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PAIRED-KANNAN CONTRACTION MAPPINGS AND FIXED POINT RESULTS

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Abstract. We introduce a novel contraction concept for mappings within metric spaces called Paired-Kannan contraction. Unlike traditional Kannan contraction mappings, which involve two points, Paired-Kannan contraction mappings extend this concept to three points. We explore their properties, noting that while these mappings may generally be discontinuous, they exhibit continuity at fixed points akin to Kannan contractions. Importantly, we establish that Paired-Kannan contraction mappings constitute distinct entities from traditional Kannan contractions. We prove a fixed point theorem for Paired Kannan contraction mappings and show that conditions like asymptotic regularity and continuity extend these theorems' applicability. Additionally, we derive two new fixed point theorems for these mappings, applicable even in non-complete metric spaces.

1. Introduction

In [15], Kannan established a result that provides a fixed point for mappings that are discontinuous. The result states that if a mapping $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$ is defined over the complete metric space $(\mathfrak{W}, \mathsf{d})$, then

$$\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) \leq \eta(\mathsf{d}(\kappa, \mathfrak{F}\kappa) + \mathsf{d}(\varrho, \mathfrak{F}\varrho)) \tag{1.1}$$

where $0 \leq \eta < \frac{1}{2}$ and $\kappa, \varrho \in \mathfrak{W}$. Then $\mathfrak{F}$ has only one fixed point.

Definition 1.1. A mapping $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$ is termed as Kannan type mapping if a constant can be found $0 \leq \eta < 1/2$ such that the inequality (1.1) is satisfied for all $\kappa, \varrho \in \mathfrak{W}$.

Further, Subrahmanyam [29] demonstrated that the result established by Kannan serves as a characterization of metric completeness. To be more precise, a metric space $\mathfrak{W}$ turns out to be complete if and only if any mapping of Kannan type on $\mathfrak{W}$ possesses a fixed point. It is worth noting that while Banach contractions are important, they don't necessarily imply metric completeness. For a specific example illustrating this concept, refer to [9], where a metric space $\mathfrak{W}$ is discussed having the property that, despite every contraction on $\mathfrak{W}$ possesses a fixed-point, the metric space itself is not complete. For further exploration into

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the topic, one might look at papers [3, 18, 27, 28, 30], which examine the differences and similarities between mappings of Kannan contractions and Banach contractions.

In fixed point theory, there are typically three types of expansions of the Kannan theorem:

- In the first type's contractive characteristics is relaxed. One can find examples of this in [4, 8, 12, 14, 19, 24, 25].
- In the second type, the topology is relaxed. Relevant literature includes [1, 2, 11, 13, 17, 20].
- The third type involves theorems related to Kannan-type multivalued mappings, as seen in [10, 21, 31].

A novel type of mappings was introduced in [7], which can be defined as mappings that contract in paired for three pairwise distinct points.

Definition 1.2 ([7])**.** Let $(\mathfrak{W}, \mathsf{d})$ be a metric space, where $|\mathfrak{W}| \geq 3$. A mapping $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$ is termed a Paired Contraction (PC) mapping on $\mathfrak{W}$ if there exists a $\delta \in [0, 1)$ such that, for any three distinct points $\kappa, \varrho, \upsilon \in \mathfrak{W}$, the inequality

$$\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) + \mathsf{d}(\mathfrak{F}\varrho, \mathfrak{F}\upsilon) \leq \delta(\mathsf{d}(\kappa, \varrho) + \mathsf{d}(\varrho, \upsilon))$$

is satisfied.

It is crucial that $\kappa, \varrho, \upsilon \in \mathfrak{W}$ are pairwise distinct. Without this condition, the definition would reduce to that of a standard contraction mapping.

For such mappings, a fixed-point theorem was proved. While the proof draws inspiration from Banach's classical theorem, the key distinction is that these mappings being defined based on three points in the space instead of two. Furthermore, a condition is required to avoid periodic points of prime period 2 in these mappings. Notably, ordinary contraction mappings are a significant subset of these mappings.

This manuscript introduces a variation of Kannan type mappings that focus on three points, which draws inspiration from [7, 22, 23], and we refer to it as Paired-Kannan Contraction (PKC). In Examples 2.6 and 3.3, we prove that the Kannan type mappings and Paired-Kannan type mappings are two different and distinct categories.

In Section 2, we have stated Paired-Kannan Contraction mappings and explore the relationship between Paired-Kannan type mappings, Kannan type mappings and Paired-Contraction mappings. Furthermore, we present Example 3.4 of Paired-Kannan type mapping that is discontinuous.

In Section 3, we propose Theorem 3.2, which is primary consequence of this article, which presents a fixed-point for PKC mappings. Notably, this result states that there can only be a maximum of two fixed point. In addition, we also establish that the PKC mappings exhibit continuity at their fixed-points.

In Section 4, we explore asymptotically regular PKC mappings. The notion of asymptotic regularity allows us to broaden the range of parameter λ in equation (2.1) from $[0, 1/2)$ to $[0, 1)$ in the fixed point theorem, as presented in Theorem 4.3. Furthermore, by imposing an additional condition of continuity, in Theorem

4.5, we derive a fixed point result for $\mathcal{F}$-Paired Kannan type mappings. Corollary 4.8 demonstrates the extension of the parameter λ range to $[0, \infty)$.

In Section 5, we have proved one result for the mapping $\mathfrak{F}$ which has approximate fixed point sequence.

In Section 6, based on Kannan's work [16], we offer two new fixed point results for PKC mappings. With the first theorem (Theorem 6.1), we eliminate the necessity of the metric space $\mathfrak{W}$ being complete. In the second theorem (Theorem 6.2), we require the mapping $\mathfrak{F}$ to be continuous throughout space, with condition (2.1) holding only on a dense subset within the space.

2. Paired-Kannan Contraction Mappings and their properties

In this section we will define Paired-Kannan contraction(PKC) mappings and establish the connections between Kannan contraction(KC), Paired- Kannan Contraction(PKC) and Paired Contractions(PC) mappings.

Definition 2.1. Let $(\mathfrak{W}, \mathsf{d})$ be a metric space, where $|\mathfrak{W}| \geq 3$. A self-mapping $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$ is said to be a PKC mapping on $\mathfrak{W}$ if there exists a parameter $\lambda \in [0, \frac{1}{2})$ such that for any three distinct points $\kappa, \varrho, \upsilon \in \mathfrak{W}$, the following inequality is satisfied:

$$\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) + \mathsf{d}(\mathfrak{F}\varrho, \mathfrak{F}\upsilon) \leq \lambda(\mathsf{d}(\kappa, \mathfrak{F}\kappa) + \mathsf{d}(\varrho, \mathfrak{F}\varrho) + \mathsf{d}(\upsilon, \mathfrak{F}\upsilon)). \tag{2.1}$$

Note: If $\mathfrak{F}$ is a PKC mapping over the metric space $(\mathfrak{W}, \mathsf{d})$. Consider the inequality (2.1) for the triplets $(\kappa, \varrho, \upsilon)$, $(\varrho, \upsilon, \kappa)$ and $(\upsilon, \kappa, \varrho)$ (in the same order) and adding right and left part of the inequalities, we get

$$\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) + \mathsf{d}(\mathfrak{F}\varrho, \mathfrak{F}\upsilon) + \mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\upsilon) \leq \frac{3\lambda}{2}(\mathsf{d}(\kappa, \mathfrak{F}\kappa) + \mathsf{d}(\varrho, \mathfrak{F}\varrho) + \mathsf{d}(\upsilon, \mathfrak{F}\upsilon)) \, . \tag{2.2}$$

Proposition 2.2. *KC mappings that have parameter $\lambda \in [0, \frac{3}{8})$ are PKC mapping.*

Proof. Consider a metric space $(\mathfrak{W}, \mathsf{d})$ where $|\mathfrak{W}| \geq 3$. Let $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$ represent a KC mapping. Take three $\kappa, \varrho, \upsilon \in \mathfrak{W}$ pairwise distinct points. Examine the inequality (1.1) for the pairs (κ, ϱ), (ϱ, υ), (κ, υ), and summing the left and right part of inequalities, we obtain:

$$\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) + \mathsf{d}(\mathfrak{F}\varrho, \mathfrak{F}\upsilon) + \mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\upsilon) \leq 2\eta(\mathsf{d}(\kappa, \mathfrak{F}\kappa) + \mathsf{d}(\varrho, \mathfrak{F}\varrho) + \mathsf{d}(\upsilon, \mathfrak{F}\upsilon)) \, . \tag{2.3}$$

To get the maximum possible value of the coefficient η so that mapping $\mathfrak{F}$ is PKC, we compare the coefficients on the right part of the inequalities (2.2) and (2.3). Then

$$2\eta = \frac{3\lambda}{2} \implies \lambda = \frac{4\eta}{3}.$$

Hence,

$$0 \leq \lambda = \frac{4\eta}{3} < \frac{1}{2} \implies 0 \leq \eta < \frac{3}{8},$$

which proves the desired argument. $\square$

Proposition 2.3. *Let $(\mathfrak{W}, \mathsf{d})$ be a metric space, where $|\mathfrak{W}| \geq 3$. If $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$ is a PKC mapping with a parameter $\lambda \in [0, \frac{1}{2})$, and let κ be an accumulation point of $\mathfrak{W}$. Assume that $\mathfrak{F}$ is continuous at κ. In this context, the following inequality*

$$\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) \leq \lambda \left(\mathsf{d}(\kappa, \mathfrak{F}\kappa) + \frac{\mathsf{d}(\varrho, \mathfrak{F}\varrho)}{2} \right) \tag{2.4}$$

is satisfied for any $\varrho \in \mathfrak{W}$.

Proof. Suppose κ is an accumulation point in $\mathfrak{W}$, and let $\varrho \in \mathfrak{W}$ be another element in the set $\mathfrak{W}$. If $\varrho = \kappa$, then there is nothing to prove. Let $\varrho \neq \kappa$. Given that κ is an accumulation point, we can assert the existence of a sequence (υ_n) where $\upsilon_n \neq \kappa$, $\upsilon_n \neq \varrho$, and all υ_n are distinct for every n. Hence, by (2.1) we conclude that

$$\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) + \mathsf{d}(\mathfrak{F}\varrho, \mathfrak{F}\upsilon_n) \leq \lambda(\mathsf{d}(\kappa, \mathfrak{F}\kappa) + \mathsf{d}(\varrho, \mathfrak{F}\varrho) + \mathsf{d}(\upsilon_n, \mathfrak{F}\upsilon_n))$$

is satisfied for each $n \in N$. Given that at κ, $\mathfrak{F}$ is continuous and $\upsilon_n \to \kappa$ as $n \to +\infty$, therefore $\mathfrak{F}\upsilon_n \to \mathfrak{F}\kappa$. We get

$$\begin{aligned}
\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) + \mathsf{d}(\mathfrak{F}\varrho, \mathfrak{F}\kappa) &\leq \lambda(\mathsf{d}(\kappa, \mathfrak{F}\kappa) + \mathsf{d}(\varrho, \mathfrak{F}\varrho) + \mathsf{d}(\kappa, \mathfrak{F}\kappa)) \\
2\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) &\leq \lambda(2\mathsf{d}(\kappa, \mathfrak{F}\kappa) + \mathsf{d}(\varrho, \mathfrak{F}\varrho)) \\
\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) &\leq \lambda \left(\mathsf{d}(\kappa, \mathfrak{F}\kappa) + \frac{\mathsf{d}(\varrho, \mathfrak{F}\varrho)}{2} \right)
\end{aligned}$$

which is the desired inequality. □

Corollary 2.4. *In a metric space $(\mathfrak{W}, \mathsf{d})$ where $|\mathfrak{W}| \geq 3$, if each point is an accumulation point of $\mathfrak{W}$ and $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$ is a PKC mapping, then $\mathfrak{F}$ is a KC mapping.*

Proof. As per proposition (2.3), the following inequalities are established:

$$\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) \leq \lambda \left(\mathsf{d}(\kappa, \mathfrak{F}\kappa) + \frac{\mathsf{d}(\varrho, \mathfrak{F}\varrho)}{2} \right), \text{ for all } \kappa, \varrho \in \mathfrak{W}, \tag{2.5}$$

$$\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) \leq \lambda \left(\mathsf{d}(\varrho, \mathfrak{F}\varrho) + \frac{\mathsf{d}(\kappa, \mathfrak{F}\kappa)}{2} \right), \text{ for all } \kappa, \varrho \in \mathfrak{W}. \tag{2.6}$$

Adding equations (2.5) and (2.6) we get

$$\begin{aligned}
2\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) &\leq \lambda \left(\frac{3}{2}\mathsf{d}(\kappa, \mathfrak{F}\kappa) + \frac{3}{2}\mathsf{d}(\varrho, \mathfrak{F}\varrho) \right) \\
\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) &\leq \frac{3\lambda}{4}(\mathsf{d}(\kappa, \mathfrak{F}\kappa) + \mathsf{d}(\varrho, \mathfrak{F}\varrho)) \\
\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) &\leq \eta(\mathsf{d}(\kappa, \mathfrak{F}\kappa) + \mathsf{d}(\varrho, \mathfrak{F}\varrho))
\end{aligned} \tag{2.7}$$

where $\eta = \frac{3\lambda}{4}$. Since $\lambda \in [0, \frac{1}{2})$, we have $\eta = \frac{3\lambda}{4} \in [0, \frac{1}{2})$, thereby concluding the argument. □

Proposition 2.5. *Let $(\mathfrak{W}, \mathsf{d})$ be a metric space, where $|\mathfrak{W}| \geq 3$. If $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$ is a PC mapping with a parameter $\lambda \in [0, \frac{3}{11})$, then $\mathfrak{F}$ is a PKC mapping.*

Proof. For any pairwise distinct points $\kappa, \varrho, \upsilon \in \mathfrak{W}$ we have

$$\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) + \mathsf{d}(\mathfrak{F}\varrho, \mathfrak{F}\upsilon) \leq \delta(\mathsf{d}(\kappa, \varrho) + \mathsf{d}(\varrho, \upsilon)) \ . \tag{2.8}$$

When triangle inequality is applied to the inequality's right-hand side (2.8), we get

$$\begin{aligned}\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) + \mathsf{d}(\mathfrak{F}\varrho, \mathfrak{F}\upsilon) &\leq \delta((\mathsf{d}(\kappa, \mathfrak{F}\kappa) + \mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) + \mathsf{d}(\mathfrak{F}\varrho, \varrho))+ \\ &\quad (\mathsf{d}(\varrho, \mathfrak{F}\varrho) + \mathsf{d}(\mathfrak{F}\varrho, \mathfrak{F}\upsilon) + \mathsf{d}(\mathfrak{F}\upsilon, \upsilon))) \\ &\leq \delta(\mathsf{d}(\kappa, \mathfrak{F}\kappa) + 2\mathsf{d}(\varrho, \mathfrak{F}\varrho) + \mathsf{d}(\upsilon, \mathfrak{F}\upsilon)) + \delta(\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) + \mathsf{d}(\mathfrak{F}\varrho, \mathfrak{F}\kappa)),\end{aligned}$$

$$\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) + \mathsf{d}(\mathfrak{F}\varrho, \mathfrak{F}\upsilon) \leq \frac{\delta}{1-\delta}(\mathsf{d}(\kappa, \mathfrak{F}\kappa) + 2\mathsf{d}(\varrho, \mathfrak{F}\varrho) + \mathsf{d}(\upsilon, \mathfrak{F}\upsilon)). \tag{2.9}$$

Similarly,

$$\mathsf{d}(\mathfrak{F}\varrho, \mathfrak{F}\upsilon) + \mathsf{d}(\mathfrak{F}\upsilon, \mathfrak{F}\kappa) \leq \frac{\delta}{1-\delta}(\mathsf{d}(\kappa, \mathfrak{F}\kappa) + \mathsf{d}(\varrho, \mathfrak{F}\varrho) + 2\mathsf{d}(\upsilon, \mathfrak{F}\upsilon)), \tag{2.10}$$

$$\mathsf{d}(\mathfrak{F}\upsilon, \mathfrak{F}\kappa) + \mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) \leq \frac{\delta}{1-\delta}(2\mathsf{d}(\kappa, \mathfrak{F}\kappa) + \mathsf{d}(\varrho, \mathfrak{F}\varrho) + \mathsf{d}(\upsilon, \mathfrak{F}\upsilon)). \tag{2.11}$$

When the left and right portions of the inequalities (2.9), (2.10) and (2.11) are summarized, we obtain

$$\mathsf{d}(\mathfrak{F}\upsilon, \mathfrak{F}\kappa) + \mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) + \mathsf{d}(\mathfrak{F}\upsilon, \mathfrak{F}\kappa) \leq \frac{2\delta}{1-\delta}(\mathsf{d}(\kappa, \mathfrak{F}\kappa) + \mathsf{d}(\varrho, \mathfrak{F}\varrho) + \mathsf{d}(\upsilon, \mathfrak{F}\upsilon)). \tag{2.12}$$

Comparing the coefficients in the inequalities (2.2) and (2.12) to get the maximum possible values of δ so that mapping $\mathfrak{F}$ is PKC. Then

$$\frac{3\lambda}{2} = \frac{2\delta}{1-\delta} \implies \lambda = \frac{4\delta}{3(1-\delta)}.$$

Therefore,

$$0 \leq \lambda = \frac{4\delta}{3(1-\delta)} < \frac{1}{2} \implies 0 \leq \delta < \frac{3}{11},$$

which brings the proof to its completion. □

We will now compose a KC mapping example that differs from PKC mapping.

Example 2.6. Consider the interval $\mathfrak{W} = [0, 1]$ with the Euclidean distance d. We examine a mapping $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$ given by $\mathfrak{F}(\kappa) = \frac{\kappa}{K}$, where $K > 1$ is a real number. Assume $\kappa \geq \varrho$, then for KC mapping we have

$$\begin{aligned}&\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) \leq \lambda(\mathsf{d}(\kappa, \mathfrak{F}\kappa) + \mathsf{d}(\varrho, \mathfrak{F}\varrho)) \\ &\implies \frac{\kappa}{K} - \frac{\varrho}{K} \leq \lambda(\kappa - \frac{\kappa}{K} + \varrho - \frac{\varrho}{K}) \\ &\implies (\kappa - \varrho) \leq \lambda(K-1)(\kappa + \varrho)\end{aligned} \tag{2.13}$$

It is obvious that the inequality (2.13) is satisfied for all $\kappa \geq \varrho$ iff $\lambda(K-1) \geq 1$. Let us suppose the following set of inequalities

$$\begin{cases} 0 \leq \lambda < \frac{1}{2} \\ \lambda(K-1) \geq 1 \end{cases} \implies \frac{1}{K-1} \leq \lambda < \frac{1}{2} \implies K > 3.$$

Thus, $\mathfrak{F}$ is KC mapping if and only if $K > 3$.

Next, let us suppose without losing generality that $\kappa > \varrho > \upsilon$. Think about inequality (2.1) for $\varrho, \kappa, \upsilon \in \mathfrak{W}$, to the same mapping $\mathfrak{F}$. This yields:

$$\begin{aligned}\left(\frac{\kappa}{K} - \frac{\varrho}{K}\right) + \left(\frac{\kappa}{K} - \frac{\upsilon}{K}\right) &\leq \lambda\left((\kappa - \frac{\kappa}{K}) + (\varrho - \frac{\varrho}{K}) + (\upsilon - \frac{\upsilon}{K})\right)\\ \Longrightarrow \frac{1}{K}(2\kappa - \varrho - \upsilon) &\leq \lambda\frac{(K-1)}{K}(\kappa + \varrho + \upsilon)\\ \Longrightarrow (\kappa - \frac{\varrho}{2} - \frac{\upsilon}{2}) &\leq \frac{\lambda(K-1)}{2}(\kappa + \varrho + \upsilon). \end{aligned} \tag{2.14}$$

It is clear that inequality (2.14) satisfies for all $\kappa > \varrho > \upsilon$ iff $\frac{\lambda(K-1)}{2} \geq 1$. Now, assume the following set of inequalities:

$$\begin{cases} 0 \leq \lambda < \frac{1}{2} \\ \frac{\lambda(K-1)}{2} \geq 1 \end{cases} \Longrightarrow \frac{2}{K-1} \leq \lambda < \frac{1}{2} \Longrightarrow K > 5.$$

Hence, for $K \in (3, 5]$, the mapping $\mathfrak{F}$ qualifies as a KC mapping but does not satisfy the criteria for a PKC mapping.

Note 2.1. In general case PKC mappings are discontinuous as well as KC mappings. Here we have presented an example to show the assertion.

Example 2.7. Consider the interval $\mathfrak{W} = [0, 1]$ equipped with Euclidean distance d. Assume the mapping $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$ given by

$$\mathfrak{F}(\kappa) = \begin{cases} \frac{\kappa}{K}, & \kappa \in [0, \frac{1}{2}]; \\ \frac{\kappa}{L}, & \kappa \in (\frac{1}{2}, 1]; \end{cases}$$

where $K, L > 1$, and $K \neq L$. It is clear that $\mathfrak{F}$ is discontinuous at $\kappa = \frac{1}{2}$. Let us establish the existence of constants K and L such that inequality (2.1) is satisfied for every set of three pairwise distinct $\kappa, \varrho, \upsilon \in \mathfrak{W}$ and for some $0 \leq \lambda < \frac{2}{3}$ implies that $\mathfrak{F}$ is a PKC mapping.

For any $\kappa > \varrho > \upsilon$, it is enough to focus on the following cases:

Case 1. $\kappa, \varrho, \upsilon \in [0, \frac{1}{2}]$: Using the above example, we obtain the restriction

$$\frac{2}{K-1} \leq \lambda < \frac{1}{2}. \tag{2.15}$$

Case 2. $\kappa, \varrho, \upsilon \in (\frac{1}{2}, 1]$: Similarly **Case 1** we will obtain the restriction

$$\frac{2}{L-1} \leq \lambda < \frac{1}{2}. \tag{2.16}$$

Without losing generality let us assume $K > L$ in next cases.

Case 3. $\varrho, \upsilon \in [0, \frac{1}{2}]$, $\kappa \in (\frac{1}{2}, 1]$: Consider inequality (2.1) in the order $\varrho, \kappa, \upsilon$. We have

$$\left(\frac{\kappa}{L} - \frac{\varrho}{K}\right) + \left(\frac{\kappa}{L} - \frac{\upsilon}{K}\right) \leq \lambda\left(\kappa - \frac{\kappa}{L} + \varrho - \frac{\varrho}{K} + \upsilon - \frac{\upsilon}{K}\right).$$

Hence,

$$0 \leq \left(\lambda - \frac{\lambda}{L} - \frac{2}{L}\right)\kappa + \left(\lambda - \frac{\lambda}{K} + \frac{1}{K}\right)\varrho + \left(\lambda - \frac{\lambda}{K} + \frac{1}{K}\right)\upsilon. \tag{2.17}$$

Case 4. $\upsilon \in [0, \frac{1}{2}]$, $\kappa, \varrho \in (\frac{1}{2}, 1]$: Considering inequality (2.1) in the order $\varrho, \kappa, \upsilon$. We obtain

$$\left(\frac{\kappa}{L} - \frac{\varrho}{L}\right) + \left(\frac{\kappa}{L} - \frac{\upsilon}{K}\right) \leq \lambda\left(\kappa - \frac{\kappa}{L} + \varrho - \frac{\varrho}{L} + \upsilon - \frac{\upsilon}{K}\right),$$

it implies that

$$0 \leq \left(\lambda - \frac{\lambda}{L} - \frac{2}{L}\right)\kappa + \left(\lambda - \frac{\lambda}{L} + \frac{1}{L}\right)\varrho + \left(\lambda - \frac{\lambda}{K} + \frac{1}{K}\right)\upsilon. \tag{2.18}$$

It is clear that there exist sufficiently large K and L and $K > L$ for any $0 < \lambda < \frac{1}{2}$, such that the inequality (2.15), (2.16), (2.17) and (2.18) holds all together. Hence, $\mathfrak{F}$ is a discontinuous PKC mapping.

3. Main Results

In a metric space $\mathfrak{W}$, a point $\kappa \in \mathfrak{W}$ is termed a periodic point of period n if applying $\mathfrak{F}$ repeatedly n times returns κ to its original position, i.e., $\mathfrak{F}^n(\kappa) = \kappa$. The smallest positive integer n that satisfies this condition is known as the prime period of κ. If $\mathfrak{F}(\mathfrak{F}(\kappa)) = \kappa$ and $\mathfrak{F}\kappa \neq \kappa$, then κ has prime period 2.

Remark 3.1. PKC mappings cannot have periodic points of prime period three. Assume that there is a point κ such that $\mathfrak{F}\kappa = \varrho$, $\varrho \neq \kappa$, $\mathfrak{F}\varrho = \upsilon$, $\upsilon \neq \varrho \neq \kappa$, and $\mathfrak{F}\upsilon = \kappa$. Then we obtain the equalities

$$\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) + \mathsf{d}(\mathfrak{F}\varrho, \mathfrak{F}\upsilon) = \mathsf{d}(\varrho, \mathfrak{F}\varrho) + \mathsf{d}(\upsilon, \mathfrak{F}\upsilon),$$

$$\mathsf{d}(\mathfrak{F}\varrho, \mathfrak{F}\upsilon) + \mathsf{d}(\mathfrak{F}\upsilon, \mathfrak{F}\kappa) = \mathsf{d}(\upsilon, \mathfrak{F}\upsilon) + \mathsf{d}(\kappa, \mathfrak{F}\kappa),$$

$$\mathsf{d}(\mathfrak{F}\upsilon, \mathfrak{F}\kappa) + \mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) = \mathsf{d}(\kappa, \mathfrak{F}\kappa) + \mathsf{d}(\varrho, \mathfrak{F}\varrho).$$

Adding above equalities, we get

$$\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) + \mathsf{d}(\mathfrak{F}\varrho, \mathfrak{F}\upsilon) + \mathsf{d}(\mathfrak{F}\upsilon, \mathfrak{F}\kappa) = \mathsf{d}(\kappa, \mathfrak{F}\kappa) + \mathsf{d}(\varrho, \mathfrak{F}\varrho) + \mathsf{d}(\upsilon, \mathfrak{F}\upsilon),$$

which contradicts the inequality (2.2).

The principal outcome of this work is the following theorem.

Theorem 3.2. *Let $(\mathfrak{W}, \mathsf{d})$ be a complete metric space with $|\mathfrak{W}| \geq 3$. Consider the mapping $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$ under the following assumptions:*

(a) *$\mathfrak{F}$ is a PKC mapping.*
(b) *There are no periodic points of prime period 2 for $\mathfrak{F}$.*

Then, there exist at least one fixed point of $\mathfrak{F}$. The number of fixed points can be at most two.

Proof. Let us take an arbitrary point $\kappa \in \mathfrak{W}$, and set the sequence $\{\kappa_n\}$ as $\kappa_0 = \kappa$, $\kappa_1 = \mathfrak{F}\kappa_0$ and $\kappa_n = \mathfrak{F}\kappa_{n-1} = \mathfrak{F}^n\kappa_0$. If for some n, κ_n is fixed point then there is nothing to prove. Assuming that κ_n is not a fixed point for any $n = 0, 1, 2, ...$, we can deduce that $\kappa_{n-1} \neq \kappa_n \neq \kappa_{n+1}$. Given the information (b), it can be deduced that $\kappa_{n+1} \neq \kappa_{n-1}$. Therefore, κ_{n-1}, κ_n and κ_{n+1} are pairwise distinct. Placing $\kappa = \kappa_{n-1}$, $\varrho = \kappa_n$ and $\upsilon = \kappa_{n+1}$ in (2.1), then we get

$$\begin{aligned}\mathsf{d}(\mathfrak{F}\kappa_{n-1}, \mathfrak{F}\kappa_n) + \mathsf{d}(\mathfrak{F}\kappa_n, \mathfrak{F}\kappa_{n+1}) \leq \\ \lambda \left\{\mathsf{d}(\kappa_{n-1}, \mathfrak{F}\kappa_{n-1}) + \mathsf{d}(\kappa_n, \mathfrak{F}\kappa_n) + \mathsf{d}(\kappa_{n+1}, \mathfrak{F}\kappa_{n+1})\right\},\end{aligned}$$

that implies

$$\mathsf{d}(\kappa_n, \kappa_{n+1}) + \mathsf{d}(\kappa_{n+1}, \kappa_{n+2}) \leq \lambda \left\{\mathsf{d}(\kappa_{n-1}, \kappa_n) + \mathsf{d}(\kappa_n, \kappa_{n+1}) + \mathsf{d}(\kappa_{n+1}, \kappa_{n+2})\right\}. \tag{3.1}$$

Again, setting $\kappa = \kappa_{n-1}$, $\varrho = \kappa_{n+1}$ and $\upsilon = \kappa_n$ in inequality (2.1). Then, we have

$$\begin{aligned}\mathsf{d}(\mathfrak{F}\kappa_{n-1}, \mathfrak{F}\kappa_{n+1}) + \mathsf{d}(\mathfrak{F}\kappa_{n+1}, \mathfrak{F}\kappa_n) \leq \\ \lambda \left\{\mathsf{d}(\kappa_{n-1}, \mathfrak{F}\kappa_{n-1}) + \mathsf{d}(\kappa_n, \mathfrak{F}\kappa_n) + \mathsf{d}(\kappa_{n+1}, \mathfrak{F}\kappa_{n+1})\right\},\end{aligned}$$

that implies

$$\mathsf{d}(\kappa_n, \kappa_{n+2}) + \mathsf{d}(\kappa_{n+1}, \kappa_{n+2}) \leq \lambda \left\{\mathsf{d}(\kappa_{n-1}, \kappa_n) + \mathsf{d}(\kappa_n, \kappa_{n+1}) + \mathsf{d}(\kappa_{n+1}, \kappa_{n+2})\right\}. \tag{3.2}$$

Adding (3.1) and (3.2) and utilising triangle inequality $\mathsf{d}(\kappa_{n+1}, \kappa_{n+2}) \leq \mathsf{d}(\kappa_n, \kappa_{n+1}) + \mathsf{d}(\kappa_n, \kappa_{n+2})$, we have

$$\begin{aligned}\mathsf{d}(\mathfrak{F}\kappa_{n-1}, \mathfrak{F}\kappa_{n+1}) + 2\mathsf{d}(\kappa_{n+1}, \kappa_{n+2}) + \mathsf{d}(\kappa_n, \kappa_{n+2}) \leq \\ 2\lambda \left\{\mathsf{d}(\kappa_{n-1}, \kappa_n) + \mathsf{d}(\kappa_n, \kappa_{n+1}) + \mathsf{d}(\kappa_{n+1}, \kappa_{n+2})\right\}. \\ 3\mathsf{d}(\kappa_{n+1}, \kappa_{n+2}) \leq 2\lambda \left\{\mathsf{d}(\kappa_{n-1}, \kappa_n) + \mathsf{d}(\kappa_n, \kappa_{n+1}) + \mathsf{d}(\kappa_{n+1}, \kappa_{n+2})\right\} \\ (3 - 2\lambda)\mathsf{d}(\kappa_{n+1}, \kappa_{n+2}) \leq 2\lambda \left\{\mathsf{d}(\kappa_{n-1}, \kappa_n) + \mathsf{d}(\kappa_n, \kappa_{n+1})\right\} \\ \mathsf{d}(\kappa_{n+1}, \kappa_{n+2}) \leq \frac{2\lambda}{3 - 2\lambda} \left\{\mathsf{d}(\kappa_{n-1}, \kappa_n) + \mathsf{d}(\kappa_n, \kappa_{n+1})\right\} \\ \mathsf{d}(\kappa_{n+1}, \kappa_{n+2}) \leq \frac{4\lambda}{3 - 2\lambda} \max \left\{\mathsf{d}(\kappa_{n-1}, \kappa_n), \mathsf{d}(\kappa_n, \kappa_{n+1})\right\}.\end{aligned}$$

Let $\alpha = \frac{4\lambda}{3-2\lambda}$. Using the relation $\lambda \in [0, \frac{1}{2})$, we get $\alpha \in [0, 1)$. As a result,

$$\mathsf{d}(\kappa_{n+1}, \kappa_{n+2}) \leq \alpha \max \left\{\mathsf{d}(\kappa_{n-1}, \kappa_n), \mathsf{d}(\kappa_n, \kappa_{n+1})\right\}. \tag{3.3}$$

Take $a_n = \mathsf{d}(\kappa_{n-1}, \kappa_n)$, $n = 1, 2, 3, ...$, and define $a = \max\{a_1, a_2\}$. Therefore, using (3.3) we get

$$a_1 \leq a,\ a_2 \leq a,\ a_3 \leq \alpha a,\ a_4 \leq \alpha a,\ a_5 \leq \alpha^2 a,\ a_6 \leq \alpha^2 a,\ a_7 \leq \alpha^3 a, ...$$

Since $\alpha < 1$, it is clear that the inequalities

$$a_1 \leq a,\ a_2 \leq a,\ a_3 \leq \alpha^{\frac{1}{2}} a,\ a_4 \leq \alpha a,\ a_5 \leq \alpha^{\frac{3}{2}} a,\ a_6 \leq \alpha^2 a,\ a_7 \leq \alpha^{\frac{2}{2}} a, ...,$$

also holds. It follows that

$$a_n \leq \alpha^{\frac{n}{2}-1} a,\ \text{for } n = 3, 4, ... \tag{3.4}$$

For $|m-n| \in \mathbf{N}$, where $|m-n| \geq 2$, applying the triangle inequality repeatedly for $n, m \geq 3$, assuming $m \geq n$ for sake of simplicity, we get

$$\begin{aligned}
\mathsf{d}(\kappa_n, \kappa_m) &\leq \mathsf{d}(\kappa_n, \kappa_{n+1}) + \mathsf{d}(\kappa_{n+1}, \kappa_{n+2}) + \ldots + \mathsf{d}(\kappa_{m-1}, \kappa_m) \\
&\leq a_{n+1} + a_{n+2} + \ldots + a_m \\
&\leq a\left(\alpha^{\frac{n+2}{2}-1} + \alpha^{\frac{n+1}{2}-1} + \ldots + \alpha^{\frac{m}{2}-1}\right) \\
&= a\alpha^{\frac{n+1}{2}-1}\left(1 + \alpha^{\frac{1}{2}} + \ldots + \alpha^{\frac{m-n-1}{2}}\right) \\
&= a\alpha^{\frac{n-1}{2}}\left(\frac{1-\sqrt{\alpha^{m-n}}}{1-\sqrt{\alpha}}\right) = \frac{a}{1-\sqrt{\alpha}}\left(\alpha^{\frac{n}{2}} - \alpha^{\frac{m}{2}}\right).
\end{aligned}$$

Since by the supposition $\alpha \in [0,1)$, therefore $\mathsf{d}(\kappa_n, \kappa_m) \to 0$ as $m, n \to \infty$. It follows, sequence $\{\kappa_n\}$ is Cauchy. Then, this sequence has a unique limit $\kappa^* \in \mathfrak{W}$ referring to the completeness of $(\mathfrak{W}, \mathsf{d})$.

We know that every triplet of consecutive elements in the sequence $\{\kappa_n\}$ consists of pairwise distinct elements, let us consider that $\kappa^* \neq \kappa_k$ for any $k \in \{1, 2, \ldots\}$. This implies that the inequality (2.1) is satisfied for the distinct elements κ^*, κ_{n-1}, and κ_n. Now, assume that $k \in \{1, 2, \ldots\}$ is the smallest index such that $\kappa^* = \kappa_k$. If there is some $m > k$ where $\kappa^* = \kappa_m$, the sequence becomes cyclic starting from k, making it non-Cauchy. Consequently, κ^*, κ_{n-1}, and κ_n must be pairwise distinct, particularly for $n \geq k+2$.

Next, we will establish that κ^* serves as a fixed-point for $\mathfrak{F}$. If there is a $\kappa_k = \kappa^*$ where $k \in \{1, 2, \ldots.\}$, then for $n \geq k+2$ and by triangle inequality with inequality (2.1) we have

$$\begin{aligned}
\mathsf{d}(\kappa^*, \mathfrak{F}\kappa^*) &\leq \mathsf{d}(\kappa^*, \kappa_n) + \mathsf{d}(\kappa_n, \mathfrak{F}\kappa^*) = \mathsf{d}(\kappa^*, \kappa_n) + \mathsf{d}(\mathfrak{F}\kappa_{n-1}, \mathfrak{F}\kappa^*) \\
&\leq \mathsf{d}(\kappa^*, \kappa_n) + \mathsf{d}(\mathfrak{F}\kappa_{n-1}, \mathfrak{F}\kappa^*) + \mathsf{d}(\mathfrak{F}\kappa^*, \mathfrak{F}\kappa_n) \\
&\leq \mathsf{d}(\kappa^*, \kappa_n) + \lambda(\mathsf{d}(\kappa^*, \mathfrak{F}\kappa^*) + \mathsf{d}(\kappa_{n-1}, \mathfrak{F}\kappa_{n-1}) + \mathsf{d}(\kappa_n, \mathfrak{F}\kappa_n)) \\
(1-\lambda)\mathsf{d}(\kappa^*, \mathfrak{F}\kappa^*) &\leq \mathsf{d}(\kappa^*, \kappa_n) + \lambda(\mathsf{d}(\kappa_{n-1}, \kappa_n) + \mathsf{d}(\kappa_n, \mathfrak{F}\kappa_{n+1})) \\
\mathsf{d}(\kappa^*, \mathfrak{F}\kappa^*) &\leq \frac{\mathsf{d}(\kappa^*, \kappa_n) + \lambda(\mathsf{d}(\kappa_{n-1}, \kappa_n) + \mathsf{d}(\kappa_n, \mathfrak{F}\kappa_{n+1}))}{(1-\lambda)}.
\end{aligned}$$

As all the right part distances are tending to zero as $n \to \infty$, this implies that

$$\mathsf{d}(\kappa^*, \mathfrak{F}\kappa^*) = 0 \implies \mathfrak{F}\kappa^* = \kappa^*.$$

Assume that there are a minimum of three fixed points κ, ϱ and υ that are pairwise distinct. Then $\mathfrak{F}\kappa = \kappa$, $\mathfrak{F}\varrho = \varrho$ and $\mathfrak{F}\upsilon = \upsilon$, which contradict to (2.2). □

Now, we will create a PKC mapping example with precisely two fixed points.

Example 3.3. Let $\mathfrak{W} = \{1, 2, 3\}$ equipped with d given by $\mathsf{d}(1,1) = \mathsf{d}(2,2) = \mathsf{d}(3,3) = 0$, $\mathsf{d}(1,2) = 1$, $\mathsf{d}(1,3) = \mathsf{d}(2,3) = 5$. Also, let the mapping $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$ given by $\mathfrak{F}(1) = 1$, $\mathfrak{F}(2) = 2$ and $\mathfrak{F}(3) = 1$. It can be seen that condition (a) of Theorem 3.2 is fulfilled and inequality (2.1) holds with $\lambda = \frac{2}{5}$.

Note 3.1. Also, it can be readily shown that $\mathfrak{F}$ is not KC mapping because the inequality (1.1) fails for any $0 \leq \eta < \frac{1}{2}$.

In the next example we reveal that condition (a) in Theorem 3.2 is essential to have fixed points.

Example 3.4. Let us take space $(\mathfrak{W}, \mathsf{d})$ as in previous example and $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$ given by $\mathfrak{F}(1) = 2,\ \mathfrak{F}(2) = 1$ and $\mathfrak{F}(3) = 1$. One can easily verify that inequality (2.1) holds for any $\frac{2}{7} \leq \lambda < \frac{1}{2}$ but $\mathfrak{F}$ does not have any fixed point.

The continuity of KC mappings at fixed points is well-known [26]. Next proposition proves this property for PKC mappings.

Proposition 3.5. *PKC mappings are continuous at fixed points.*

Proof. If $(\mathfrak{W}, \mathsf{d})$ is a metric space, where $|\mathfrak{W}| \geq 3$. Suppose $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$ is a PKC mapping and κ^* is a fixed point of $\mathfrak{F}$. Now, consider a sequence $\{\kappa_n\}$ such that $\kappa_n \to \kappa^*$. To prove continuity at κ^*, we will show that $\mathfrak{F}\kappa_n \to \mathfrak{F}\kappa^*$.

If $\kappa_n \neq \kappa_{n+1}$ and $\kappa_n \neq \kappa^*$ for any positive integer n, then by (2.1) we obtain

$$\begin{aligned} \mathsf{d}(\mathfrak{F}\kappa_n, \mathfrak{F}\kappa^*) + \mathsf{d}(\mathfrak{F}\kappa^*, \mathfrak{F}\kappa_{n+1}) &\leq \lambda(\mathsf{d}(\kappa^*, \mathfrak{F}\kappa^*) + \mathsf{d}(\kappa_n, \mathfrak{F}\kappa_n) + \mathsf{d}(\kappa_{n+1}, \mathfrak{F}\kappa_{n+1})) \\ &= \lambda(\mathsf{d}(\kappa_n, \mathfrak{F}\kappa_n) + \mathsf{d}(\kappa_{n+1}, \mathfrak{F}\kappa_{n+1})). \end{aligned}$$

Using triangle inequality, we get

$$\begin{gathered} \mathsf{d}(\mathfrak{F}\kappa_n, \mathfrak{F}\kappa^*) + \mathsf{d}(\mathfrak{F}\kappa^*, \mathfrak{F}\kappa_{n+1}) \leq \\ \lambda(\mathsf{d}(\kappa_n, \mathfrak{F}\kappa^*) + \mathsf{d}(\mathfrak{F}\kappa^*, \mathfrak{F}\kappa_n) + \mathsf{d}(\kappa_{n+1}, \mathfrak{F}\kappa^*) + \mathsf{d}(\mathfrak{F}\kappa^*, \mathfrak{F}\kappa_{n+1})) \\ \leq \frac{\lambda}{1-\lambda}(\mathsf{d}(\kappa_n, \kappa^*) + \mathsf{d}(\kappa_{n+1}, \kappa^*)). \end{gathered}$$

Since $\mathsf{d}(\kappa_n \kappa^*) \to 0$ and $\mathsf{d}(\kappa_{n+1}, \kappa^*) \to 0$ we obtain

$$d(\mathfrak{F}\kappa_n, \mathfrak{F}\kappa^*) + \mathsf{d}(\mathfrak{F}\kappa^*, \mathfrak{F}\kappa_{n+1}) \to 0,$$

implies $\mathsf{d}(\mathfrak{F}\kappa^*, \mathfrak{F}\kappa_n) \to 0$.

If $\kappa_n \neq \kappa^*$ for every n, but for some n, $\kappa_n = \kappa_{n+1}$ is feasible, then there exists a subsequence $\{\kappa_{n_k}\}$ such that $\kappa_{n_k} \neq \kappa_{n_{k+1}}$ for all k, and $\kappa_{n_k} \to \kappa^*$. Now, we can show that $\mathfrak{F}\kappa_{n_k} \to \mathfrak{F}\kappa^*$ as above. By inserting corresponding repeating consecutive terms in $\mathfrak{F}\kappa_{n_k}$ we can obtain $\mathfrak{F}\kappa_n$. It follows that $\mathfrak{F}\kappa_n \to \mathfrak{F}\kappa^*$.

If $\kappa_n = \kappa^*$ for each $n > N$, it follows that $\mathfrak{F}\kappa_n \to \mathfrak{F}\kappa^*$.

If $\{\kappa_n\}$ is an arbitrary sequence which does not belong to previous cases. Consider a subsequence $\{\kappa_{n_k}\}$ achieved from $\{\kappa_n\}$ after removing element κ^* (if it exists). Clearly $\kappa_{n_k} \to \kappa^*$. It is just shown that such $\mathfrak{F}\kappa_{n_k} \to \mathfrak{F}\kappa^*$. We can easily obtain $\mathfrak{F}\kappa_n$ from $\mathfrak{F}\kappa_{n_k}$ by inserting element $\mathfrak{F}\kappa^* = \kappa^*$ on some places. It is evident that $\mathfrak{F}\kappa_n \to \mathfrak{F}\kappa^*$. □

4. Asymptotic Regularity

Asymptotic regularity comes into play when dealing with mappings that may not have fixed points under all conditions. It allows us to consider mappings that might not be strictly contractive (i.e., they don't necessarily shrink distances between points) but still exhibit a form of controlled behaviour over sequences. The notion of asymptotic regularity extends the family of mappings for which fixed-point theorems holds.

Definition 4.1 ([6])**.** Let $(\mathfrak{W}, \mathsf{d})$ be a metric space. A function $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$ is considered asymptotically regular if it meets the condition below for all $\kappa \in \mathfrak{W}$:

$$\lim_{n\to\infty} \mathsf{d}(\mathfrak{F}^{n+1}(\kappa), \mathfrak{F}^{n}(\kappa)) = 0. \tag{4.1}$$

Remark 4.2. Suppose $(\mathfrak{W}, \mathsf{d})$ represent a metric space, and let $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$ be a self-map. We create a sequence $\{\kappa_n\}$ where κ_0 belongs to $\mathfrak{W}$, $\kappa_1 = \mathfrak{F}\kappa_0$, $\kappa_2 = \mathfrak{F}\kappa_1$, and so forth. If $\mathfrak{F}$ is asymptotically regular and $\{\kappa_n\}$ fails to converge to a fixed point of $\mathfrak{F}$, then all points κ_k for $k \geq 0$ are distinct from one another. If they were not, the sequence $\{\kappa_n\}$ would become cyclic after some point, which would contradict the condition (4.1).

Theorem 4.3. *Let $(\mathfrak{W}, \mathsf{d})$ be a complete metric space with $|\mathfrak{W}| \geq 3$. Suppose $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$ is an asymptotically regular PKC mapping with a parameter $\lambda \in [0, 1)$. Then, $\mathfrak{F}$ has at least one fixed point. The maximum number of fixed points can be two.*

Proof. Take $\kappa_0 \in \mathfrak{W}$ and define the sequence $\{\kappa_n\}$ by $\kappa_1 = \mathfrak{F}\kappa_0$, $\kappa_2 = \mathfrak{F}\kappa_1$, and so on. Assume $\{\kappa_n\}$ does not include any fixed-point of $\mathfrak{F}$. Now, we will demonstrate that the sequence $\{\kappa_n\}$ is Cauchy. It is sufficient for the proof that $\mathsf{d}(\kappa_n, \kappa_m) \to 0$ as $m, n \to \infty$. If $|m-n| = 1$, which is a direct result of the concept of asymptotic regularity. When $|m - n| \geq 2$, the terms κ_n, κ_{m-1}, and κ_m are pairwise distinct according to Remark 4.2. By using asymptotic regularity, triangle inequality, and inequality (2.1), assuming $m \geq n$ for sake of simplicity, we obtain

$$\begin{aligned}
\mathsf{d}(\kappa_n, \kappa_m) &\leq \mathsf{d}(\kappa_n, \kappa_{m-1}) + \mathsf{d}(\kappa_{m-1}, \kappa_m) \\
&\leq \lambda(\mathsf{d}(\kappa_n, \mathfrak{F}\kappa_n) + \mathsf{d}(\kappa_{m-1}, \mathfrak{F}\kappa_{m-1}) + \mathsf{d}(\kappa_m, \mathfrak{F}\kappa_m)) \\
&= \lambda(\mathsf{d}(\mathfrak{F}^n\kappa_0, \mathfrak{F}^{n+1}\kappa_0) + \mathsf{d}(\mathfrak{F}^{m-1}\kappa_0, \mathfrak{F}^m\kappa_0) + \mathsf{d}(\mathfrak{F}^m\kappa_0, \mathfrak{F}^{m+1}\kappa_0)) \\
&\to 0 \text{ as } m, n \to \infty.
\end{aligned}$$

As a result, $\{\kappa_n\}$ is a Cauchy sequence. The argument's remaining portion derived from Theorem 3.2. □

In the subsequent discussion, we establish that assuming continuity for the mappings $\mathfrak{F}$ enables the derivation of fixed point theorems that apply to a wider range of mappings compared to PKC mappings, even when the coefficient λ is within the interval $[0, 1)$.

We now present the following definitions and offer a more comprehensive version of PKC mappings. Let $\mathbb{F}$ denote a set of functions $\mathcal{F} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ attaining the requirements listed below:

(i) $\mathcal{F}(0, 0, 0) = 0$;
(ii) $\mathcal{F}$ is continuous at $(0, 0, 0)$.

Definition 4.4. Consider $(\mathfrak{W}, \mathsf{d})$ as a metric space where $|\mathfrak{W}| \geq 3$. A mapping $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$ is defined as an $\mathcal{F}$-PKC mapping on $\mathfrak{W}$ if a function $\mathcal{F} \in \mathbb{F}$exists so that, for any three pairwise distinct points $\kappa, \varrho, \upsilon \in \mathfrak{W}$, the following inequality holds:

$$\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) + \mathsf{d}(\mathfrak{F}\varrho, \mathfrak{F}\upsilon) \leq \mathcal{F}(\mathsf{d}(\kappa, \mathfrak{F}\kappa), \mathsf{d}(\varrho, \mathfrak{F}\varrho), \mathsf{d}(\upsilon, \mathfrak{F}\upsilon)). \tag{4.2}$$

Theorem 4.5. *In a complete metric space $(\mathfrak{W}, \mathsf{d})$ with at least three points, if $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$ is a continuous, asymptotically regular $\mathcal{F}$-PKC mapping, then $\mathfrak{F}$ possesses fixed points. Moreover, the number of fixed points of $\mathfrak{F}$ does not exceed two.*

Proof. Let $\kappa_0 \in \mathfrak{W}$ and define the sequence $\{\kappa_n\}$ by $\kappa_1 = \mathfrak{F}\kappa_0$, $\kappa_2 = \mathfrak{F}\kappa_1$, and so on. Assume that the $\{\kappa_n\}$ does not include any fixed point of $\mathfrak{F}$. Now, we will demonstrate that the sequence $\{\kappa_n\}$ is Cauchy. It is sufficient for the proof that $\mathsf{d}(\kappa_n, \kappa_m) \to 0$ as $m, n \to \infty$. If $|m - n| = 1$, this is a direct consequence of the concept of asymptotic regularity. When $|m-n| \geq 2$, the terms κ_n, κ_{m-1}, and κ_m are pairwise distinct according to Remark 4.2. By using asymptotic regularity, triangle inequality, and inequality (4.2), assuming $m \geq n$ for sake of simplicity, we obtain

$$\begin{aligned}\mathsf{d}(\kappa_n, \kappa_m) &\leq \mathsf{d}(\kappa_n, \kappa_{m-1}) + \mathsf{d}(\kappa_{m-1}, \kappa_m)\\ &\leq \mathcal{F}(\mathsf{d}(\kappa_n, \mathfrak{F}\kappa_n), \mathsf{d}(\kappa_{m-1}, \mathfrak{F}\kappa_{m-1}), \mathsf{d}(\kappa_m, \mathfrak{F}\kappa_m))\\ &= \mathcal{F}(\mathsf{d}(\kappa_n, \kappa_{n+1}), \mathsf{d}(\kappa_{m-1}, \kappa_m), \mathsf{d}(\kappa_m, \kappa_{m+1})) \to 0\end{aligned}$$

when $m, n \to \infty$. Thus, $\{\kappa_n\}$ is Cauchy sequence. This sequence has a limit $\kappa^* \in \mathfrak{W}$ according to the completeness of $(\mathfrak{W}, \mathsf{d})$. Then, we obtain

$$\mathsf{d}(\mathfrak{F}\kappa^*, \kappa^*) \leq \mathsf{d}(\mathfrak{F}\kappa^*, \kappa_n) + \mathsf{d}(\kappa_n, \kappa^*) = \mathsf{d}(\mathfrak{F}\kappa^*, \mathfrak{F}\kappa_{n-1}) + \mathsf{d}(\kappa_n, \kappa^*).$$

Given that $\mathfrak{F}$ is continuous, $\mathfrak{F}\kappa^* = \kappa^*$ can be obtained by letting $n \to \infty$. The remaining part of the argument is based on the reasoning similar to that found in the last section of the argument of Theorem 3.2. □

It must be noted that, we refer $\mathcal{B}$ as the set of functions $\beta : [0, \infty) \to [0, \infty)$ satisfying the subsequent inequality:

$$\limsup_{t \to 0} \beta(t) < \infty. \tag{4.3}$$

Definition 4.6. Let $(\mathfrak{W}, \mathsf{d})$ be a metric space, where $|\mathfrak{W}| \geq 3$. A self-mapping $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$ is defined as a $\mathcal{B}$-PKC mapping on $\mathfrak{W}$ if there exist functions $\beta_1, \beta_2, \beta_3 \in \mathcal{B}$ such that for any three pairwise distinct points $\kappa, \varrho, \upsilon \in \mathfrak{W}$, the inequality:

$$\begin{gathered}\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) + \mathsf{d}(\mathfrak{F}\varrho, \mathfrak{F}\upsilon) \leq\\ \beta_1(\mathsf{d}(\kappa, \mathfrak{F}\kappa))\mathsf{d}(\kappa, \mathfrak{F}\kappa) + \beta_2(\mathsf{d}(\varrho, \mathfrak{F}\varrho))\mathsf{d}(\varrho, \mathfrak{F}\varrho) + \beta_3(\mathsf{d}(\upsilon, \mathfrak{F}\upsilon))\mathsf{d}(\upsilon, \mathfrak{F}\upsilon)\end{gathered} \tag{4.4}$$

holds true.

Corollary 4.7. *If $(\mathfrak{W}, \mathsf{d})$ is a complete metric space with $|\mathfrak{W}| \geq 3$. Assume that $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$ is continuous, asymptotically regular $\mathcal{B}$-PKC mapping. Then, $\mathfrak{F}$ possesses fixed-points, and the number of fixed-points does not exceed two.*

Proof. By setting $\mathcal{F}(\kappa, \varrho, \upsilon) = \beta_1(\kappa)\kappa + \beta_2(\varrho)\varrho + \beta_3(\upsilon)\upsilon$, then $\mathcal{F}(0,0,0) = 0$ and $\lim_{\kappa,\varrho,\upsilon\to 0} \mathcal{F}(\kappa, \varrho, \upsilon) = 0$. Then, the claim proceeds from Theorem 4.5, as for $i = 1, 2, 3$, $\limsup_{t\to 0} \beta_i(t) < \infty$. □

On taking $\beta_1(t) = \beta_1(t) = \beta_1(t) = \lambda \geq 0$ in Theorem 4.3, we get a PKC mapping characterized by the coefficient $\lambda \in [0, \infty)$. This leads directly to the following corollary.

Corollary 4.8. *If* $(\mathfrak{W}, \mathsf{d})$ *is a complete metric space where* $|\mathfrak{W}| \geq 3$*. Assume that* $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$ *is continuous, asymptotically regular PKC mapping with a parameter* $\lambda \in [0, \infty)$*. Then,* $\mathfrak{F}$ *possesses fixed points, and the number of fixed points does not exceed two.*

Remark 4.9. It is possible to remove continuity condition for the mapping $\mathfrak{F}$ in Corollary 4.8. Instead of requiring continuity, Corollary 4.8 remains valid under orbital continuity, κ_0-orbitally continuity, almost orbitally continuity, weakly orbitally continuity, k-continuity or $\mathfrak{F}$-orbitally lower semi-continuity. For further discussion on these weaker continuity concepts, refer to [5].

Example 4.10. Consider the set $\mathfrak{W} = [0, 1]$ with euclidean distance function d. Consider a mapping $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$ given as $\mathfrak{F}\kappa = \frac{\kappa}{K}$ for some real $K > 1$. It was shown in Example 2.6 that inequality (2.1) holds for all $\kappa > \varrho > \upsilon$ iff $\lambda \geq \frac{2}{K-1}$. Therefore, if $K \in (3, \infty)$ ($K \in (1, \infty)$), then $\mathfrak{F}$ is a PKC mapping with a parameter $\lambda \in (0, 1)$ ($\lambda \in (0, \infty)$). Therefore, mappings $\mathfrak{F}(\kappa) = \alpha\kappa$ where $\alpha \in (0, \frac{1}{3})$ ($\alpha \in (0, 1)$) are PKC mappings with a parameter $\lambda \in (0, 1)$ ($\lambda \in (0, \infty)$).

Next proposition is obvious.

Proposition 4.11. *For a finite non-empty metric space* $\mathfrak{W}$ *with a self-mapping* $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$*. The mapping* $\mathfrak{F}$ *is asymptotically regular if and only if there are no points of prime period* $n \geq 2$*.*

Corollary 4.12. *In a finite non-empty metric space* $\mathfrak{W}$ *with a self-mapping* $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$*, if* $\mathfrak{F}$ *does not have any periodic points with a prime period* $n \geq 2$*, then* $\mathfrak{F}$ *must have fixed points.*

Proof. Clearly $\mathfrak{W}$ is complete and $\mathfrak{F}$ is continuous, and, according to Proposition 4.11, $\mathfrak{F}$ is also asymptotically regular. Let $\mathfrak{F}$ not have any fixed points. Hence, $\mathsf{d}(\kappa, \mathfrak{F}\kappa) + \mathsf{d}(\varrho, \mathfrak{F}\varrho) + \mathsf{d}(\upsilon, \mathfrak{F}\upsilon) \neq 0$ for all $\kappa, \varrho, \upsilon \in \mathfrak{W}$. It is evident that $\mathfrak{F}$ is a PKC mapping with the coefficient:

$$\lambda = \max_{\kappa,\varrho,\upsilon\in\mathfrak{W}} \frac{\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) + \mathsf{d}(\mathfrak{F}\varrho, \mathfrak{F}\upsilon)}{\mathsf{d}(\kappa, \mathfrak{F}\kappa) + \mathsf{d}(\varrho, \mathfrak{F}\varrho) + \mathsf{d}(\upsilon, \mathfrak{F}\upsilon)},$$

the maximum is calculated across all pairwise distinct points $\kappa, \varrho, \upsilon \in \mathfrak{W}$. Applying Corollary 4.8 we get a contradiction. □

Note 4.1. In the context of periodic points, fixed points are considered periodic points with prime period $n = 1$. Thus, we arrive to the subsequent consequence.

Corollary 4.13. *In a finite non-empty metric space* $\mathfrak{W}$*, a self-mapping* $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$ *possesses has a periodic point.*

Proof. Let $\kappa_0 \in \mathfrak{W}$. Consider the sequence $\kappa_n = \mathfrak{F}(\kappa_{n-1})$, $n \geq 1$. By applying Pigeonhole principle, a value of n (where $1 \neq n \neq |\mathfrak{W}|$) occurs for which κ_n matches a preceding term of the sequence. Let k denote the minimal index for which κ_k repeats, and i be an index less than k where $\kappa_i = \kappa_k$. Then κ_i turns into a periodic point, whose prime period is $k - i$. □

5. Approximate Fixed-Point sequence and F-PKC Mappings

In this section, we will establish a result concerning the mapping $\mathfrak{F}$, which is characterized by having an approximate fixed-point sequence. To begin, we will introduce and define the concept of an approximate fixed-point:

Definition 5.1. Let $(\mathfrak{W}, \mathsf{d})$ be a metric space and $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$. A sequence $\{\kappa_n\} \subset \mathfrak{W}$ is defined as an approximate fixed-point sequence of $\mathfrak{F}$ if $\mathsf{d}(\kappa_n, \mathfrak{F}\kappa_n) \to 0$ as $n \to \infty$.

Next example illustrates that $\mathfrak{F}$ has an approximate fixed-point sequence but it does not satisfy asymptotic regularity.

Example 5.2. Consider the interval $\mathfrak{W} = [0, 1]$ equipped with standard metric d. Define the mapping $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$ for which $\mathfrak{F}(\kappa) = 1 - \kappa$ to each value of $\kappa \in [0, 1]$. Subsequently,

$$\mathfrak{F}^2(\kappa) = 1 - \mathfrak{F}(\kappa) = 1 - (1 - \kappa) = \kappa \text{ where } \kappa \in \mathfrak{W},$$

which indicates that all points except $\kappa = \frac{1}{2}$ are periodic points and have prime period 2. The point $\kappa = \frac{1}{2}$ itself act as a fixed-point of $\mathfrak{F}$.

Additionally, the sequence $\{\kappa_n\}_{n\geq 2}$ defined by $\kappa_n = \frac{1}{2} + \frac{1}{n}$ serves as an approximate fixed-point sequence of $\mathfrak{F}$. Indeed,

$$\mathsf{d}(\kappa_n, \mathfrak{F}\kappa_n) = \left| \frac{1}{2} + \frac{1}{n} - \left(1 - \left(\frac{1}{2} + \frac{1}{n}\right)\right) \right| = \frac{2}{n} \to 0 \text{ as } n \to \infty.$$

Theorem 5.3. *Let $(\mathfrak{W}, \mathsf{d})$ denote a complete metric space with $|\mathfrak{W}| \geq 3$. Consider $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$ as a continuous $\mathcal{F}$-PKC mapping possessing an approximate fixed-point sequence $\{\kappa_n\} \subset \mathfrak{W}$ where $\mathsf{d}(\kappa_n, \mathfrak{F}\kappa_n) \to 0$ as $n \to \infty$. Then, $\mathfrak{F}$ must have fixed points, and it has no more than two fixed points.*

Proof. Utilising the definition of $\mathcal{F}$-PKC mapping and triangle inequality, assuming $m \geq n$ for sake of simplicity, we get

$$\begin{aligned} \mathsf{d}(\kappa_n, \kappa_m) \leq & \mathsf{d}(\kappa_n, \mathfrak{F}\kappa_n) + \mathsf{d}(\mathfrak{F}\kappa_n, \mathfrak{F}\kappa_{m-1}) + \mathsf{d}(\mathfrak{F}\kappa_{m-1}, \mathfrak{F}\kappa_m) + \mathsf{d}(\mathfrak{F}\kappa_m, \kappa_m) \\ \leq & \mathsf{d}(\kappa_n, \mathfrak{F}\kappa_n) + \mathcal{F}(\mathsf{d}(\kappa_n, \mathfrak{F}\kappa_n), \mathsf{d}(\kappa_{m-1}, \mathfrak{F}\kappa_{m-1}), \mathsf{d}(\kappa_m, \mathfrak{F}\kappa_m)) \\ & + \mathsf{d}(\mathfrak{F}\kappa_m, \kappa_m), \end{aligned}$$

which implies $\mathsf{d}(\kappa_n, \kappa_m) \to 0$ when $m, n \to \infty$. Hence, the sequence $\{\kappa_n\}$ is Cauchy. The remaining part of the proof proceeds as in Theorem 3.2. □

6. Fixed-Point Results in Incomplete Metric Spaces

The theorem presented here is similar to Theorem 1 in [16], with the notable differences being the exclusion of the completeness requirement for the metric space and the inclusion of two extra assumptions, denoted as (c) and (d).

Theorem 6.1. *Consider the metric space* $(\mathfrak{W}, \mathsf{d})$*, where* $|\mathfrak{W}| \geq 3$*. Take a mapping* $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$ *that satisfies the following criteria:*

(a) $\mathfrak{F}$ *has no periodic points with prime period 2.*
(b) $\mathfrak{F}$ *is a PKC mapping.*
(c) $\mathfrak{F}$ *is continuous at* $\kappa^* \in \mathfrak{W}$*.*
(d) *There exists* $\kappa_0 \in \mathfrak{W}$ *such that the sequence* $\{\kappa_n\}$*, where* $\kappa_n = \mathfrak{F}\kappa_{n-1}$ *for* $n = 1, 2, ...,$ *has a convergent subsequence* $\{\kappa_{n_k}\}$ *to* κ^**.*

Then κ^* *is a fixed-point of* $\mathfrak{F}$*. Moreover,* $\mathfrak{F}$ *exhibit no more than two fixed-point.*

Proof. Given that $\kappa_{n_k} \to \kappa^*$ and $\mathfrak{F}$ is continuous at κ^*, it follows that $\mathfrak{F}\kappa_{n_k} = \kappa_{n_k+1} \to \mathfrak{F}\kappa^*$. Note that while κ_{n_k+1} is a subsequence of κ_n, it is not necessarily a subsequence of κ_{n_k}. Let us consider, for the sake of contradiction, that $\kappa^* \neq \mathfrak{F}\kappa^*$. Let us examine two balls:

$$B_1 = B_1(\kappa^*, r) \text{ and } B_2 = B_2(\kappa^*, r),$$

where $r < \frac{1}{3}\mathsf{d}(\kappa^*, \mathfrak{F}\kappa^*)$. Therefore, there exists $N \in \mathbb{N}$ such that for any $i > N$, we get

$$\kappa_{n_i} \in B_1 \text{ and } \kappa_{n_i+1} \in B_2.$$

Implies that,

$$\mathsf{d}(\kappa_{n_i}, \kappa_{n_i+1}) > r \text{ for } i > N. \tag{6.1}$$

If there is no fixed points for the mapping $\mathfrak{F}$ in the sequence $\{\kappa_n\}$, then we can apply the arguments presented in Theorem 3.2. For $n = 3, 4, ...,$ by (3.4), we get

$$\mathsf{d}(\kappa_{n-1}, \kappa_n) \leq \alpha^{\frac{n}{2}-1} a,$$

where $a = \max\{\mathsf{d}(\kappa_0, \kappa_1), \mathsf{d}(\kappa_1, \kappa_2)\}$ and $\alpha = \frac{4\lambda}{3-2\lambda} \in [0, 1)$. Therefore,

$$\mathsf{d}(\kappa_{n_i}, \kappa_{n_i+1}) \leq \alpha^{\frac{n_i+1}{2}-1} a.$$

Which tends to 0 as $i \to \infty$, that leads to a contradiction to (6.1), and implies that $\mathfrak{F}\kappa^* = \kappa^*$.

From the final paragraph of Theorem 3.2, it follows that there can be at most two fixed points. □

In the next theorem corresponding to Theorem 2 in [16], we consider $\mathfrak{F}$ as a PKC mapping defined on an everywhere dense subset of $\mathfrak{W}$. This map $\mathfrak{F}$ is continuous on $\mathfrak{W}$, although this continuity is not necessarily restricted to the point κ^*.

Theorem 6.2. *Consider the metric space* $(\mathfrak{W}, \mathsf{d})$*, where* $|\mathfrak{W}| \geq 3$*. Take a mapping* $\mathfrak{F} : \mathfrak{W} \to \mathfrak{W}$ *which is continuous on* $\mathfrak{W}$ *and fulfils the following requirements:*

(a) $\mathfrak{F}$ *has no periodic points of prime period 2.*
(b) *On* $(\mathcal{M}, \mathsf{d})$*,* $\mathfrak{F}$ *is a PKC mapping, where* $\mathcal{M}$ *is everywhere dense subset of* $\mathfrak{W}$*.*

(c) *There is an element* $\kappa_0 \in \mathfrak{W}$ *such that the sequence* $\{\kappa_n\}$, *where* $\kappa_n = \mathfrak{F}\kappa_{n-1}$ *for* $n = 1, 2, ...$, *possesses a convergent subsequence* $\{\kappa_{n_k}\}$ *converging to* κ^*.

Then $\mathfrak{F}\kappa^* = \kappa^*$ *that is* κ^* *is fixed point. Moreover,* $\mathfrak{F}$ *exhibit no more than two fixed-point.*

Proof. Based on Theorem 6.1, the proof will be given, provided that $\mathfrak{F}$ is demonstrated to be PKC mapping on $\mathfrak{W}$. Consider three pairwise distinct points $\kappa, \varrho, \upsilon$ from $\mathfrak{W}$. To demonstrate that $\mathfrak{F}$ is a PKC mapping, we will examine three distinct cases:

Case 1. $\kappa, \varrho \in \mathcal{M}$ and $\upsilon \in \mathfrak{W}/\mathcal{M}$;
Since $\mathcal{M}$ is an everywhere dense subset of $\mathfrak{W}$, we get a sequence $\{\upsilon_n\} \subset \mathcal{M}$ such that $\upsilon_n \to \upsilon$, $\upsilon_n \neq \kappa$, $\upsilon_n \neq \varrho$ for every n, and $\upsilon_i \neq \upsilon_j$ for $i \neq j$. Consequently,

$$\begin{aligned}
\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) + \mathsf{d}(\mathfrak{F}\varrho, \mathfrak{F}\upsilon) &\leq \mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) + \mathsf{d}(\mathfrak{F}\varrho, \mathfrak{F}\upsilon_n) + \mathsf{d}(\mathfrak{F}\upsilon_n, \mathfrak{F}\upsilon) \\
&\leq \lambda(\mathsf{d}(\kappa, \mathfrak{F}\kappa) + \mathsf{d}(\varrho, \mathfrak{F}\varrho) + \mathsf{d}(\upsilon_n, \mathfrak{F}\upsilon_n)) + \mathsf{d}(\mathfrak{F}\upsilon_n, \mathfrak{F}\upsilon) \\
&\leq \lambda(\mathsf{d}(\kappa, \mathfrak{F}\kappa) + \mathsf{d}(\varrho, \mathfrak{F}\varrho) + \mathsf{d}(\upsilon_n, \upsilon) + \mathsf{d}(\upsilon, \mathfrak{F}\upsilon) + \mathsf{d}(\mathfrak{F}\upsilon, \mathfrak{F}\upsilon_n)) + \mathsf{d}(\mathfrak{F}\upsilon_n, \mathfrak{F}\upsilon) \\
&\leq \lambda(\mathsf{d}(\kappa, \mathfrak{F}\kappa) + \mathsf{d}(\varrho, \mathfrak{F}\varrho) + \mathsf{d}(\upsilon, \mathfrak{F}\upsilon)) + \lambda \mathsf{d}(\upsilon_n, \upsilon) + (1+\lambda)\mathsf{d}(\mathfrak{F}\upsilon_n, \mathfrak{F}\upsilon).
\end{aligned}$$

As $n \to \infty$, we have $\mathsf{d}(\upsilon_n, \upsilon) \to 0$ and $\mathsf{d}(\mathfrak{F}\upsilon_n, \mathfrak{F}\upsilon) \to 0$. Therefore, inequality (2.1) is established.

Case 2. $\kappa \in \mathcal{M}$ and $\varrho, \upsilon \in \mathfrak{W}/\mathcal{M}$;
Let $\{\varrho_n\}$, $\{\upsilon_n\}$ be sequences in $\mathcal{M}$ such that $\varrho_n \to \varrho$ and $\upsilon_n \to \upsilon$. (Here and in next case we will consider the points $\kappa, \varrho, \upsilon$ and all points of the sequences including limit points of the sequences are pairwise distinct.) Then, by applying inequality (2.1) and triangle inequality on $\mathcal{M}$, it follows that

$$\begin{aligned}
&\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) + \mathsf{d}(\mathfrak{F}\varrho, \mathfrak{F}\upsilon) \leq \\
&\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho_n) + \mathsf{d}(\mathfrak{F}\varrho_n, \mathfrak{F}\varrho) + \mathsf{d}(\mathfrak{F}\varrho, \mathfrak{F}\varrho_n) + \mathsf{d}(\mathfrak{F}\varrho_n, \mathfrak{F}\upsilon_n) + \mathsf{d}(\mathfrak{F}\upsilon_n, \mathfrak{F}\upsilon) \\
&\leq \lambda(\mathsf{d}(\kappa, \mathfrak{F}\kappa) + \mathsf{d}(\varrho_n, \mathfrak{F}\varrho_n) + \mathsf{d}(\upsilon_n, \mathfrak{F}\upsilon_n)) + 2\mathsf{d}(\mathfrak{F}\varrho_n, \mathfrak{F}\varrho) + \mathsf{d}(\mathfrak{F}\upsilon_n, \mathfrak{F}\upsilon) \\
&\leq \lambda(\mathsf{d}(\kappa, \mathfrak{F}\kappa) + \mathsf{d}(\varrho, \mathfrak{F}\varrho) + \mathsf{d}(\upsilon, \mathfrak{F}\upsilon)) + \lambda(\mathsf{d}(\varrho_n, \varrho) + \mathsf{d}(\upsilon_n, \upsilon)) \\
&\quad +(2+\lambda)\mathsf{d}(\mathfrak{F}\varrho_n, \mathfrak{F}\varrho) + (1+\lambda)\mathsf{d}(\mathfrak{F}\upsilon_n, \mathfrak{F}\upsilon).
\end{aligned}$$

Once again, the inequality (2.1) is obtained by letting $n \to \infty$.

Case 3. $\kappa, \varrho, \upsilon \in \mathfrak{W}/\mathcal{M}$, and let $\{\kappa_n\}$, $\{\varrho_n\}$ and $\{\upsilon_n\}$ are sequences in $\mathcal{M}$ such that $\kappa_n \to \kappa$, $\varrho_n \to \varrho$ and $\upsilon_n \to \upsilon$. Consequently,

$$\begin{aligned}
&\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\varrho) + \mathsf{d}(\mathfrak{F}\varrho, \mathfrak{F}\upsilon) \leq \\
&\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\kappa_n) + \mathsf{d}(\mathfrak{F}\kappa_n, \mathfrak{F}\varrho_n) + \mathsf{d}(\mathfrak{F}\varrho_n, \mathfrak{F}\varrho) + \mathsf{d}(\mathfrak{F}\varrho, \mathfrak{F}\varrho_n) \\
&+\mathsf{d}(\mathfrak{F}\varrho_n, \mathfrak{F}\upsilon_n) + \mathsf{d}(\mathfrak{F}\upsilon_n, \mathfrak{F}\upsilon) \\
&\leq \lambda(\mathsf{d}(\kappa_n, \mathfrak{F}\kappa_n) + \mathsf{d}(\varrho_n, \mathfrak{F}\varrho_n) + \mathsf{d}(\upsilon_n, \mathfrak{F}\upsilon_n)) + \mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\kappa_n) \\
&+2\mathsf{d}(\mathfrak{F}\varrho, \mathfrak{F}\varrho_n) + \mathsf{d}(\mathfrak{F}\upsilon, \mathfrak{F}\upsilon_n) \\
&\leq \lambda(\mathsf{d}(\kappa, \mathfrak{F}\kappa) + \mathsf{d}(\varrho, \mathfrak{F}\varrho) + \mathsf{d}(\upsilon, \mathfrak{F}\upsilon)) + \lambda(\mathsf{d}(\kappa_n, \kappa) + \mathsf{d}(\varrho_n, \varrho) + \mathsf{d}(\upsilon_n, \upsilon)) \\
&+(1+\lambda)\mathsf{d}(\mathfrak{F}\kappa, \mathfrak{F}\kappa_n) + (2+\lambda)\mathsf{d}(\mathfrak{F}\varrho, \mathfrak{F}\varrho_n) + (1+\lambda)\mathsf{d}(\mathfrak{F}\upsilon, \mathfrak{F}\upsilon_n)
\end{aligned}$$

Once more, allowing $n \to \infty$, the inequality (2.1) is obtained.

Hence, $\mathfrak{F}$ is a PKC mappings. The remaining part of the proof proceeds as in Theorem 6.1. □

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