int_0^T \Big(\int_\Omega \theta\varphi_t + \int_\Omega \nabla\theta\nabla\varphi - \int_\Omega \nabla u \nabla\varphi_t \Big) = -\int_\Omega \theta_0\varphi(0) + \int_\Omega \nabla u_1 \nabla\varphi(0)$$

and

$$-\int_0^T \int_\Sigma \phi\chi_t + \int_\Sigma \nabla\phi\nabla\chi = -\int_\Omega u_t\chi + \int_\Sigma \phi_1\chi(0).$$

Hence, we have proved that (u,ψ,θ,ϕ) is a weak solution of the problem studied.

Step 4: Now we prove the uniqueness. Assume that there exist two solutions $(u^1,\psi^1,\theta^1,\phi^1)$ and $(u^2,\psi^2,\theta^2,\phi^2)$ in $L^2\big([0,T], (H_0^2(\Omega))^2 \times H_0^1(\Omega) \times H^1(\Sigma)\big)$ such that, for some $c > 0$ being sufficiently small, we have $\|u^1\| \le c$ and $\|u^2\| \le c$. This implies that $u^{12} = u^1 - u^2$, $\theta^{12} = \theta^1 - \theta^2$ and $\phi^{12} = \phi^1 - \phi^2$ satisfies the following problem:

$$(\mathbb{P}_2) \begin{cases} P_\alpha u_{tt}^{12} + \Delta^2 u^{12} - \mu\big(\phi_t^{12}\big)_{|x_3=0} = G(u^1,\psi^1) - G(u^2,\psi^2) & \textit{in} \quad \Omega \times [0,T], \\ \theta_t^{12} - \Delta\theta^{12} = \Delta u_t^{12} & \textit{in} \quad \Omega \times [0,T], \\ (\phi^{12})_{tt} = \Delta\phi^{12} & \textit{in} \quad \Sigma \times [0,T], \\ u^{12} = \partial_\nu u^{12} = \theta^{12} = 0 & \textit{on} \quad \partial\Omega \times [0,T], \\ (u^{12})_{|t=0} = 0, ((u^{12})_t)_{|t=0} = 0, (\theta^{12})_{|t=0} = 0 & \textit{in} \quad \Omega, \\ (\phi^{12})_{|t=0} = 0, \quad ((\phi^{12})_t)_{|t=0} = 0 & \textit{in} \quad \Sigma, \\ \partial_{x_3}\phi^{12} = u_t^{12} & \textit{on} \quad \Omega \times [0,T], \\ \partial_\nu \phi^{12} = 0 & \textit{on} \quad \partial\Sigma/\Omega \times [0,T]. \end{cases}$$

Which means that $(u^{12},\theta^{12},\phi^{12})$ is a solution of the problem $(\mathbb{P}_2)$. Proposition 2.7, Theorem 2.6 and Theorem 2.1 ensure that there exists $c_1 > 0$ such that

$$\begin{aligned} \|(u^{12},\theta^{12})\|_* + \mu\|\phi^{12}\|_{**} &\le e^T \int_0^T \big\|G(u^1,\psi^1) - G(u^2,\psi^2)\big\|_{L^2(\Omega)}^2, \\ &\le e^T c_1^2 \int_0^T \big\|u^1 - u^2\big\|^2. \end{aligned}$$

Since c_1 is small and thus $0 < c_3 = Te^T c_1^2 < 1$, it follows that

$$\int_0^T \big\|(u^{12},\theta^{12})\big\|_* + \mu\|\phi^{12}\|_{**} \le c_3 \int_0^T \big\|(u^{12},\theta^{12})\big\|_* + \mu\|\phi^{12}\|_{**},$$

which, with $0 < c_3 < 1$, we then deduce that $u^1 = u^2$, $\psi^1 = \psi^2$, $\theta^1 = \theta^2$ and $\phi^1 = \phi^2$.

We conclude that the dynamic von Karman equation has one and only one weak solution (u,ψ,θ,ϕ) in $L^2\Big([0,T], (H_0^2(\Omega))^2 \times H_0^1(\Omega) \times H^1(\Sigma)\Big)$. The demonstration of the theorem is finished. □

Proposition 3.2. *Let* $(u, \psi, \theta, \phi) \in L^2\Big([0,T], (H_0^2(\Omega))^2 \times H_0^1(\Omega) \times H^1(\Sigma)\Big)$*. Then the following equalities:*

$$\widetilde{E}\big(u(t), u_t(t), \psi\big) + \frac{1}{2}\left|\theta\right|_2^2 - \int_0^t \left|\nabla\theta_t\right|_2^2 + \frac{\mu}{2}\big(\left|\phi_t\right|_2^2 + \left|\nabla\phi\right|_2^2\big) = \widetilde{E}_1(u_0, u^1, \psi_0) + \frac{1}{2}\left|\theta_0\right|_2^2 + \frac{\mu}{2}\big(\left|\phi_1\right|_2^2 + \left|\nabla\phi_0\right|_2^2\big),$$

with

$$\widetilde{E}\big(u(t), u_t(t), \psi\big) = \frac{1}{2}\big(\left|u_t\right|_2^2 + \left\|u\right\|^2 + \alpha\left|\nabla u_t\right|_2^2 \big) + \frac{1}{4}\int_\Omega \big(\left|\Delta\phi\right|^2 - 2\left[u, F_0\right] u - 4pu\big)$$

and

$$\widetilde{E}_1(u_0, u_1, \psi_0) = \frac{1}{2}\big(\left|u_1\right|_2^2 + \alpha\left|\nabla u_1\right|_2^2 + \left\|u_0\right\|^2 \big) + \frac{1}{4}\int_\Omega \big(\left|\Delta\phi_0\right|^2 - 2\left[u_0, F_0\right] u_0 - 4pu_0\big)$$

holds for any $0 \le t \le T$*, where* $\psi_0 \in H_0^2(\Omega)$ *is the unique solution of the equation* $\Delta^2\psi_0 = -\left[u_0, u_0\right]$.

Proof. Thanks to Theorem 2.6, for any $\forall 0 \le t \le T$ and $f = G(u,\psi) = [u, \psi + F_0] + p(x_1, x_2)$, (u, θ, ϕ) satisfies the following variational problems:

$$\int_\Omega u_{tt}u_t + \alpha\int_\Omega \nabla u_{tt}\nabla u_t + \int_\Omega \Delta u\Delta u_t + \int_\Omega \Delta\theta u_t - \mu\int_\Omega \phi_t u_t = \int_\Omega G(u,\psi)u_t,$$

$$\int_\Omega \theta_t\theta - \int_\Omega (\nabla\theta)^2 = \int_\Omega \Delta u_t\theta \quad \text{and} \quad \int_\Sigma \phi_{tt}\phi_t + \int_\Sigma \nabla\phi\nabla\phi_t = -\int_\Omega u_t\phi_t,$$

this implies that

$$\frac{d}{2dt}\big(\left|u_t\right|_2^2 + \left\|u\right\|^2 + \alpha\left|\nabla u_t\right|_2^2 \big) + \frac{d}{2dt}\left|\theta\right|_2^2 + \left|\nabla\theta\right|_2^2 + \frac{\mu d}{2dt}\big(\left|\phi_t\right|_2^2 + \left|\nabla\phi\right|_2^2 \big) = \int_\Omega G(u,\psi)u_t, \tag{3.4}$$

First we have

$$\int_0^t \int_\Omega p(x_1, x_2)u_t = \int_\Omega p(x_1, x_2)u(t) - \int_\Omega p(x_1, x_2)u_0.$$

Otherwise, see [5], one has, with $\Delta^2\psi = [u, u]$,

$$\begin{aligned}
\int_0^t \int_\Omega \left[u, \psi + F_0\right] u_t &= \int_0^t \int_\Omega \left[u, \psi\right] u_t + \int_0^t \int_\Omega \left[u, F_0\right] u_t, \\
&= \frac{1}{2}\int_0^t \int_\Omega \frac{d}{dt}(\left[u, u\right] \psi) + \frac{1}{2}\int_0^t \int_\Omega \frac{d}{dt}(\left[u, F_0\right] u), \\
&= -\frac{1}{4}\int_\Omega \left|\Delta\psi\right|^2 + \frac{1}{4}\int_\Omega \left|\Delta\psi_0\right|^2 + \frac{1}{2}\int_\Omega \left[u, u\right] F_0 - \frac{1}{2}\int_\Omega \left[u_0, u_0\right] F_0
\end{aligned}$$

and

$$\begin{aligned}
\int_0^t \int_\Omega \Delta\theta u_t = \int_0^t \int_\Omega \theta\Delta u_t &= \frac{k}{2}\int_0^t \frac{d}{dt}\left|\theta\right|_2^2 - \int_0^t \left|\nabla\theta\right|_2^2 \\
&= \frac{k}{2}\left|\theta\right|_2^2 - \frac{k}{2}\left|\theta_0\right|_2^2 - \int_0^t \left|\nabla\theta\right|_2^2.
\end{aligned}$$

If we integrate 3.4 with $t \in [0,T]$, we conclude that

$$\widetilde{E}\big(u(t), u_t(t), \psi\big) + \frac{1}{2}\,|\theta|_2^2 - \int_0^t |\nabla\theta_t|_2^2 + \frac{\mu}{2}\big(|\phi_t|_2^2 + |\nabla\phi|_2^2\big) = \widetilde{E}_1(u_0, u_1, \psi_0) + \frac{1}{2}\,|\theta_0|_2^2 + \frac{\mu}{2}\big(|\phi_1|_2^2 + |\nabla\phi_0|_2^2\big).$$

□

4. Numerical Application

In this section, we present a numerical resolution of the previous theoretical study using the finite difference method.

4.1. **Preliminaries.** We take $\Omega =]0,1[\times]0,1[\subset \mathbb{R}^2$, $\Sigma =]0,1[\times]0,1[\times]0,1[\subset \mathbb{R}^3$ and let $T > 0$. For solving numerically the problem $(\mathbb{P}_0)$, We apply the finite difference method by considering a uniform mesh of width h. Let us denote by Ω_h the set of all mesh points within the Ω with internal points: $x_i = ih$, $y_j = jh$, $i, j = 1, ...N-1$, $h = \frac{1}{N+1}$, $\Delta t = \frac{1}{T}$. On the other hand $\overline{\Omega}_h$ the set of boundary mesh points and u_h the finite-difference that approximates u. In [4], the author examined a numerical analysis of the convergence and stability of conservative finite difference schemes applied to the dynamic von Karman plate equations. To numerically approximate the unique weak solution of our problem, we employ the discrete von-Karman evolution model outlined in [4, 12].

$$(1)\begin{cases} (\delta_t^2 - \alpha\delta_t^2(\delta_x^2+\delta_y^2))u_{ij}^n + (\delta_x^2+\delta_y^2)\theta_{ij}^n + \Delta_h^2 u_{ij}^n = [\ u_{ij}^n\ v_{ij}^n + F_{ij}\ ] + p_{ij} & in \quad \Omega_h, \\ \delta_t\theta_{ij}^n - (\delta_x^2+\delta_y^2)\theta_{ij}^n - \delta_t(\delta_x^2+\delta_y^2)u_{ij}^n = 0 & in \quad \Omega_h, \\ \Delta_h^2 v_{ij}^n = -[\ u_{ij}^n\ u_{ij}^n\ ] & in \quad \Omega_h, \\ u_{ij}^0 = (\varphi_0)_{ij},\ \delta_t u_{ij}^0 = (\varphi_1)_{ij},\ \theta_{ij}^0 = (\theta_0)_{ij} & in \quad \Omega_h, \\ u_{ij}^n = v_{ij}^n = 0 & on \quad \overline{\Omega}_h, \\ \partial_\nu u_{ij}^n = \partial_\nu v_{ij}^n = 0 & on \quad \overline{\Omega}_h, \end{cases}$$

and we also approach the problem $(\mathbb{S})$ using the following finite-difference system:

$$(2)\begin{cases} \delta_t^2\phi_{ijk}^n = \Delta_h\phi_{ijk}^n & in \quad \Sigma_h, \\ \phi_{ijk}^0 = (\phi_0)_{ijk}, \quad \delta_t\phi_{ijk}^0 = (\phi_1)_{ijk} & in \quad \Sigma_h, \\ \partial_z\phi_{ij0}^n = \delta_t u_{ij0}^n & on \quad \overline{\Sigma_h}\cap\overline{\Omega_h}, \\ (\partial_\nu\phi)_{ijk}^n = 0 & on \quad \overline{\Sigma_h}/\overline{\Omega_h}, \end{cases}$$

with the following discrete differential operators:

$$\delta_t u_{ij}^n = \frac{u_{ij}^{n+1} - u_{ij}^n}{\Delta t}, \qquad \delta_t^2 u_{ij}^n = \frac{u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1}}{(\Delta t)^2},$$

$$\delta_x^2 u_{ij}^n = \frac{u_{i+1j}^n - 2u_{ij}^n + u_{i-1j}^n}{(h)^2}, \qquad \delta_y^2 u_{ij}^n = \frac{u_{ij+1}^n - 2u_{ij}^n + u_{ij-1}^n}{(h)^2},$$

$$\delta_{xy}^2 u_{ij}^n = \frac{u_{i+1j+1}^n - u_{i+1j-1}^n - u_{i-1j+1}^n + u_{i-1j-1}^n}{(2h)^2},$$

$$\Delta_h^2 u_{ij}^n = h^{-4}\,[\ u_{ij-2} + u_{ij+2} + u_{i-2j} + u_{i+2j} - 8(u_{ij-1} + u_{ij+1} + u_{i-1j} + u_{i+1j}) + 2(u_{i-1j-1} + u_{i-1j+1} + u_{i+1j-1} + u_{i+1j+1}) - 20u_{ij}\ ],$$

$[\ u^n_{ij},\ v^n_{ij}\ ] = \delta^2_x u^n_{ij}\delta^2_y v^n_{ij} - 2\delta^2_{xy}u^n_{ij}\delta^2_{xy}v^n_{ij} + \delta^2_y u^n_{ij}\delta^2_x v^n_{ij}.$

In summary, we have effectively transformed the above problem to a numerical resolution three steps:

First step: We approximate the velocity of potential gas using the finite difference method for the following semilinear wave equations, [4].

$$(*)\begin{cases} \phi_{tt} = \Delta\phi & in \quad \Sigma\times[0,T]\,, \\ \phi_{|t=0} = \phi_0, \quad (\phi_t)_{|t=0} = \phi_1 & in \quad \Sigma, \\ \partial_{x_3}\phi = u_t & on \quad \Omega\times[0,T]\,, \\ \partial_\nu\phi = 0 & on \quad \partial\Sigma/\Omega\times[0,T]\,, \end{cases}$$

Second step: We utilize the numerical procedure of 13-point formula of finite difference discussed in [7]. We used this method to illustrate the weak solution to the following problem:

$$(**)\begin{cases} \Delta^2 v = f_1 & in \quad \Omega, \\ v = g_1 & on \quad \Gamma, \\ \partial_\nu v = g_2 & on \quad \Gamma. \end{cases}$$

Third step: In this step, we use the discrete model of von Karman evolution $(*)$ for illustrating the the solution of the thermoelastic model coupled with the dynamic von-Karman evolution.

4.2. **Non-coupled approach.** In [7], Gubta Showcased a numerical analysis of finite-difference method about the numerical resolution of the Biharmonic equation. Such method is known as the non-coupled method of 13-point, can be summed up by the following result:

Proposition 4.1. *The* 13*-point approximation of the Biharmonic equation for approaching the unique solution w of the problem (P) is defined by:*

$$(1)\begin{cases} L_h w_{ij} = \dfrac{1}{h^4}\Big[w_{ij-2} + w_{ij+2} + w_{i-2j} + w_{i+2j} - 8(w_{ij-1} + w_{ij+1} + w_{i-1j} \\ \quad +w_{i+1j}) + 2(w_{i-1j-1} + w_{i-1j+1} + w_{i+1j-1} + w_{i+1j+1}) - 20w_{ij}\Big] = f_1(x_i, y_j) \end{cases}$$

where $w_{ij} = w(x_i, y_j)$, for $i, j = 1, 2, ..., N-1$.

If the mesh point (x_i, y_j) is adjacent to the boundary $\overline{\Omega}_h$, then the undefined values of w_h are conventionally calculated by the following approximation of $\partial_\nu w$:

$$\begin{aligned} w_{i-2,j} &= \tfrac{1}{2}w_{i+1,j} - w_{ij} + \tfrac{3}{2}w_{i-1,j} - h(\partial_x w)_{i-1,j}, \\ w_{i,j-2} &= \tfrac{1}{2}w_{i,j+1} - w_{ij} + \tfrac{3}{2}w_{i,j-1} - h(\partial_y w)_{i,j-1}, \\ w_{i+2,j} &= \tfrac{1}{2}w_{i+1,j} - w_{ij} + \tfrac{3}{2}w_{i-1,j} - h(\partial_x w)_{i+1,j}, \\ w_{i,j+2} &= \tfrac{1}{2}w_{i,j+1} - w_{ij} + \tfrac{3}{2}w_{i,j-1} - h(\partial_y w)_{i,j+1}. \end{aligned}$$

4.3. **Numerical test.** Let us consider the following body forces:

$F_0(x,y) = 0.001xe^{-x^2-y^2}cos^2(\pi x), \qquad p(x,y) = 0.001x(x-1)^2(y-1)^2e^{-x^2-y^2},$

$\varphi_0 = 1510^{-8}(\sin(\pi x)\sin(\pi y))^2, \qquad \varphi_1 = 1510^{-8}x^2y^2(x-y-1)^2(y-1)^2e^{-x^2-y^2},$

$\theta_0 = 10^{-2}x^2y^3(x-1)^2(y-1)^3(e^{-x^2} - e^{-y^2}).$

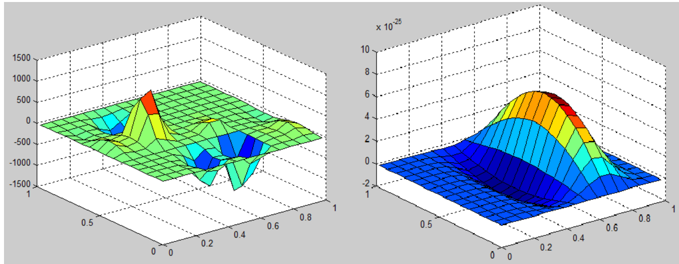

FIGURE 1. Displacement of plate and Thermal value θ, $T = 0.6s$.

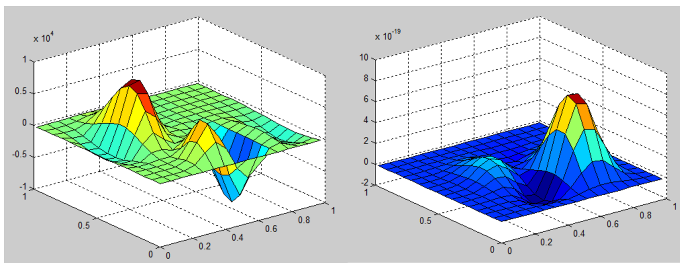

FIGURE 2. Displacement of plate and Thermal value θ, $T = 5s$.

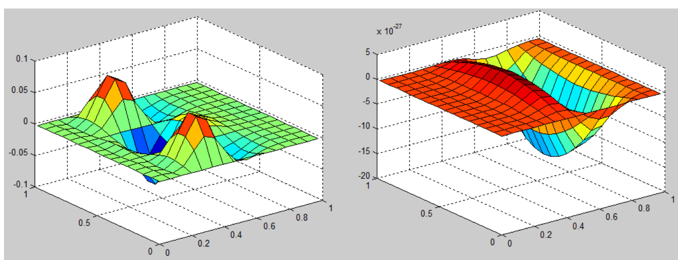

FIGURE 3. Displacement of plate and Thermal value θ, $T = 23s$.

5. Conclusion

In this paper, we have presented an iterative method to obtain a unique weak solution to the von Karman dynamic equations coupled to the thermoelastic equation with a flexible phenomenon of small nonlinear vibration displacement in nonlinear oscillation of the elastic plate, with rotational inertia and not clamped boundary conditions subject to thermal dissipation. Our approach is in fact a good tool for justifying the theoretical results and we then use the method of finite difference for approaching the unique solution of the theoretical problem. These results have potential for application in the fields of physics. Future work can be devoted to the study for the models of dynamic von Karman equations with thermal dissipation and for free boundary conditions of the shell could.

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References

1. Akhlil, K. (2018). The Laplacian with Robin boundary conditions involving signed measures. Gulf Journal of Mathematics., 6(1), 12-19. https://doi.org/10.56947/gjom.v6i1.120
2. Antman, S.S. and Von-Karman, T. (2006). A Panorama of Hungarian Mathematics in the Twentieth Centuray I, Bolyai Soc. Math. Studies., 14, 373-382. https://doi.org/10.1007/978-3-540-30721-1
3. Bassonon Y. C. and Ouedraogo A. (2024). Discontinuous Galerkin method for linear parabolic equations with L^1-data. Gulf Journal of Mathematics. 16(2), 122-134. https://doi.org/10.56947/gjom.v16i2.1874
4. Bilbao, S. (2007). A family of conservative finite difference schemes for the dynamical von Karman plate equations, Numer. Meth. Partial. Diff. Eqns., 24(1), 193-218. https://doi.org/10.1002/num.20260
5. Chueshov, I. and Lasiecka, I. (2010). Von Karman Evolution, Well-posedness and Long Time Dynamics, New York, Springer. https://doi.org/10.1007/978-0-387-87712-9
6. Ciarlet, P. G. and Rabier, R. (1980). Les Equations de von Karman, Lecture Notes in Mathematics, 826, New York, Springer. https://doi.org/10.1007/BFb0091528
7. Gubta, M. M. and Manohar, R. P. (1979). Direct solution of Biharmonic equation using non coupled approach, J. Computational Physics., 33(2), 236-248. https://doi.org/10.1016/0021-9991(79)90018-4
8. Lions, J. L. (2002). Quelques Methodes de Résolution des Problèmes aux Limites non Linéaires. Paris, Dunod. https://ulysse.univ-lorraine.fr/permalink/33UDL_INST/1ft321i/alma991004374059705596
9. Lions, J. L. and Magenes, E. (1968). Problèmes aux Limites non Homogènes et Applications, 1, Gauthier-Villars, Paris, Dunod. https://search.worldcat.org/fr/title/25525823
10. Oudaani, J. (2022). Existence, uniqueness weak solution for a dynamic Full von Karman System of thermoelasticity, Moroccan Journal of Pure and Applied Analysis. 8(3), 375 - 400. https://doi.org/10.2478/mjpaa-2022-0026
11. Oudaani, J. and Raïssouli, M. (2022). Wave Equations coupled by Von Karman Models with Rotational Forces, Int. J. Open Problems Compt. Math., 15(2), 1-23.
12. Pereira, D. C., Raposo, C. A. and Avila, A. J. (2014). Numerical solution and exponential decay to von Karman system with frictional damping, Int. J. Math. Information Sci., 8(4), 1575-1582. http://dx.doi.org/10.12785/amis/080411

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