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A SYSTEMATIC LADDER-OPERATOR APPROACH TO ORTHOGONAL POLYNOMIALS

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Abstract. An elementary and systematic approach to the problem of factorizing classes of second-order linear homogeneous equations, with corresponding classes of orthogonal polynomials as their solutions, is developed. The factorized formats lead to ladder-operator representations of the solution of each class of second-order linear homogeneous equation and present a *different* (more straightforward and *explicit*) resolution of the mathematical problem leading to the factorization approach of, in particular, Jafarizadeh and Fakhri [3, 4]. The main solution process involves solving coupled Riccati equations along with a third 'consistency' equation *directly*, to find, without further assumption, all of the components of each factorization representation. As a bonus, one of the assumptions of the original Jafarizadeh and Fakhri approach is eliminated. In some cases, a basic technique of Cotfas [1, 2] is also invoked to produce simplified derivations. The main results are presented in tabular form and a brief discussion of previous/other approaches given.

1. Introduction and preliminaries

Consider the second-order linear ordinary differential equation (with *non-negative* integer n)

$$p(z)\ y_n''(z) + q(z)\ y_n'(z) + \lambda_n\ y_n(z) = 0 \tag{1.1}$$

where

$$p(z) = \frac{p''}{2}z^2 + p'(0)z + p(0) \ \text{ and } \ q(z) = q'z + q(0) \tag{1.2}$$

and we *assume* that

$$\lambda_n = -\frac{n}{2}[(n-1)p'' + 2q'] \tag{1.3}$$

As usual, the dashes denote differentiation with respect to the function argument (in this case the variable z). The class of ordinary differential equation determined by (1.1) in conjunction with (1.3) is well-known [7] to have (classical) *orthogonal* polynomial solutions provided that the condition (1.3) holds.

In addition to this, (1.1), with (1.2) and (1.3) in mind, is also the starting point for two other classes of ordinary differential equation and their solution types. So, if we introduce the *weighting function* $w(z)$ as the solution of the Pearson equation

$$p(z)w'(z) + [p'(z) - q(z)]w(z) = 0 \tag{1.4}$$

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then the new functions [6] (with the $K_n$normalization factors)

$$\psi_n(z) = K_n^{-1}\sqrt{w(z)}y_n(z) \tag{1.5}$$

form an *orthonormal* set, for non-negative integer n. From (1.1) and (1.5), we see that the $\psi_n(z)$ satisfy the ordinary differential equation [6]

$$p(z)\psi_n''(z) + p'(z)\psi_n'(z) + \lambda_n(z)\psi_n(z) = 0 \tag{1.6}$$

where

$$\lambda_n\,(z) = \lambda_n - \frac{1}{2}(q' - p'') - \frac{1}{4}\frac{(q-p')^2}{p} \tag{1.7}$$

Further, the so-called *associated equation* for (1.1), that is

$$p(z)y_{n,m}''(z) + q(z)y_{n,m}'(z) + \lambda_{n,m}(z)y_{n,m}(z) = 0 \tag{1.8}$$

has solution set [1, 2, 3]

$$y_{n,m}(z) = (-1)^m p^{\frac{m}{2}}(z)y_n^{(m)}(z) \tag{1.9}$$

for integer $0 \leq m \leq n$ provided

$$\lambda_{n,m}(z) = \lambda_n - \lambda_m - \frac{\frac{m}{2}\left(p''p + p'[q + (\frac{m}{2} - 1)p']\right)}{p} \tag{1.10}$$

Equations, (1.1), (1.6) and (1.8) have considerable application in the solution of problems in mathematical physics, especially the solution of the Schroedinger equation [7].

Now, a well-known approach to the solution of (1.1), (1.6) or (1.8), is to (effectively) 'factorize' the second-order linear differential operator on the left-hand side of (1.1) or (1.6) or (1.8): the 'factorization approach' to the solution of (1.1), (1.6) or (1.8). To put this situation into sharper perspective, we give a brief overview of the factorization methodology for equation (1.1), the methodology applied to the equations (1.6) and (1.8) being essentially identical.

The basic idea [3] is to develop an 'equivalent' factorization representation of (1.1), a term explained in detail in subsequent sections, by defining the 'ladder-operators'

$$[p(z)\frac{d}{dz} + \varphi_n^+(z)]\; y_n(z) = y_{n+1}(z) \tag{1.11}$$

and

$$[p(z)\frac{d}{dz} - \varphi_n^-(z)]\; y_n(z) = E_n y_{n-1}(z) \tag{1.12}$$

with the E_n *independent* of the variable z.

To appreciate the point of the factorizations (1.11) and (1.12), we assume that we have found the functions φ_n^+, φ_n^- and E_n. Then, if we set $n = 0$ in (1.12), the resulting differential equation is easy to solve, provided $E_0 = 0$. In fact, as we show below, it is indeed possible to *determine* the factorizations (1.11) and (1.12) so that $E_0 = 0$ *identically*, that is, without further assumption. In this case we have

$$p(z)\frac{dy_0}{dz} - \varphi_0^-(z)y_0(z) = 0 \tag{1.13}$$

Given $y_0(z)$ from (1.3), then all other $y_n(z)$, $n \geq 1$, follow from the repeated application of the ladder-operator (1.11) (and a choice of integration constant). In other words, using (1.11), we step-up through the solutions of (1.1) one at a

time, in direct analogy with the use of a step-ladder. Similar remarks, in reverse, apply to (1.12) – hence the name for the operators.

There is a considerable body of literature on the factorization approach to equations (1.1), (1.6) and (1.8). For example, Lorente [6], Cotfas [1, 2], Jafarizadeh and Fakhri [3, 4], Yanez et al [13] and the 'Mexican school' [9, 8, 11, 12]. The main aim of the present work, is to reconsider the *specific* factorization methodology of Jafarizadeh and Fakhri [3, 4] and present a *different* (more straightforward and *explicit*) resolution of the mathematical problem leading to the factorization representation of (1.1), (1.6) and, additionally [6], (1.8). (However, special mention must be made, again, of the work of the 'Mexican school' [9, 8, 11, 12], as there is an obvious overlap with their work and the work of Jafarizadeh and Fakhri [3, 4] and that presented here.)

In particular, we will set-up and solve the factorization problems so that the parameters in the factorization formulae (e.g., the $\varphi's$and the $E's$ above):

(a) depend only on the coefficients $p(z)$ and $q(z)$ of (1.1);
(b) are determined *explicitly* and *entirely* by the solution process itself;
(c) are obtained in as identical and/or as straightforward a manner as possible, in all cases.

In fact, the critical mathematical problem, which must be solved to obtain the factorization representation(s), is the solution of certain forms of the Riccati equation. This is where the solution process described and applied below, differs from that of [3, 4]. The solution process presented in sections 2, 3, 4 and 5 below, solves the Riccati equations directly (along with a third 'consistency' equation) to find, e.g., φ_n^+, φ_n^- and E_n above, without further assumption. The method of [3, 4] is an 'indirect' method and, in addition, requires a further assumption (*implying* $E_0 = 0$ above). However, that aside, we will leave the comparison of the present methodology with other approaches till the conclusions and discussion section, section 6 below.

The paper is organized as follows. In section 2 we develop the simplified factorization method for equation (1.1) while the task of developing a factorization representation of (1.6), with this improved methodology, is dealt with in section 3. Following this, in sections 4 ('n'- factorization) and 5 ('m'-factorization), the new solution process is applied to the associated equation, (1.8). The more important standard examples of the factorization processes are summarized in tables 1 to 4. In the discussion and conclusions section, section 6, we present a *brief* comparison/discussion of the approach of this paper with that of the other workers in this field. We turn now to the solution of the basic problem itself, that is, the derivation of the factorized forms of (1.1), (1.6) and (1.8).

2. The basic factorization formalism

We require the enforced factorizations of (1.1) in the forms [3, 4]

$$p(z)[p(z)y_n'' + q(z)y_n' + \lambda_n y_n] = 0 \tag{2.1}$$

and

$$p(z)[p(z)y_{n-1}'' + q(z)y_{n-1}' + \lambda_{n-1} y_{n-1}] = 0 \tag{2.2}$$

as

$$[p(z)\frac{d}{dz}+\varphi_{n-1}^{+}][p(z)\frac{d}{dz}-\varphi_{n}^{-}]\ y_n = E_n y_n \tag{2.3}$$

and

$$[p(z)\frac{d}{dz}-\varphi_{n}^{-}][p(z)\frac{d}{dz}+\varphi_{n-1}^{+}]\ y_{n-1} = E_n y_{n-1} \tag{2.4}$$

respectively, by defining the ladder-operators (note the change $\varphi_{n-1}^{+} \to \varphi_{n}^{+}$)

$$[p(z)\frac{d}{dz}+\varphi_{n}^{+}]\ y_n = y_{n+1} \tag{2.5}$$

and

$$[p(z)\frac{d}{dz}-\varphi_{n}^{-}]\ y_n = E_n y_{n-1} \tag{2.6}$$

with the E_n independent of the variable z.

The problem is to find φ_{n-1}^{+}, φ_{n}^{-} and E_n. Expanding (2.3) and (2.4), we get

$$p^2 y_n'' + p(p' + \varphi_{n-1}^{+} - \varphi_{n}^{-})y_n' - (p(\varphi_{n}^{-})' + \varphi_{n-1}^{+}\varphi_{n}^{-})y_n = E_n y_n \tag{2.7}$$

and

$$\begin{aligned} &p^2 y_{n-1}'' + p(p' + \varphi_{n-1}^{+} - \varphi_{n}^{-})y_{n-1}' \\ &+ (p(\varphi_{n-1}^{+})' - \varphi_{n-1}^{+}\varphi_{n}^{-})y_{n-1} = E_n y_{n-1} \end{aligned} \tag{2.8}$$

Comparing (2.7) with (2.1) and (2.8) with (2.2), we get three equations for the three unknowns, φ_{n-1}^{+}, φ_{n}^{-} and E_n. The three equations are

$$\varphi_{n-1}^{+} - \varphi_{n}^{-} + p' - q = 0 \tag{2.9}$$

and

$$p(\varphi_{n-1}^{+})' - \varphi_{n-1}^{+}\varphi_{n}^{-} - E_n - p\lambda_{n-1} = 0 \tag{2.10}$$

and

$$p(\varphi_{n}^{-})' + \varphi_{n-1}^{+}\varphi_{n}^{-} + E_n + p\lambda_n = 0 \tag{2.11}$$

To solve (2.9), (2.10) and (2.11), we first show that $(\varphi_{n-1}^{+})'$ and $(\varphi_{n}^{-})'$ are *independent* of z. Differentiating (2.9) we get the first of two equations for $(\varphi_{n-1}^{+})'$ and $(\varphi_{n}^{-})'$ that is

$$(\varphi_{n-1}^{+})' - (\varphi_{n}^{-})' = q' - p'' \tag{2.12}$$

Next, adding (2.10) and (2.11) together, we find a second equation for $(\varphi_{n-1}^{+})'$and $(\varphi_{n}^{-})'$, that is

$$(\varphi_{n-1}^{+})' + (\varphi_{n}^{-})' = \lambda_{n-1} - \lambda_n = (n-1)p'' + q' \tag{2.13}$$

on applying equation (1.2). Finally, solving (2.12) and (2.13) for $(\varphi_{n-1}^{+})'$and $(\varphi_{n}^{-})'$, we get

$$(\varphi_{n-1}^{+})' = \frac{(n-2)}{2}p'' + q' \tag{2.14}$$

and

$$(\varphi_{n}^{-})' = \frac{n}{2}p'' \tag{2.15}$$

Since, as we see from (2.14) and (2.15), $(\varphi_{n-1}^{+})'$ and $(\varphi_{n}^{-})'$, are indeed independent of z, it follows that φ_{n-1}^{+} and $\varphi_{n}^{-}$are *linear* functions *and* $(\varphi_{n-1}^{+})'' = (\varphi_{n}^{-})'' = 0$. This last fact ($(\varphi_{n-1}^{+})'' = (\varphi_{n}^{-})'' = 0$) enables us to solve (2.10) and (2.11) for φ_{n-1}^{+} and φ_{n}^{-}, as follows. If we differentiate through (2.10) and (2.11), we eliminate the product term $\varphi_{n-1}^{+}\varphi_{n}^{-}$ and get two equations for $(\varphi_{n-1}^{+})'$ and $(\varphi_{n}^{-})'$. That is, after differentiating (2.10) and (2.11), we end up with

$$p'(\varphi_{n-1}^{+})' - \varphi_{n-1}^{+}(\varphi_n^{-})' - (\varphi_{n-1}^{+})'\varphi_n^{-} - p'\lambda_{n-1} = 0 \tag{2.16}$$

and

$$p'(\varphi_n^{-})' + \varphi_{n-1}^{+}(\varphi_n^{-})' + (\varphi_{n-1}^{+})'\varphi_n^{-} + p'\lambda_n = 0 \tag{2.17}$$

In conjunction with equation (2.9), equations (2.16) and (2.17) may be solved, by elimination, to leave φ_{n-1}^{+} and φ_n^{-} as

$$\varphi_{n-1}^{+} = \frac{(\varphi_{n-1}^{+})'q - \lambda_{n-1}p'}{(\varphi_{n-1}^{+})' + (\varphi_n^{-})'} = \frac{[(n-1)p' + q][(n-2)p'' + 2q']}{2[(n-1)p'' + q']} \tag{2.18}$$

and

$$\varphi_n^{-} = -\frac{(\varphi_n^{-})'q + \lambda_n p'}{(\varphi_{n-1}^{+})' + (\varphi_n^{-})'} = \frac{n\{[(n-1)p'' + q']p' + (p'q' - p''q)\}}{2[(n-1)p'' + q']} \tag{2.19}$$

We have now, through the relations (2.18) and (2.19), φ_{n-1}^{+} and φ_n^{-} given entirely in terms of known quantities. The determination of the constant E_n is now a matter of substituting into equations (2.10) or (2.11), to find, after some algebra, that

$$E_n = \frac{\begin{array}{r} n(2q' + (n-2)p'')[2(q' + (n-1)p'')^2 p \\ - (q + (n-1)p')([2q' + (n-1)p'']p' - p''q)] \end{array}}{4(q' + (n-1)p'')^2} \tag{2.20}$$

Note, in particular, that $E_0 = 0$. (The fact that E_n is indeed independent of z, can be verified by multiplying-out the top line of the fraction in (2.20), when the z and z^2 terms are found to *cancel-out*.)

Examples of the factorization format, for many important equations, are summarized in Table 1 (a, b and c are *constants*). The starting equation for the ladder-operator approach is particularly simple, that is (see (1.12) and Table 1)

$$\frac{dy_0}{dz} = 0 \tag{2.21}$$

which is true in general, and shows that $y_0(z) = C$, with C constant.

Finally, we can develop a *three-term recurrence* formula. Eliminating the derivative terms from equations (2.5) and (2.6) leaves

$$y_{n+1} = (\varphi_n^{+} + \varphi_n^{-})y_n + E_n y_{n-1} \tag{2.22}$$

which is just the three-term recurrence relation mentioned above.

3. The orthonormal factorization formalism

We require the enforced factorizations of (1.6) in the forms

$$p(z)[p(z)\psi_n'' + p'(z)\psi_n' + \lambda_n(z)\psi_n] = 0 \tag{3.1}$$

and

$$p(z)[p(z)\psi_{n-1}'' + p'(z)\psi_{n-1}' + \lambda_{n-1}(z)\psi_{n-1}] = 0 \tag{3.2}$$

as

$$[p(z)\frac{d}{dz} + \phi_{n-1}^{+}][p(z)\frac{d}{dz} - \phi_n^{-}]\ \psi_n = \varepsilon_n\psi_n \tag{3.3}$$

and

$$[p(z)\frac{d}{dz} - \phi_n^{-}][p(z)\frac{d}{dz} + \phi_{n-1}^{+}]\ \psi_{n-1} = \varepsilon_n\psi_{n-1} \tag{3.4}$$

respectively, by defining the ladder-operators (note the change $\phi_{n-1}^{+} \to \phi_n^{+}$)

$$[p(z)\frac{d}{dz} + \phi_n^+]\ \psi_n = \psi_{n+1} \tag{3.5}$$

and

$$[p(z)\frac{d}{dz} - \phi_n^-]\ \psi_n = \varepsilon_n \psi_{n-1} \tag{3.6}$$

with the ε_n independent of the variable z. The problem is to find $\phi_{n-1}^+,\ \phi_n^-$ and ε_n.

Expanding (3.3) and (3.4), we get

$$p^2\psi_n'' + p(p' + \phi_{n-1}^+ - \phi_n^-)\psi_n' - (p(\phi_n^-)' + \phi_{n-1}^+\phi_n^-)\psi_n = \varepsilon_n\psi_n \tag{3.7}$$

and

$$\begin{aligned} p^2\psi_{n-1}'' + p([p' + \phi_{n-1}^+ - \phi_n^-)\psi_{n-1}' \\ + (p(\phi_{n-1}^+)' - \phi_{n-1}^+\phi_n^-)\psi_{n-1} = \varepsilon_n\psi_{n-1} \end{aligned} \tag{3.8}$$

Comparing (3.7) with (3.1) and (3.8) with (3.2), we get three equations for the three unknowns, $\phi_{n-1}^+,\ \phi_n^-$ and ε_n. The three equations are

$$\phi_{n-1}^+ - \phi_n^- \ = 0 \tag{3.9}$$

and

$$p(\phi_{n-1}^+)' - \phi_{n-1}^+\phi_n^- \ - \varepsilon_n - p\lambda_{n-1}(z) = 0 \tag{3.10}$$

and

$$p(\phi_n^-\ )' + \phi_{n-1}^+\phi_n^- \ + \varepsilon_n + p\lambda_n(z) = 0 \tag{3.11}$$

Equations (3.9), (3.10) and (3.11) can be solved in an identical manner to equations (2.9), (2.10) and (2.11), to find that

$$\phi_n^- = \phi_{n-1}^+ = \varphi_{n-1}^+ + \frac{1}{2}(p' - q) = \varphi_n^- - \frac{1}{2}(p' - q) \tag{3.12}$$

and

$$\varepsilon_n = E_n \tag{3.13}$$

Note, in particular, that $\varepsilon_0 = 0$ again. Examples of the above factorization format, for many important equations, are summarized in Table 2. Since $\varepsilon_0 = 0$ is a *result* in this approach, we can find ψ_0, via (3.6), from

$$[p(z)\frac{d}{dz} - \phi_0^-]\ \psi_0 = 0 \tag{3.14}$$

and hence $\psi_n,\ n \geq 1$, from (3.5).

As before, we can develop, now, another *three-term recurrence* formula. Eliminating the derivative terms from equations (3.5) and (3.6) leaves

$$\psi_{n+1} = (\phi_n^+ + \phi_n^-)\psi_n + \varepsilon_n\psi_{n-1} \tag{3.15}$$

which is indeed another three-term recurrence relation.

All this is as it should be. However, a *simplification* is possible here, as we have a *given* relation between ψ_n and y_n in the form of (1.5). In fact, if we drop the K_n normalization factors and substitute (1.5) into (2.5) and (2.6), with (1.4) in mind, then we find that

$$[p(z)\frac{d}{dz} + \varphi_n^+ + \frac{1}{2}(p' - q)]\ \psi_n = \psi_{n+1} \tag{3.16}$$

and

$$[p(z)\frac{d}{dz} - \varphi_n^- + \frac{1}{2}(p' - q)]\ \psi_n = E_n\psi_{n-1} \tag{3.17}$$

from which our results follow, by comparison with (3.5) and (3.6). Compare this with [6].

4. The Associated n-Factorization Formalism

We move on now and consider two possible factorizations of the associated equation, (1.8). First, we determine the enforced 'n-factorizations' [3, 4] of the associated equation (1.8) in the forms

$$p(z)[p(z)y''_{n,m} + q(z)y'_{n,m} + \lambda_{n,m}(z)y_{n,m}] = 0 \tag{4.1}$$

and

$$p(z)[p(z)y''_{n-1,m} + q(z)y'_{n-1,m} + \lambda_{n-1,m}(z)y_{n-1,m}] = 0 \tag{4.2}$$

as

$$[p(z)\frac{d}{dz} + \varphi^{+}_{n-1,m}][p(z)\frac{d}{dz} - \varphi^{-}_{n,m}]\ y_{n,m} = E_{n,m}y_{n,m} \tag{4.3}$$

and

$$[p(z)\frac{d}{dz} - \varphi^{-}_{n,m}][p(z)\frac{d}{dz} + \varphi^{+}_{n-1,m}]\ y_{n-1,m} = E_{n,m}y_{n-1,m} \tag{4.4}$$

respectively, by defining the ladder-operators (note the change $\varphi^{+}_{n-1,m} \to \varphi^{+}_{n,m}$)

$$[p(z)\frac{d}{dz} + \varphi^{+}_{n,m}]\ y_{n,m} = y_{n+1,m} \tag{4.5}$$

and

$$[p(z)\frac{d}{dz} - \varphi^{-}_{n,m}]\ y_{n,m} = E_{n,m}y_{n-1,m} \tag{4.6}$$

with the $E_{n,m}$ independent of the variable z. The problem is to find $\varphi^{+}_{n-1,m}$, $\varphi^{-}_{n,m}$ and $E_{n,m}$. Expanding (4.4) and (4.5), we get

$$\begin{aligned} p^2y''_{n,m} + p(p' + \varphi^{+}_{n-1,m} - \varphi^{-}_{n,m})y'_{n,m} \\ - (p(\varphi^{-}_{n,m})' + \varphi^{+}_{n-1,m}\varphi^{-}_{n,m})y_{n,m} = E_{n,m}y_{n,m} \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} p^2y''_{n-1,m} + p(p' + \varphi^{+}_{n-1,m} - \varphi^{-}_{n,m})y'_{n-1,m} \\ + (p(\varphi^{+}_{n-1,m})' - \varphi^{+}_{n-1,m}\varphi^{-}_{n,m})y_{n-1,m} = E_{n,m}y_{n-1,m} \end{aligned} \tag{4.8}$$

Comparing (4.7) with (4.1) and (4.8) with (4.2), we get three equations for the three unknowns, $\varphi^{+}_{n-1,m}$, $\varphi^{-}_{n,m}$ and $E_{n,m}$.The three equations are

$$\varphi^{+}_{n-1.m} - \varphi^{-}_{n,m}\ + p' - q = 0 \tag{4.9}$$

and

$$p(\varphi^{+}_{n-1,m})' - \varphi^{+}_{n-1,m}\varphi^{-}_{n,m}\ - E_{n,m} - p\lambda_{n-1,m} = 0 \tag{4.10}$$

and

$$p(\varphi^{-}_{n,m}\ )' + \varphi^{+}_{n-1,m}\varphi^{-}_{n,m}\ + E_{n,m} + p\lambda_{n,m} = 0 \tag{4.11}$$

Equations (4.8), (4.9) and (4.10) can be solved in an identical manner to equations (2.9), (2.10) and (2.11), to find that

$$\varphi^{+}_{n-1,m} = \varphi^{+}_{n-1} - \frac{m[q'p' - p''q]}{2\ [(n-1)p'' + q']} \tag{4.12}$$

and

$$\varphi^{-}_{n,m}\ = \varphi^{-}_{n} - \frac{m[q'p' - p''q]}{2\ [(n-1)p'' + q']} \tag{4.13}$$

with

$$E_{n,m} = E_n\ - \frac{\begin{aligned} m[(2n-m-2)p'' + 2q'][2(q' + (n-1)p'')^2p \\ -(q + (n-1)p')([2q' + (n-1)p'']p' - p''q)] \end{aligned}}{4(q' + (n-1)p'')^2} \tag{4.14}$$

Note, in this case, that $E_{m,m} = 0$ *identically* and, again, $\varphi^+_{n-1,m}$ and $\varphi^-_{n,m}$ are manifestly *linear functions* of z . As with E_n, the fact that $E_{n,m}$ is indeed independent of z, can be verified by multiplying-out the top line of the fraction in (4.14), when the z and z^2 terms are found to *cancel-out*. Examples of the above factorization format, for many important equations, are summarized in Table 3. Since $E_{m,m} = 0$, is a *result* in this approach, we can find $y_{m,m}$, via (4.6), from

$$[p(z)\frac{d}{dz} - \varphi^-_{m,m}]\ y_{m,m} = 0 \tag{4.15}$$

and hence $y_{m+1,m}$, $0 \le m \le n$,from (4.5).

Finally, we can develop, now, another *three-term recurrence* formula. Eliminating the derivative terms from equations (4.5) and (4.6) leaves

$$y_{n+1,m} = (\varphi^+_{n,m} + \varphi^-_{n,m})y_{n,m} + E_{n,m}y_{n-1,m} \tag{4.16}$$

which is indeed another three-term recurrence relation.

5. The Associated m-Factorization Formalism

We consider, now, the enforced 'm-factorizations' [3, 4] of the associated equation, (1.8), in the standard 'twin' format

$$p(z)y''_{n,m} + q(z)y'_{n,m} + \lambda_{n,m}(z)y_{n,m} = 0 \tag{5.1}$$

and

$$p(z)y''_{n,m-1} + Q(z)y'_{n,m-1} + \lambda_{n,m-1}(z)y_{n,m-1} = 0 \tag{5.2}$$

as

$$[p^{\frac{1}{2}}(z)\frac{d}{dz} + \phi^+_{n,m-1}][p^{\frac{1}{2}}(z)\frac{d}{dz} - \phi^-_{n,m}]\ y_{n,m} = \varepsilon_{n,m}y_{n,m} \tag{5.3}$$

and

$$[p^{\frac{1}{2}}(z)\frac{d}{dz} - \phi^-_{n,m}][p^{\frac{1}{2}}(z)\frac{d}{dz} + \phi^+_{n,m-1}]\ y_{n,m-1} = \varepsilon_{n,m}y_{n,m-1} \tag{5.4}$$

respectively, by defining the ladder-operators (note the change $\phi^+_{n,m-1} \to \phi^+_{n,m}$)

$$[p^{\frac{1}{2}}(z)\frac{d}{dz} + \phi^+_{n,m}]\ y_{n,m} = y_{n,m+1} \tag{5.5}$$

and

$$[p^{\frac{1}{2}}(z)\frac{d}{dz} - \phi^-_{n,m}]\ y_{n,m} = \varepsilon_{n,m}y_{n,m-1} \tag{5.6}$$

with the $\varepsilon_{n,m}$ independent of the variable z. The problem is to find $\phi^+_{n,m-1}$, $\phi^-_{n,m}$ and $\varepsilon_{n,m}$. Expanding (5.3) and (5.4), we get

$$\begin{aligned} py''_{n,m} + [\tfrac{1}{2}p' + p^{\frac{1}{2}}(\phi^+_{n,m-1} - \phi^-_{n,m})]y'_{n,m} \\ - [p^{\frac{1}{2}}(\phi^-_{n,m})' + \phi^+_{n,m-1}\phi^-_{n,m}]\ y_{n,m} = \varepsilon_{n,m}y_{n,m} \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} py''_{n,m-1} + [\tfrac{1}{2}p' + p^{\frac{1}{2}}(\phi^+_{n,m-1} - \phi^-_{n,m})]y'_{n,m-1} \\ + (p^{\frac{1}{2}}(\phi^+_{n,m-1})' - \phi^+_{n,m-1}\phi^-_{n,m})y_{n,m-1} = \varepsilon_{n,m}y_{n,m-1} \end{aligned} \tag{5.8}$$

Comparing (5.7) with (5.1) and (5.8) with (5.2), we get three equations for the three unknowns, $\phi^+_{n,m-1}$, $\phi^-_{n,m}$ and $\varepsilon_{n,m}$. The three equations are

$$p^{\frac{1}{2}}(\phi^+_{n.m-1} - \phi^-_{n,m})\ + \frac{1}{2}p' - q = 0 \tag{5.9}$$

and

$$p^{\frac{1}{2}}(\phi^+_{n,m-1})' - \phi^+_{n,m-1}\phi^-_{n,m}\ - \varepsilon_{n,m} - \lambda_{n,m-1} = 0 \tag{5.10}$$

and

$$p^{\frac{1}{2}}(\phi^-_{n,m})' + \phi^+_{n,m-1}\phi^-_{n,m} + \varepsilon_{n,m} + \lambda_{n,m} = 0 \tag{5.11}$$

At first sight we have a rather more difficult problem that the previous cases. However, as in section 3, in this case we have the additional information, from (1.5), that (again, compare the following analysis with Cotfas [1, 2])

$$y_{n,m}(z) = (-1)^m p^{\frac{m}{2}}(z) y_n^{(m)}(z) \tag{5.12}$$

In fact, plugging (5.12) into (5.5), differentiating and simplifying, we find that

$$\phi^+_{n,m-1} = -\frac{m}{2}p^{-\frac{1}{2}}p' \equiv -m(p^{\frac{1}{2}})' \tag{5.13}$$

If we now substitute from (5.13) for $\phi^+_{n,m-1}$ in (5.9), we may solve for $\phi^-_{n,m}$ as

$$\phi^-_{n,m} = -p^{-\frac{1}{2}}\left(\frac{m-1}{2}p' + q\right) \tag{5.14}$$

Finally, substituting for $\phi^+_{n,m-1}$ and $\phi^-_{n,m}$ in (5.11), we find that

$$\varepsilon_{n,m} = \lambda_m - \lambda_n = \frac{1}{2}(n-m)[(m+n-1)p'' + q'] \tag{5.15}$$

The rest of the factorization formalism, equations (5.10) and (5.11), become, then, consistency conditions or a check on the factorization solution, that is, equations (5.13), (5.14) and (5.15). As before, we have determined the ϕ in terms of $p(z)$ and $q(z)$, while, from (5.15), we see that $\varepsilon_{m,m} = 0$, *identically.*

Examples of the above factorization format, for many important equations, are summarized in Table 4. We have now, with $\varepsilon_{m,m} = 0$, the standard ladder-operator sequence, beginning with, from (5.6)

$$[p^{\frac{1}{2}}(z)\frac{d}{dz} - \phi^-_{m,m}]\, y_{m,m} = 0 \tag{5.16}$$

from which we get $y_{m,m}(z)$, $0 \le m \le n$, with all other solutions of (1.8) coming through the (repeated) application of (5.5).

In this instance, our *three term recurrence* relation is

$$y_{n+1,m} = (\phi^+_{n,m} + \phi^-_{n,m})y_{n,m} + \varepsilon_{n,m}y_{n-1,m} \tag{5.17}$$

which follows, as usual, from the elimination of the derivative term from (5.5) and (5.6).

6. Discussion and conclusions

In the previous sections of this paper we have re-examined a basic and important problem in mathematical physics, that is, the development of a factorization formalism for the solution of equations (1.1), (1.6) and (1.8) (with the coefficients of (1.1), (1.6) and (1.8) subject to certain restrictions). The basic advance of the current approach to this problem is the *direct* solution of the 'determining equations' for the factorization representation: equations (2.9) to (2.11), (3.9) to (3.11), (4.9) to (4.11) and (5.10) to (5.12). The solution process is both 'cleaner' and simpler than the original method of Jafarizadeh and Fakhri [3, 4], described in more detail below, in that the solution process is *explicit* and two important points are derived rather than assumed: we prove the '$\varphi's$' are linear and the '$E's$' vanish as necessary. Further, in sections 3 and 5, in the cases of the normalized equation (1.6) and the 'm-factorization' of the associated equation (1.8), the factorization process is simplified to the extent of being *almost* trivial,

something which cannot be said of the original method. One further small, but practical point, is that the resulting factorization formalism is given in terms of the coefficients of the original second-order linear differential equations *directly*. Although this *is* a small point, it makes applying the factorization formula, to any particular instance of a differential equation, somewhat simpler.

In fact, an examination of the Jafarizadeh and Fakhri approach [3] shows (in our notation) that they consider only *two* equations

$$\varphi_{n-1}^{+} - \varphi_{n}^{-} + p' - q = 0 \tag{6.1}$$

and

$$p(\varphi_{n}^{-})' + \varphi_{n-1}^{+}\varphi_{n}^{-} + E_n + p\lambda_n = 0 \tag{6.2}$$

for example, in *three* unknowns, with the missing equation

$$p(\varphi_{n-1}^{+})' - \varphi_{n-1}^{+}\varphi_{n}^{-} - E_n - p\lambda_{n-1} = 0 \tag{6.3}$$

lying 'dormant' in their formalism. Of course the critical point in the present approach is that *both* (6.2) and (6.3), along with (6.1), are required to prove that the '$\varphi' s$' are *linear* functions.

Without (6.3), it proves necessary to adopt a different, more awkward solution process and to make further assumptions to follow this through [3]. (See, also, [5] and a paper by Rainville [10], for a discussion of elements of the technique used by Jafarizadeh and Fakhri [3].) Matters get more involved still if the Jafarizadeh and Fakhri [4] methodology is used in the development of the 'm-factorization' of the associated equation (1.8), instead of the much simplified analysis, following Cotfas [1, 2], presented in section 5 (Jafarizadeh and Fakhri do not consider equation (1.6)).

In addition to Jafarizadeh and Fakhri [3, 4], there has been, of course, considerable attention given to the factorization problem associated with equations (1.1), (1.6) and (1.8) by many other workers, but we will only discuss just a few of the alternative approaches to tackling this problem that have been proposed (apologies to the *many* authors omitted). The works quoted below are considered more relevant to the present approach.

Beginning with Pina [9], the 'Mexican school' [9, 8, 11, 12] have tackled the basic factorization and ladder-operator problem by a *direct* factorization of (1.1) involving taking $\sqrt{p(x)}$. The formulae obtained in this factorization methodology differ from that obtained here, but, of course, the factorizations of the various specific cases in Table 1 are the same (as far as they go). Overall the work of the 'Mexican school' constitutes a 'parallel' tradition to that of Jafarizadeh and Fakhri and represents a significant contribution to the theory of special functions.

Building directly on the work of Nikiforov and Uvarov [7], Lorente [6] has described an approach to the derivation of 'ladder-operator factorizations' for 'hypergeometric-type' equations, essentially equations (1.1) and (1.6). The starting point of the Nikiforov and Uvarov approach [7] (and hence that of Lorente [6]) is, essentially, the well-known Rodrigues formula solution to (1.1), that is (actually Nikiforov and Uvarov [7] deal with a complex form of (6.4))

$$y_n(z) = \frac{K_n}{w(z)} \left(\frac{d}{dz}\right)^n [w(z)p^n(z)] \tag{6.4}$$

with K_n independent of z.

In detail, Lorente produces 'ladder operators' for the solutions of (1.1) and (1.6) from the recurrence relations of Nikiforov and Uvarov [7] and the ladder operators are then used to expose a ladder-operator factorization for (1.1) and (1.6) [6]. This technique for constructing the ladder-operator factorization of (1.1) (and (1.6)), which is effectively the opposite of the method promoted in the present work, should be compared with that of Kaufman [5], who also started with 'ladder-operator relations' and proceeded to construct second-order ODE in the manner of Lorente. Kaufman [5], however, considered only special cases of ladder-operators, which he considered as *given*. Nikiforov and Uvarov [7] derive the recurrence relations through a general analysis via contour integration – the results quoted and applied by Lorente appearing as special cases of this more general analysis.

In fact Nikiforov and two of his colleagues [13] have also derived the ladder-operator relations and the three term recurrence relation, (2.22) in our case, for equation (1.1). This analysis is again based on the work presented in Nikiforov and Uvarov [7] and is dependent on the manipulation of contour integrals. Of course, all of the works mentioned above come to the same conclusions, with (essentially) the same formulae, regardless of the difference in the starting point or overall methodology, though not all authors develop all of the possibilities of their formalism. For example, Jafarizadeh and Fakhri [3, 4] make no mention of the three term recurrence relation and Yanez et al [13] do not develop their formalism for equations (1.4) or for equation (1.6) and so on!

Another development of ladder-operators, independent of complex variable analysis is due to Cotfas [1, 2], whom we have invoked already. Cotfas [1, 2] has been particular active in the analysis of equation (1.8) and its applications in quantum mechanics. In particular, Cotfas [1, 2] obtained the m-ladder-operator representation of (1.8) directly from (1.9), *independent* of the factorization formalism of section 5. Naturally, the factorization formalism is consistent with Cotfas' results.

Finally, we note that we have developed particular solutions to the Riccati equations that underpin our various factorization results. More general solutions can be developed and applied to factorization techniques: see [11] for details.

TABLE 1. Important ODE and Their Factorization Formulae

Equation	*EquationForm*	*FactorizedForm*
Legendre	$(1-z^2)P_n'' - 2zP_n' + n(n+1)P_n = 0$	$[(1-z^2)\frac{d}{dz} - nz][(1-z^2)\frac{d}{dz} + nz]P_n = -n^2P_n$
Hermite	$H_n'' - 2zH_n' + 2nH_n = 0$	$(\frac{d}{dz} - z)(\frac{d}{dz})H_n = -2nH_n$
Laguerre	$zL_n'' + (1-z)L_n' + nL_n = 0$	$(z\frac{d}{dz} + n - z)(z\frac{d}{dz} - n)L_n = -n^2L_n$
ChebyshevI	$(1-z^2)T_n'' - zT_n' + n^2T_n = 0$	$[(1-z^2)\frac{d}{dz} - (n-1)z][(1-z^2)\frac{d}{dz} + nz]T_n = -n(n-1)T_n$
ChebyshevII	$(1-z^2)U_n'' - 3zU_n' + n(n+2)U_n = 0$	$[(1-z^2)\frac{d}{dz} - (n+1)z][(1-z^2)\frac{d}{dz} + nz]U_n = -n(n+1)U_n$
Ultrasherical	$(1-z^2)C_n'' - (2a+1)zC_n' + n(n+2a)C_n = 0$	$[(1-z^2)\frac{d}{dz} - (n+2a-1)z][(1-z^2)\frac{d}{dz} + nz]C_n = -n(n+2a-1)C_n$
GeneralizedLaguerre	$zL_{n,a}'' + (a+1-z)L_{n,a}' + nL_{n,a} = 0$	$(z\frac{d}{dz} + a + n - z)(z\frac{d}{dz} - n)y_n = -n(n+a)y_n$
Bessel	$z^2J_n'' + (az+b)J_n' - n(n+a-1)J_n = 0$	$(z^2\frac{d}{dz} + (a+n-2)z + \frac{(a+n-2)b}{(a+2n-2)})\times (z^2\frac{d}{dz} - nz + \frac{nb}{(a+2n-2)})J_n = \frac{n(a+n-2)b^2}{(a+2n-2)^2}J_n$
Romanovsky	$(a^2+z^2)R_n'' + (c+2(b+1)z)\,R_n' - n(n+2b+1)R_n = 0$	$[(a^2+z^2)\frac{d}{dz} + (n+2b)z + \frac{(n+2b)c}{2(n+b)}]\times [(a^2+z^2)\frac{d}{dz} - nz + \frac{cn}{2(n+b)}]R_n = \frac{n(2b+n)[4a^2(b+n)^2+c^2]}{4(b+n)^2}R_n$
Jacobi	$(1-z^2)P_{n,a,b}'' + [b-a-(a+b+2)z]\,P_{n,a,b}' + n(a+b+n+1)P_{n,a,b} = 0$	$[(1-z^2)\frac{d}{dz} - (a+b+n)z - \frac{(a+b+n)(a-b)}{(a+b+2n)}] \times [(1-z^2)\frac{d}{dz} + nz - \frac{n(a-b)}{(a+b+2n)}]\,P_{n,a,b} = -\frac{4n(a+n)(b+n)(a+b+n)}{(a+b+2n)^2}P_{n,a,b}$

TABLE 2. Important ODE and their Normalized Factorization Formulae

Equation	*EquationForm*	*FactorizedForm*
Legendre	$(1-z^2)\Pi_n'' - 2z\Pi_n + \ n(n+1)\Pi_n = 0$	$[(1-z^2)\frac{d}{dz} - nz][(1-z^2)\frac{d}{dz} + nz]\ \Pi_n = -n^2\Pi_n$
Hermite	$\mathrm{H}_n'' + (1-z^2+2n)\mathrm{H}_n = 0$	$[\frac{d}{dz} - z][\frac{d}{dz} + z]\ \mathrm{H}_n = -2n\mathrm{H}_n$
Laguerre	$z\Lambda_n'' + \Lambda_n' + [n + \frac{1}{2} - \frac{1}{4}z]\Lambda_n = 0$	$[z\frac{d}{dz} - \frac{1}{2}z + \frac{1}{2}n][z\frac{d}{dz} + \frac{1}{2}z - \frac{1}{2}n]\ \Lambda_n = -n^2\Lambda_n$
ChebyshevI	$(1-z^2)\Theta_n'' - 2z\Theta_n' + [n^2 - \frac{1}{2} - \frac{1}{4}\frac{z^2}{1-z^2}]\Theta_n = 0$	$[(1-z^2)\frac{d}{dz} - (n-\frac{1}{2})z][(1-z^2)\frac{d}{dz} + (n-\frac{1}{2})z]\ \Theta_n = -n(n-1)\Theta_n$
ChebyshevII	$(1-z^2)\Upsilon_n'' - 2z\Upsilon_n' + [n(n+2) + \frac{1}{2} - \frac{1}{4}\frac{z^2}{1-z^2}]\Upsilon_n = 0$	$[(1-z^2)\frac{d}{dz} - (n+\frac{1}{2})z][(1-z^2)\frac{d}{dz} + (n+\frac{1}{2})z]\ \Upsilon_n = -n(n+1)\Upsilon_n$
Ultrasherical	$(1-z^2)\Gamma_n'' - 2z\Gamma_n' + [\ n(n+2a) + \frac{1}{2}(2a-1) - \frac{1}{4}\frac{(2a-1)^2z^2}{1-z^2}]\Gamma_n = 0$	$[(1-z^2)\frac{d}{dz} - \frac{1}{2}(2n+2a-1)z]\times [(1-z^2)\frac{d}{dz} + \frac{1}{2}(2n+2a-1)z]\ \Gamma_n = -n(n+2a-1)\Gamma_n$
GeneralizedLaguerre	$z\Lambda_{n,a}'' + \Lambda_{n,a}' + [n - \frac{1}{4}(z - 2a - 2 + \frac{a^2}{z}]\Lambda_{n,a} = 0$	$[z\frac{d}{dz} - \frac{1}{2}z + \frac{1}{2}(2n+a)][z\frac{d}{dz} + \frac{1}{2}z - \frac{1}{2}(2n+a)]\Lambda_{n,a} = -n(n+a)\Lambda_{n,a}$
Bessel	$z^2\mathrm{I}_n'' + 2z\mathrm{I}_n' - \ [n(n+a-1) + \frac{1}{2}(a-2) + \frac{1}{4}\frac{[(a-2)z+b]^2}{z^2}]\mathrm{I}_n = 0$	$[z^2\frac{d}{dz} + \frac{1}{2}(2n+a-2)z + \frac{b(a-2)}{2(2n+a-2)}]\times [z^2\frac{d}{dz} - \frac{1}{2}(2n+a-2)z - \frac{b(a-2)}{2(2n+a-2)}]\ \mathrm{I}_n = \frac{n(n+a-2)b^2}{(2n+a-2)^2}\mathrm{I}_n$
Romanovsky	$(a^2+z^2)\mathrm{P}_n'' + 2z\ \mathrm{P}_n' - [n(n+2b+1) + b + \frac{b^2z^2}{a^2+z^2}]\mathrm{P}_n = 0$	$[(a^2+z^2)\frac{d}{dz} + (n+b)z][(a^2+z^2)\frac{d}{dz} - (n+b)z]\ \mathrm{P}_n = \frac{n(2b+n)[4a^2(b+n)^2+c^2]}{4(b+n)^2}\mathrm{P}_n$
Jacobi	$(1-z^2)\Pi_{n,a,b}'' - 2z\Pi_{n,a,b}' + [n(a+b+n+1) + \frac{1}{2}(a+b) - \frac{1}{4}\frac{(b-a-(a+b)z)^2}{1-z^2}]\Pi_{n,a,b} = 0$	$[(1-z^2)\frac{d}{dz} - \frac{1}{2}(2n+a+b)z + \frac{b^2-a^2}{2(2n+a+b)}]\times [(1-z^2)\frac{d}{dz} + \frac{1}{2}(2n+a+b)z - \frac{b^2-a^2}{2(2n+a+b)}]\ \Pi_{n,a,b} = -\frac{4n(a+n)(b+n)(a+b+n)}{(a+b+2n)^2}\Pi_{n,a,b}$

TABLE 3. Important Associated ODE and Their n-Factorization Formulae

Equation	*EquationForm*	*FactorizedForm*
Legendre	$(1-z^2)P''_{n,m} - 2z\ P'_{n,m} + \left[n(n-1) - \frac{m^2}{1-z^2}\right]\ P_{n,m} = 0$	$[(1-z^2)\frac{d}{dz} - nz][(1-z^2)\frac{d}{dz} + nz]P_{n,m} = -(n-m)^2 P_{n,m}$
Hermite	$H''_{n,m} - 2z\ H'_{n,m} + 2(n-m)H_{n,m} = 0$	$(\frac{d}{dz} - z)(\frac{d}{dz})H_{n,m} = -2(n-m)H_{n,m}$
GeneralizedLaguerre	$zL''_{n,m} + (a+1-z)L'_{n,m} + \frac{1}{2}[(2n-m) - \frac{m(2a+m)}{2z}]\ L_{n,m} = 0$	$(z\frac{d}{dz} + a + n - \frac{m}{2} - z)\times (z\frac{d}{dz} - n + \frac{m}{2})L_{n,a} = -n(n+a)L_{n,m}$
Bessel	$z^2 J''_{n,m,a,b} + [(\mathrm{a}+2)\mathrm{z} + \mathrm{b}]\ J'_{n,m,a,b} - [n(a+n+1) + \frac{mb}{z}]\ J_{n,m,a,b} = 0$	$z^2 J''_{n,m,a,b} + [(\mathrm{a}+2)\mathrm{z} + \mathrm{b}]\ J'_{n,m,a,b} - [n(a+n+1) + \frac{mb}{z}]\ J_{n,m,a,b} = 0$
Jacobi	$(1-z^2)P''_{n,m,a,b} + [b-a-(a+b+2)z]\ \ P'_{n,m,a,b} + \left[n(n+a+b+1) - m(m+a+b) - \frac{m[(b-a)-(m+a+b)z]\ z}{1-z^2}\right]\ P_{n,m,a,b} = 0$	$[(1-z^2)\frac{d}{dz} - (a+b+n)z - \frac{(a+b+n+m)(a-b)}{(a+b+2n)}]\times [(1-z^2)\frac{d}{dz} + nz - \frac{(n-m)(a-b)}{(a+b+2n)}]\ P_{n,m,a,b} = -\frac{4(n-m)(a+n)(b+n)(a+b+n-m)}{(a+b+2n)^2}P_{n,m,a,b}$

TABLE 4. Important Associated ODE and Their m-Factorization Formulae

Equation	*EquationForm*	*FactorizedForm*
Legendre	$(1-z^2)P''_{n,m} - 2z\ P'_{n,m} + \left[n(n-1) - \frac{m}{1-z^2}\right]\ P_{n,m} = 0$	$[\sqrt{(1-z^2)}\frac{d}{dz} + \frac{m}{\sqrt{(1-z^2)}}][\sqrt{(1-z^2)}\frac{d}{dz} - \frac{(m+1)z}{\sqrt{(1-z^2)}}]\ P_{n,m} = (m-n)(m+n+1)P_{n,m}$
Hermite	$H''_{n,m} - 2z\ H'_{n,m} + 2(n-m)H_{n,m} = 0$	$[\frac{d}{dz}][\frac{d}{dz} - 2z]\ H_{n,m} = 2(m-n)H_{n,m}$
GeneralizedLaguerre	$zL''m + (a+1-z)L'_{n,m} + \frac{1}{2}[(2n-m) - \frac{m(2a+m)}{2z}]\ L_{n,m} = 0$	$[\sqrt{z}\frac{d}{dz} - \frac{m}{2\sqrt{z}}][\sqrt{z}\frac{d}{dz} + \frac{1}{2\sqrt{z}}\,(m+2a+1-2z)]\ L_{n,m} = (m-n)L_{n,m}$
Bessel	$z^2 J''_{n,m,a,b} + [(\mathrm{a}+2)\mathrm{z} + \mathrm{b}]\ J'_{n,m,a,b} - [n(a+n+1) + \frac{mb}{z}]J_{n,m,a,b} = 0$	$[z\frac{d}{dz} - m][z\frac{d}{dz} + \frac{b}{z} + (m+a+1)]J_{n,m,a,b} = -(m-n)(n+m+a+1)J_{n,m,a,b}$
Jacobi	$(1-z^2)P''_{n,m,a,b} + [b-a-(a+b+2)z]\ P'_{n,m,a,b} + \begin{bmatrix} n(n+a+b+1) \\ -m(m+a+b) - \frac{m[(b-a)-(m+a+b)z]\ z}{1-z^2} \end{bmatrix}\ P_{n,m,a,b} = 0$	$[\sqrt{(1-z^2)}\frac{d}{dz} + \frac{mz}{\sqrt{(1-z^2)}}]\times [\sqrt{(1-z^2)}\frac{d}{dz} + \frac{1}{\sqrt{(1-z^2)}}\,(b-a-(m+a+b+1)z)]P_{n,m,a,b} = (m-n)(m+a+b+1)P_{n,m,a,b}$

References

1. N. Cotfas, *Shape invariant hypergeometric type operators with application to quantum mechanics*, Cent. Eur. J. Phys **4** (2006), 318-330.
2. N. Cotfas, *Systems of orthogonal polynomials defined by hypergeometric type equations with application to quantum mechanics*, Cent. Eur. J. Phys **2** (2004), 456-466.
3. M. A, Jafarizadeh and H. Fakhri,*Supersymmetry and shape invariance in differential equations of mathematical physics*, Phys. Lett. A **230** (1997), 164-170.
4. M. A, Jafarizadeh and H. Fakhri,*Parasupersymmetry and shape invariance in differential equations of mathematical physics and quantum mechanics*, Ann. Phys. (NY) **262** (1998), 260-276.
5. B. Kaufman, *Special Functions of Mathematical Physics from the Viewpoint of Lie Algebra* J. Math. Phys. **7** (1966), 447-458.
6. M. Lorente, *Raising and lowering operators, factorization and differential/difference operators of hypergeometric type* , J. Phys. A: Math. Gen. **34** (2001), 569-588.
7. A. F. Nikiforov and V. B. Uvarov, Special Functions of Mathematical Physics, Birkhauser, 1988.
8. A. Perez-Lorenzana, *On the factorization method and ladder operators*, Rev. Mex. Fis. **42** (1996), 1060-1069.
9. E. Pina, *Algunas propiedades de los operadores de escalera*, Rev. Mex. Fis. **41** (1995), 913-924.
10. E. D. Rainville, *Necessary conditions for polynomial solutions of certain Riccati equations*, Am. Math. Mon. **43** (1936), 473-476.
11. M. A. Reyes, D. Jimenez and H. C. Rosu, *One-parameter isospectral special functions*, Rev. Mex. Fis. **49** (2003), 358-363.
12. H. C. Rosu, M. Reyes and O. Obregon, *Recurrence-shift relations for the polynomial functions of Aldaya, Bisquert, and Navarro-Salas*, Rev. Mex. Fis. **43** (1997), 224-231.
13. R. J. Yanez, J. s¿ Dehesa and A. F. Nikiforov *The three-term recurrence relation and the differentiation formulas for hypergeometric-type functions*, J. Math. Anal. App. **188** (1994), 855-866.

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