

TOPOLOGICAL DEGREE METHOD FOR A NEW CLASS OF Φ -HILFER FRACTIONAL DIFFERENTIAL LANGEVIN EQUATION

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ABSTRACT. This article's goal is to examine the existence and uniqueness of the solution for a novel class of Φ -Hilfer fractional differential Langevin equations. The suggested work depends on several basic definitions of topological degree theory and fractional calculus. Employing topological degree method, we were able to obtain the existence result, and we establish the uniqueness result using Banach's fixed-point theorem. We provide an example to illustrate our theoretical finding.

1. INTRODUCTION

Recently, fractional differential equations have garnered significant attention due to their effectiveness in modeling various phenomena across multiple scientific disciplines, including biology, physics, chemistry, mechanics and additional fields, for example, refer to the works [17, 23, 1, 7, 13, 21, 14, 24, 12, 27].

Paul Langevin stated the Langevin equation of the following form $\mathcal{M} \frac{d^2 w}{d\tau^2} = -\lambda \frac{dw}{d\tau} + \mathfrak{h}(\tau)$, in 1908, where $\frac{dw}{d\tau}$ is the velocity of the particle, and $\mathcal{M}$ is its mass and a noise term $\mathfrak{h}(\tau)$. Mathematicians employed fractional derivative to eliminate the noise term, Consequently, it is crucial to study Langevin equations employing Φ -Hilfer fractional derivatives, for more details, see [28, 8, 11].

Although fractional integrals and derivatives have many definitions, the most widely used ones in literature are Riemann-Liouville and Caputo fractional derivatives. Hilfer fractional derivative, which has an order of ϱ and a parameter of $\eta \in [0, 1]$, is an extension of these derivatives that Hilfer [16] proposed. For a more comprehensive understanding of Hilfer and Φ -Hilfer fractional derivatives, we recommend consulting the works in [3, 22, 15, 29]. Moreover, various versions of fixed point theorems are frequently employed to prove the existence and uniqueness of solutions for different types of fractional differential equations. By using coincidence degree theory, Isaia [17] demonstrated a new fixed theorem

Date: Received: Aug 19, 2024; Accepted: Sep 12, 2024.

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2020 *Mathematics Subject Classification.* Primary 26A33; Secondary 34A08.

Key words and phrases. Φ -Hilfer fractional derivative, Langevin equations, Φ -Hilfer fractional differential Langevin equations, Topological degree theory, Fractional differential Langevin equations, Fixed point theorem.

for condensing operators. Researchers used this fixed point theorem of Isaia to prove the existence results for several kinds of nonlinear differential equations [2, 6, 19, 23].

Through the use of Banach contraction principle and topological degree theory for condensing maps, Baitiche et al. [7] recently explored the existence results for ψ -Caputo fractional differential equations. Faree and Panchal [13] examined the existence and uniqueness of solution by using topological structures under certain pertinent criteria for Caputo fractional differential equation. Based on Krasnoselskii, Banach fixed-point theorem and Leray-Schauder's alternative, Hilal et al. [15], discussed the existence and uniqueness of solution for a class of Langevin equations and inclusions. In 2023, Lmou et al. [25] investigated the existence and uniqueness of solution for a new class of ψ -Hilfer-type fractional differential equation with two fractional derivatives of different order by using topological degree theory, Building upon this previous work we extend its theoretical findings to study the Langevin equation, due to the importance of this equation in the literature, and also this work shed light on the use of topological degree method to study this equation.

Inspired by the preceding works, we unite these concepts to examine the existence and uniqueness results for the following problem:

$$\begin{cases} {}^{\mathcal{H}}\mathfrak{D}^{\varrho_1, \eta_1; \Phi}({}^{\mathcal{H}}\mathfrak{D}^{\varrho_2, \eta_2; \Phi} + \lambda)w(\tau) = g(\tau, w(\tau)), \tau \in \Upsilon := [\tilde{a}, \tilde{b}], \\ w(\tilde{a}) = 0, w(\tilde{b}) = \sum_{i=1}^m \iota_i \mathfrak{J}^{\beta_i; \Phi} w(v_i). \end{cases} \quad (1.1)$$

Where ${}^{\mathcal{H}}\mathfrak{D}^{\varrho_j, \eta_j; \Phi}$, is the Φ -Hilfer fractional derivative of order ϱ_j , $0 < \varrho_j \leq 1$, and parameter η_j , $0 \leq \eta_j \leq 1$, $j = 1, 2$, $\tilde{a} \geq 0$, $\lambda \in \mathbb{R}$, $\mathfrak{J}^{\beta_i; \Phi}$ is the Φ -Riemann-Liouville fractional integral of order $\beta_i > 0$, $\iota_i \in \mathbb{R}$, $i = 1, \dots, n$, $\tilde{a} < v_1 < \dots < v_m < \tilde{b}$, $\mathfrak{g} \in \mathfrak{C}(\Upsilon \times \mathbb{R}, \mathbb{R})$.

The originality of this work is studying an innovative case of fractional derivative named the Φ -Hilfer fractional derivative [30], this kind of fractional derivative generalize the well-known fractional derivatives (Riemman-Liouville, Caputo, Φ -Riemman-Liouville, Hilfer-Hadamard, Katugampola derivative), to a variety of values of function Φ and parameters η_j , $j = 1, 2$ such as

- ★ (1.1) simplifies to Riemman-Liouville problem when $\Phi(\tau) = \tau$ and $\eta_j = 0$.
- ★ (1.1) simplifies Caputo problem when $\Phi(\tau) = \tau$ and $\eta_j = 1$.
- ★ (1.1) simplifies to Φ -Riemman-Liouville problem when $\eta_j = 0$.
- ★ (1.1) simplifies to Hilfer problem when $\Phi(\tau) = \tau$.
- ★ (1.1) simplifies to Hilfer-Hadamard problem when $\Phi(\tau) = \log(\tau)$.
- ★ (1.1) simplifies to Katugampola problem when $\Phi(\tau) = \tau^\rho$.

The remainder of the work is arranged as follows : In section 2, we cover some theorems, notations, lemmas, and definitions from fractional calculus as well as significant findings from the topological degree approach that will be applied throughout this research. In section 3, we discuss the existence result for the problem (1.1) applying the topological degree method, and we verify the

uniqueness of solution by using Banach's contraction principle . In section 4, we give an example to support the main result.

2. PRELIMINARIES

Throughout the work, this section will be used to provide definitions, lemmas, and useful notations.

Let X be a Banach space and Γ_X the class of non-empty and bounded subsets of X . The space $\mathfrak{C}(\Upsilon, \mathbb{R})$ represents the Banach space of all continuous functions from Υ to $\mathbb{R}$, equipped with the norm $\|g\| = \sup_{\tau \in \Upsilon} \{|g(\tau)|\}$. We define $\mathfrak{C}^m(\Upsilon, \mathbb{R})$ as the set of m -times absolutely continuous functions given by $\mathfrak{C}^m(\Upsilon, \mathbb{R}) = \left\{ g : \Upsilon \longrightarrow \mathbb{R}; g^{(m-1)} \in \mathfrak{C}(\Upsilon, \mathbb{R}) \right\}$. The closed ball is denoted by $\mathcal{B}_\rho(0)$.

Definition 2.1. [20] Consider the interval $(\tilde{a}, \tilde{b})$, where $-\infty \leq \tilde{a} < \tilde{b} \leq +\infty$, be a finite or infinite interval of the half-axis $(0, +\infty)$, and let $\varrho > 0$. Let $\Phi(\tau)$ be a positive increasing function on $(\tilde{a}, \tilde{b}]$, that possesses a continuous derivative $\Phi'(\tau)$ on $(\tilde{a}, \tilde{b})$. The Φ -Riemann-Liouville fractional integral of a function g with respect to another function Φ on $[\tilde{a}, \tilde{b}]$ is given by

$$\mathfrak{I}_{\tilde{a}+}^{\varrho; \Phi} g(\tau) = \frac{1}{\Gamma(\varrho)} \int_{\tilde{a}}^{\tau} \Phi'(\tau) (\Phi(\tau) - \Phi(\nu))^{\varrho-1} g(\nu) d\nu, \quad (2.1)$$

where $\Gamma(\cdot)$ denoted the Gamma function.

Definition 2.2. [20] Let $\Phi'(\tau) > 0$ and $\varrho > 0$, $m \in \mathbb{N}$. The Φ -Riemann-Liouville derivative of a function g with respect to another function Φ of order ϱ is defined by

$$\mathfrak{D}_{\tilde{a}+}^{\varrho; \Phi} g(\tau) = \left(\frac{1}{\Phi'(\tau)} \frac{d}{d\tau} \right)^m \mathfrak{I}_{\tilde{a}+}^{m-\varrho; \Phi} g(\tau), \quad (2.2)$$

$$= \frac{1}{\Gamma(m-\varrho)} \left(\frac{1}{\Phi'(\tau)} \frac{d}{d\tau} \right)^m \int_{\tilde{a}}^{\tau} \Phi'(\tau) (\Phi(\tau) - \Phi(\nu))^{m-\varrho-1} g(\nu) d\nu, \quad (2.3)$$

where $m-1 < \varrho < m$, $m = [\varrho] + 1$, and $[\varrho]$ denotes the integer part of the real number ϱ .

Definition 2.3. [20] Let $m-1 < \varrho < m$ with $m \in \mathbb{N}$, and let $[\tilde{a}, \tilde{b}]$ be an interval where $-\infty \leq \tilde{a} < \tilde{b} \leq +\infty$. Suppose $g, \Phi \in \mathfrak{C}^m(\Upsilon, \mathbb{R})$ are two functions such that Φ is increasing and $\Phi'(\tau) > 0$ for all $\tau \in \varsigma > 0$. The Φ -Hilfer fractional derivative of a function g of order ϱ and type $0 \leq \eta \leq 1$, is defined by

$$\begin{aligned} {}^H \mathfrak{D}_{\tilde{a}+}^{\varrho, \eta; \Phi} g(\tau) &= \mathfrak{I}_{\tilde{a}+}^{\eta(m-\varrho); \Phi} \left(\frac{1}{\Phi'(\tau)} \frac{d}{d\tau} \right)^m \mathfrak{I}_{\tilde{a}+}^{(1-\eta)(m-\varrho); \Phi} g(\tau), \\ &= \mathfrak{I}_{\tilde{a}+}^{\delta-\varrho; \Phi} \mathfrak{D}_{\tilde{a}+}^{\delta; \Phi} g(\tau), \end{aligned} \quad (2.4)$$

where $m-1 < \varrho < m$ and $m = [\varrho] + 1$, with $\delta = \varrho + \eta(m-\varrho)$.

Lemma 2.4. [20] *Let $\varrho, \eta > 0$, the following semigroup property holds*

$$\mathfrak{I}_{\tilde{a}+}^{\varrho;\Phi} \mathfrak{I}_{\tilde{a}+}^{\eta;\Phi} g(\tau) = \mathfrak{I}_{\tilde{a}+}^{\varrho+\eta;\Phi} g(\tau), \tau > \tilde{a}. \quad (2.5)$$

Proposition 2.5. [20] *Let $\tilde{a} \geq 0$, $z > 0$ and $\tau > \tilde{a}$, then*

$$\begin{aligned} \text{(i)} \quad & \mathfrak{I}_{\tilde{a}+}^{\varrho;\Phi} (\Phi(\nu) - \Phi(\tilde{a}))^{z-1}(\tau) = \frac{\Gamma(z)}{\Gamma(z+\varrho)} (\Phi(\nu) - \Phi(\tilde{a}))^{z+\varrho-1}(\tau). \\ \text{(ii)} \quad & {}^H \mathfrak{D}_{\tilde{a}+}^{\varrho;\Phi} (\Phi(\nu) - \Phi(\tilde{a}))^{z-1}(\tau) = \frac{\Gamma(z)}{\Gamma(z+\varrho)} (\Phi(\nu) - \Phi(\tilde{a}))^{z-\varrho-1}(\tau), \quad m-1 < \varrho < m, \\ & z > m. \end{aligned}$$

Lemma 2.6. [20] *If $g \in \mathfrak{C}^m(\Upsilon, \mathbb{R})$, $m-1 < \varrho < m$, $0 \leq \eta \leq 1$ and $\delta = \varrho + \eta(m-\varrho)$, then*

$$\mathfrak{I}_{\tilde{a}+}^{\varrho;\Phi} ({}^H \mathfrak{D}_{\tilde{a}+}^{\varrho;\Phi} g)(\tau) = g(\tau) - \sum_{k=1}^m \frac{(\Phi(\tau) - \Phi(\tilde{a}))^{\delta-k}}{\Gamma(\delta-k+1)} g_{\Phi}^{[m-k]} I_{\tilde{a}+}^{(1-\eta)(m-\varrho);\Phi} g(\tilde{a}), \quad (2.6)$$

for all $\tau \in \Upsilon$, where $g_{\Phi}^{[m-k]}(\tau) := \left(\frac{1}{\Phi'(\tau)} \frac{d}{d\tau} \right)^{m-k} g(\tau)$.

Definition 2.7. [9] The Kuratowski measure of non-compactness $\kappa : \Gamma_X \rightarrow \mathbb{R}_+$ is defined by:

$$\kappa(\tilde{B}) = \inf \{ \varsigma > 0 : \tilde{B} \text{ covered by a limited number of sets with diameter } \leq \varsigma \}. \quad (2.7)$$

Proposition 2.8. [9] κ satisfies the following properties

- 1- $\kappa(\tilde{A}) = 0$ if and only if $\tilde{A}$ is relatively compact,
- 2- $\tilde{A} \subset \tilde{B} \Rightarrow \kappa(\tilde{A}) \leq \kappa(\tilde{B})$,
- 3- $\kappa(\tilde{A}) = \kappa(\overline{\tilde{A}}) = \kappa(\text{conv}(\tilde{A}))$, where $\overline{\tilde{A}}$ and $\text{conv}(\tilde{A})$ denote the closure and the convex hull of $\tilde{A}$ respectively,
- 4- $\kappa(\tilde{A} + \tilde{B}) \leq \kappa(\tilde{A}) + \kappa(\tilde{B})$,
- 5- $\kappa(\lambda \tilde{A}) = |\lambda| \kappa(\tilde{A})$, $\lambda \in \mathbb{R}$.

Definition 2.9. Consider a continuous bounded map $\mathcal{G} : \tilde{A} \rightarrow X$. The operator $\mathcal{G}$ is called κ -Lipschitz if there exists $p \geq 0$ such that

$$\kappa(\mathcal{G}(\tilde{B})) < p \kappa(\tilde{B}), \text{ for every } \tilde{B} \subset \tilde{A}. \quad (2.8)$$

Moreover, if $p < 1$, then $\mathcal{G}$ is considered a strict κ -contraction.

Definition 2.10. $\mathcal{G} : \tilde{A} \rightarrow X$ is said to be κ -condensing if

$$\kappa(\mathcal{G}(\tilde{B})) < \kappa(\tilde{B}), \quad (2.9)$$

for any bounded and nonprecompact subset $\tilde{B}$ of $\tilde{A}$, such that $\kappa(\tilde{B}) > 0$.

Definition 2.11. The function $\mathcal{G} : \tilde{A} \rightarrow X$ is Lipschitz if there exists $p > 0$ satisfies

$$\|\mathcal{G}(w) - \mathcal{G}(v)\| \leq p \|w - v\|, \text{ for all } w, v \in \tilde{A}. \quad (2.10)$$

Moreover, if $p < 1$, then $\mathcal{G}$ is a strict contraction.

Proposition 2.12. [17, 9] If $\mathcal{G}, \mathcal{Z} : \tilde{A} \longrightarrow X$, are κ -Lipschitz mapping with constants p_1 and p_2 respectively, then $\mathcal{G} + \mathcal{Z} : \tilde{A} \longrightarrow X$ is also a κ -Lipschitz mapping with constant $p_1 + p_2$.

Proposition 2.13. [17, 9] If $\mathcal{G} : \tilde{A} \longrightarrow X$, is compact, then $\mathcal{G}$ is κ -Lipschitz mapping with constant $p = 0$.

Proposition 2.14. [17, 9] If $\mathcal{G} : \tilde{A} \longrightarrow X$, is a Lipschitz mapping with constant p , then $\mathcal{G}$ is a κ -Lipschitz mapping with the same constant p .

Theorem 2.15. [17] Let $\mathcal{C} : \tilde{A} \longrightarrow X$ be κ -condensing and define

$$\Lambda_\varepsilon = \{w \in X : w = \varepsilon \mathcal{C}w, \text{ for some } 0 \leq \varepsilon \leq 1\}. \quad (2.11)$$

If Λ_ε is a bounded set in X , then there exists $\rho > 0$, such that $\Lambda_\varepsilon \in \mathcal{B}_\rho(0)$. Consequently, the degree

$$\deg(I - \varepsilon \mathcal{C}, \mathcal{B}_\rho(0), 0) = 1, \text{ for all } \varepsilon \in [0, 1]. \quad (2.12)$$

As a result, there is at least one fixed point in $\mathcal{C}$, and the set of all fixed points is included in $\mathcal{B}_\rho(0)$.

3. MAIN RESULT

Definition 3.1. A function $w \in \mathfrak{C}(\Upsilon, \mathbb{R})$ is said to be a solution of problem (1.1), if w satisfies the equation ${}^{\mathcal{H}}\mathfrak{D}^{\varrho_1, \eta_1; \Phi}({}^{\mathcal{H}}\mathfrak{D}^{\varrho_2, \eta_2; \Phi} + \lambda)w(\tau) = g(\tau, w(\tau))$, a.e on Υ and the conditions $w(\tilde{a}) = 0$, $w(\tilde{b}) = \sum_{i=1}^m \iota_i \mathfrak{I}^{\beta_i; \Phi} w(v_i)$.

Lemma 3.2. Let $\tilde{a} \geq 0$, $0 < \varrho_j \leq 1$, $0 \leq \eta_j \leq 1$, $\delta_j = \varrho_j + \eta_j - \varrho_j \eta_j$, $j = 1, 2$ and $h \in \mathfrak{C}(\Upsilon, \mathbb{R})$. Then, the following problem has a solution given by the function w :

$$\begin{cases} {}^{\mathcal{H}}\mathfrak{D}^{\varrho_1, \eta_1; \Phi}({}^{\mathcal{H}}\mathfrak{D}^{\varrho_2, \eta_2; \Phi} + \lambda)w(\tau) = h(\tau), \tau \in \Upsilon := [\tilde{a}, \tilde{b}], \\ w(\tilde{a}) = 0, w(\tilde{b}) = \sum_{i=1}^m \iota_i \mathfrak{I}^{\beta_i; \Phi} w(v_i), \tilde{a} < v_i < \tilde{b}, \end{cases} \quad (3.1)$$

if and only if

$$\begin{aligned} w(\tau) = & \mathfrak{I}^{\varrho_1 + \varrho_2; \Phi} h(\tau) - \lambda \mathfrak{I}^{\varrho_2; \Phi} w(\tau) + \frac{(\Phi(\tau) - \Phi(\tilde{a}))^{\delta_1 + \varrho_2 - 1}}{\Delta \Gamma(\delta_1 + \varrho_2)} \left[\mathfrak{I}^{\varrho_1 + \varrho_2; \Phi} h(\tilde{b}) \right. \\ & \left. - \lambda \mathfrak{I}^{\varrho_2; \Phi} w(\tilde{b}) - \sum_{i=1}^m \iota_i \mathfrak{I}^{\varrho_1 + \varrho_2 + \beta_i; \Phi} h(v_i) + \lambda \sum_{i=1}^m \iota_i \mathfrak{I}^{\varrho_2 + \beta_i; \Phi} w(v_i) \right], \end{aligned} \quad (3.2)$$

where

$$\Delta = \sum_{i=1}^m \iota_i \frac{(\Phi(v_i) - \Phi(\tilde{a}))^{\delta_1 + \varrho_2 + \beta_i - 1}}{\Gamma(\delta_1 + \varrho_2 + \beta_i)} - \frac{(\Phi(\tilde{b}) - \Phi(\tilde{a}))^{\delta_1 + \varrho_2 - 1}}{\Gamma(\delta_1 + \varrho_2)} \neq 0. \quad (3.3)$$

Proof. We may formulate the problem (3.1) as

$$I^{\delta_1 - \varrho_1; \Phi} \mathfrak{D}^{\delta_1; \Phi} \left[{}^{\mathcal{H}}\mathfrak{D}^{\varrho_2, \eta_2; \Phi} + \lambda \right] w(\tau) = h(\tau), \tau \in \Upsilon. \quad (3.4)$$

Where $\delta_1 = \varrho_1 + \eta_1 - \varrho_1\eta_1$. By applying the Φ -Riemann-Liouville fractional integral of order ϱ_1 to both sides of equation (3.4), and utilizing Lemma 2.6, we get

$${}^{\mathcal{H}}\mathfrak{D}^{\varrho_2, \eta_2; \Phi} w(\tau) + \lambda w(\tau) = \mathfrak{J}^{\varrho_1; \Phi} h(\tau) + \frac{l_0}{\Gamma(\delta_1)} ((\Phi(\tau) - \Phi(\tilde{a}))^{\delta_1-1}), \quad (3.5)$$

where a constant l_0 is used. The equation (3.5) may thus be expressed as

$$\mathfrak{J}^{\delta_2 - \varrho_2; \Phi} ({}^{\mathcal{H}}\mathfrak{D}^{\delta_2; \Phi} w(\tau)) = \mathfrak{J}^{\varrho_1; \Phi} h(\tau) - \lambda w(\tau) + \frac{l_0}{\Gamma(\delta_1)} ((\Phi(\tau) - \Phi(\tilde{a}))^{\delta_1-1}), \quad (3.6)$$

here $\delta_2 = \varrho_2 + \eta_2 - \varrho_2\eta_2$. By applying the Φ -Riemann-Liouville fractional integral of order ϱ_2 to both sides of equation (3.6), and utilizing Lemma 2.6, we have

$$\begin{aligned} w(\tau) &= \mathfrak{J}^{\varrho_1 + \varrho_2; \Phi} h(\tau) - \lambda \mathfrak{J}^{\varrho_2; \Phi} w(\tau) + \frac{l_0}{\Gamma(\delta_1 + \varrho_2)} ((\Phi(\tau) - \Phi(\tilde{a}))^{\delta_1 + \varrho_2 - 1} \\ &\quad + \frac{l_1}{\Gamma(\delta_2)} ((\Phi(\tau) - \Phi(\tilde{a}))^{\delta_2 - 1}), \end{aligned} \quad (3.7)$$

where a constant l_1 is used. After that, we get that $l_1 = 0$ by applying the boundary condition $x(\tilde{a}) = 0$ in (3.7). Subsequently, we obtain

$$w(\tau) = \mathfrak{J}^{\varrho_1 + \varrho_2; \Phi} h(\tau) - \lambda \mathfrak{J}^{\varrho_2; \Phi} w(\tau) + \frac{l_0}{\Gamma(\delta_1 + \varrho_2)} ((\Phi(\tau) - \Phi(\tilde{a}))^{\delta_1 + \varrho_2 - 1}). \quad (3.8)$$

Using $w(\tilde{b}) = \sum_{i=1}^m \iota_i w(v_i)$ as the boundary condition, we derive in (3.8)

$$l_0 = \frac{1}{\Delta} \left[\mathfrak{J}^{\varrho_1 + \varrho_2; \Phi} h(\tilde{b}) - \lambda \mathfrak{J}^{\varrho_2; \Phi} w(\tilde{b}) - \sum_{i=1}^m \iota_i \mathfrak{J}^{\varrho_1 + \varrho_2 + \beta_i; \Phi} h(v_i) + \lambda \sum_{i=1}^m \iota_i \mathfrak{J}^{\varrho_2 + \beta_i; \Phi} w(v_i) \right]. \quad (3.9)$$

The integral equation in (3.2) may be obtained by substituting the value of l_0 in (3.8). Direct calculation yields the result. $\square$

To make the calculations easier, we employ the following notation.

$$\begin{aligned} \Theta_1 &= \frac{(\Phi(\tilde{b}) - \Phi(\tilde{a}))^{\varrho_1 + \varrho_2}}{\Gamma(\varrho_1 + \varrho_2 + 1)} \\ &\quad + \frac{(\Phi(\tilde{b}) - \Phi(\tilde{a}))^{\delta_1 + \varrho_2 - 1}}{|\Delta| \Gamma(\delta_1 + \varrho_2)} \left[\frac{(\Phi(\tilde{b}) - \Phi(\tilde{a}))^{\varrho_1 + \varrho_2}}{\Gamma(\varrho_1 + \varrho_2 + 1)} + \sum_{i=1}^m |\iota_i| \frac{(\Phi(v_i) - \Phi(\tilde{a}))^{\varrho_1 + \varrho_2 + \beta_i}}{\Gamma(\varrho_1 + \varrho_2 + \beta_i + 1)} \right], \end{aligned} \quad (3.10)$$

$$\begin{aligned} \Theta_2 &= |\lambda| \left\{ \frac{(\Phi(\tilde{b}) - \Phi(\tilde{a}))^{\varrho_2}}{\Gamma(\varrho_2 + 1)} \right. \\ &\quad \left. + \frac{(\Phi(\tilde{b}) - \Phi(\tilde{a}))^{\delta_1 + \varrho_2 - 1}}{|\Delta| \Gamma(\delta_1 + \varrho_2)} \left[\frac{(\Phi(\tilde{b}) - \Phi(\tilde{a}))^{\varrho_2}}{\Gamma(\varrho_2 + 1)} + \sum_{i=1}^m |\iota_i| \frac{(\Phi(v_i) - \Phi(\tilde{a}))^{\varrho_2 + \beta_i}}{\Gamma(\varrho_2 + \beta_i + 1)} \right] \right\}. \end{aligned} \quad (3.11)$$

Assume that

($\mathcal{H}_1$): A constant $\mathcal{L}_g > 0$ exists in order to

$$|g(\tau, w) - g(\tau, v)| \leq \mathcal{L}_g |w - v|, \text{ for each } \tau \in \Upsilon \text{ and } w, v \in \mathfrak{C}(\Upsilon, \mathbb{R}). \quad (3.12)$$

($\mathcal{H}_2$): Two constants $\mathcal{K}_g, \mathcal{P}_g > 0$, and $\alpha \in (0, 1)$ exist such that

$$|g(\tau, w)| \leq \mathcal{K}_g |w|^\alpha + \mathcal{P}_g \text{ for every } \tau \in \Upsilon \text{ and } w \in \mathfrak{C}(\Upsilon, \mathbb{R}). \quad (3.13)$$

The definition of the operators $\mathcal{G}, \mathcal{Z} : \mathfrak{C}(\Upsilon, \mathbb{R}) \longrightarrow \mathfrak{C}(\Upsilon, \mathbb{R})$ comes from Lemma 3.2. through

$$\begin{aligned} \mathcal{G}w(\tau) &= \mathfrak{J}^{\varrho_1 + \varrho_2; \Phi} g(\tau, w(\tau)) + \frac{(\Phi(\tau) - \Phi(\tilde{a}))^{\delta_1 + \varrho_2 - 1}}{\Delta\Gamma(\delta_1 + \varrho_2)} \\ &\times \left[\mathfrak{J}^{\varrho_1 + \varrho_2; \Phi} g(b, w(\tilde{b})) - \sum_{i=1}^m \iota_i \mathfrak{J}^{\varrho_1 + \varrho_2 + \beta_i; \Phi} g(v_i, w(v_i)) \right], \tau \in \Upsilon, \end{aligned} \quad (3.14)$$

and

$$\mathcal{Z}w(\tau) = -\lambda \mathfrak{J}^{\varrho_2; \Phi} w(\tau) + \frac{(\Phi(\tau) - \Phi(\tilde{a}))^{\delta_1 + \varrho_2 - 1}}{\Delta\Gamma(\delta_1 + \varrho_2)} \left[-\lambda \mathfrak{J}^{\varrho_2; \Phi} w(\tilde{b}) + \lambda \sum_{i=1}^m \iota_i \mathfrak{J}^{\varrho_2 + \beta_i; \Phi} w(v_i) \right], \quad (3.15)$$

Next, the fractional integral formula (3.2) may be expressed in the following way:

$$\mathcal{C}w(\tau) = \mathcal{G}w(\tau) + \mathcal{Z}w(\tau), \tau \in \Upsilon. \quad (3.16)$$

Theorem 3.3. *Assuming that ($\mathcal{H}_1$)–($\mathcal{H}_2$) fulfill their requirements, there exists at least one solution to the problem (1.1) such that $w \in \mathfrak{C}(\Upsilon, \mathbb{R})$, provided that $\Theta_2 < 1$. Additionally, the problem (1.1) solution set is confined in $\mathfrak{C}(\Upsilon, \mathbb{R})$.*

To prove Theorem 3.3, we will demonstrate it through a series of lemmas.

Lemma 3.4. *$\mathcal{Z}$ is κ -Lipschitz with the constant Θ_2 , where Θ_2 is given by (3.11).*

Proof. Considering that $w, v \in \mathfrak{C}(\Upsilon, \mathbb{R})$, we obtain

$$\begin{aligned}
|\mathcal{Z}w(\tau) - \mathcal{Z}v(\tau)| &\leq |\lambda| \mathfrak{I}^{\varrho_2; \Phi} |w(\tau) - v(\tau)| + \frac{(\Phi(\tau) - \Phi(\tilde{a}))^{\delta_1 + \varrho_2 - 1}}{|\Delta| \Gamma(\delta_1 + \varrho_2)} \left[|\lambda| \mathfrak{I}^{\varrho_2; \Phi} |w(\tilde{b}) - v(\tilde{b})| \right. \\
&\quad \left. + |\lambda| \sum_{i=1}^m |\iota_i| \mathfrak{I}^{\varrho_2 + \beta_i; \Phi} |w(v_i) - v(v_i)| \right], \\
&\leq |\lambda| \frac{(\Phi(\tilde{b}) - \Phi(\tilde{a}))^{\varrho_2}}{\Gamma(\varrho_2 + 1)} |w(\tau) - v(\tau)| + \frac{(\Phi(\tilde{b}) - \Phi(\tilde{a}))^{\delta_1 + \varrho_2 - 1}}{|\Delta| \Gamma(\delta_1 + \varrho_2)} \\
&\quad \times \left[|\lambda| \frac{(\Phi(\tilde{b}) - \Phi(\tilde{a}))^{\varrho_2}}{\Gamma(\varrho_2 + 1)} |w(\tilde{b}) - v(\tilde{b})| \right. \\
&\quad \left. + |\lambda| \sum_{i=1}^m |\iota_i| \frac{(\Phi(v_i) - \Phi(\tilde{a}))^{\varrho_2 + \beta_i}}{\Gamma(\varrho_2 + \beta_i + 1)} |w(v_i) - v(v_i)| \right], \\
&\leq |\lambda| \left\{ \frac{(\Phi(\tilde{b}) - \Phi(\tilde{a}))^{\varrho_2}}{\Gamma(\varrho_2 + 1)} + \frac{(\Phi(\tilde{b}) - \Phi(\tilde{a}))^{\delta_1 + \varrho_2 - 1}}{|\Delta| \Gamma(\delta_1 + \varrho_2)} \right. \\
&\quad \left. \times \left[\frac{(\Phi(\tilde{b}) - \Phi(\tilde{a}))^{\varrho_2}}{\Gamma(\varrho_2 + 1)} + \sum_{i=1}^m |\iota_i| \frac{(\Phi(v_i) - \Phi(\tilde{a}))^{\varrho_2 + \beta_i}}{\Gamma(\varrho_2 + \beta_i + 1)} \right] \right\} \|w - v\|, \\
&\leq \Theta_2 \|w - v\|.
\end{aligned}$$

If we take the supremum over τ , we get

$$\|\mathcal{Z}w - \mathcal{Z}v\| \leq \Theta_2 \|w - v\|.$$

Finally, $\mathcal{Z}$ is Lipschitz with the constant Θ_2 , and according to Proposition 2.14, $\mathcal{Z}$ is also κ -Lipschitz with the same constant Θ_2 . $\square$

Lemma 3.5. $\mathcal{G}$ is continuous and fulfills the following inequality.

$$\|\mathcal{G}w\| \leq \Theta_1 (\mathcal{K}_g \|w\|^\alpha + \mathcal{P}_g), \quad (3.17)$$

where Θ_1 given by (3.10).

Proof. Let $w_m, w \in \mathfrak{C}(\Upsilon, \mathbb{R})$ such that w_m converges to w in $\mathfrak{C}(\Upsilon, \mathbb{R})$. This means there exists $\mu > 0$ such that $\|w_m\| \leq \mu$ for all $m \geq 1$. Additionally, by taking limits, we have $\|w\| \leq \mu$. Utilizing the continuity of g and condition $(\mathcal{H}_2)$, for every $\tau \in \Upsilon$, we obtain:

$$\begin{aligned}
|g(\tau, w_m(\tau)) - g(\tau, w(\tau))| &\leq |g(\tau, w_m(\tau))| + |g(\tau, w(\tau))| \\
&\leq 2(\mathcal{K}_g \mu^\alpha + \mathcal{P}_g).
\end{aligned}$$

For $s \in [0, \tau], \tau \in \Upsilon$, the function $s \rightarrow 2(\mathcal{K}_g \mu^\alpha + \mathcal{P}_g)$ is integrable. By using the Lebesgue Dominated Convergence theorem, we obtain

$$\begin{aligned}
 |\mathcal{G}w_m(\tau) - \mathcal{G}w(\tau)| &\leq \mathfrak{I}^{\varrho_1+\varrho_2;\Phi} |g(\tau, w_m(\tau)) - g(\tau, w(\tau))| + \frac{(\Phi(\tau) - \Phi(\tilde{a}))^{\delta_1+\varrho_2-1}}{|\Delta|\Gamma(\delta_1 + \varrho_2)} \\
 &\quad \times \left[\mathfrak{I}^{\varrho_1+\varrho_2;\Phi} |g(b, w_m(\tilde{b})) - g(b, w(\tilde{b}))| \right. \\
 &\quad \left. + \sum_{i=1}^m |\iota_i| \mathfrak{I}^{\varrho_1+\varrho_2+\beta_i;\Phi} |g(v_i, w_m(v_i)) - g(v_i, w(v_i))| \right]
 \end{aligned}$$

consequently, as $m \rightarrow \infty$, $\|\mathcal{G}w_m - \mathcal{G}w\| \rightarrow 0$, suggests that $\mathcal{G}$ is continuous. In addition, for every $\tau \in \Upsilon$ we get

$$\begin{aligned}
 |\mathcal{G}w(\tau)| &\leq \mathfrak{I}^{\varrho_1+\varrho_2;\Phi} |g(\tau, w(\tau))| + \frac{(\Phi(\tau) - \Phi(\tilde{a}))^{\delta_1+\varrho_2-1}}{|\Delta|\Gamma(\delta_1 + \varrho_2)} \\
 &\quad \times \left[\mathfrak{I}^{\varrho_1+\varrho_2;\Phi} |g(b, w(\tilde{b}))| + \sum_{i=1}^m |\iota_i| \mathfrak{I}^{\varrho_1+\varrho_2+\beta_i;\Phi} |g(v_i, w(v_i))| \right], \\
 &\leq \left\{ \frac{(\Phi(\tilde{b}) - \Phi(\tilde{a}))^{\varrho_1+\varrho_2}}{\Gamma(\varrho_1 + \varrho_2 + 1)} + \frac{(\Phi(\tilde{b}) - \Phi(\tilde{a}))^{\delta_1+\varrho_2-1}}{|\Delta|\Gamma(\delta_1 + \varrho_2)} \right. \\
 &\quad \left. \times \left[\frac{(\Phi(\tilde{b}) - \Phi(\tilde{a}))^{\varrho_1+\varrho_2}}{\Gamma(\varrho_1 + \varrho_2 + 1)} + \sum_{i=1}^m |\iota_i| \frac{(\Phi(v_i) - \Phi(\tilde{a}))^{\varrho_1+\varrho_2+\beta_i}}{\Gamma(\varrho_1 + \varrho_2 + \beta_i + 1)} \right] \right\} (\mathcal{K}_g \|w\|^\alpha + \mathcal{P}_g),
 \end{aligned}$$

implies that $\|\mathcal{G}w\| \leq \Theta_1(\mathcal{K}_g \|w\|^\alpha + \mathcal{P}_g)$. $\square$

Lemma 3.6. *As a result of $\mathcal{G}$ being compact, $\mathcal{G}$ is κ -Lipschitz with zero constant.*

Proof. In order to demonstrate the compactness of $\mathcal{G}$, let $\mathcal{Q}$ be a bounded set such that $\mathcal{Q} \subset \mathcal{B}_\rho$, it remains to demonstrate the relative compactness of $\mathcal{G}(\mathcal{Q})$ in $\mathfrak{C}(\Upsilon, \mathbb{R})$. To this end, let $w \in \mathcal{Q} \subset \mathcal{B}_\rho$. Using (3.17), we may therefore get

$$\|\mathcal{G}w\| \leq \Theta_1(\mathcal{K}_g \rho^\alpha + \mathcal{P}_g) := v, \quad (3.18)$$

therefore $\mathcal{G}(\mathcal{Q}) \subset \mathcal{B}_v$, as a result $\mathcal{G}(\mathcal{Q})$ is bounded.

To establish the equicontinuity of $\mathcal{G}$, let $\tau_1, \tau_2 \in \Upsilon$ such that $\tau_1 < \tau_2$, and let $w \in \mathcal{Q}$, then we get

$$\begin{aligned}
|\mathcal{G}w(\tau_2) - \mathcal{G}w(\tau_1)| &\leq \mathfrak{I}^{\varrho_1+\varrho_2;\Phi} |g(\tau_2, w(\tau_2)) - g(\tau_1, w(\tau_1))| \\
&+ \frac{(\Phi(\tau_2) - \Phi(\tilde{a}))^{\delta_1+\varrho_2-1} - \Phi(\tau_1) - \Phi(\tilde{a}))^{\delta_1+\varrho_2-1}}{|\Delta|\Gamma(\delta_1 + \varrho_2)} \\
&\times \left[\mathfrak{I}^{\varrho_1+\varrho_2;\Phi} |g(b, w(\tilde{b}))| + \sum_{i=1}^m |\iota_i| \mathfrak{I}^{\varrho_1+\varrho_2+\beta_i;\Phi} |g(v_i, w(v_i))| \right], \\
&\leq \frac{(\mathcal{K}_g \rho^\alpha + \mathcal{P}_g)}{\Gamma(\varrho_1 + \varrho_2)} \left| \int_{\tilde{a}}^{\tau_1} \Phi'(s) \left((\Phi(\tau_2) - \Phi(\nu))^{\varrho_1+\varrho_2-1} - \right. \right. \\
&(\Phi(\tau_1) - \Phi(\nu))^{\varrho_1+\varrho_2-1} \Big) ds + \int_{\tau_1}^{\tau_2} \Phi'(s) (\Phi(\tau_2) - \Phi(\nu))^{\varrho_1+\varrho_2-1} ds \Big| \\
&+ \frac{(\Phi(\tau_2) - \Phi(\tilde{a}))^{\delta_1+\varrho_2-1} - \Phi(\tau_1) - \Phi(\tilde{a}))^{\delta_1+\varrho_2-1}}{|\Delta|\Gamma(\delta_1 + \varrho_2)} \\
&\times \left[\frac{(\Phi(\tilde{b}) - \Phi(\tilde{a}))^{\varrho_1+\varrho_2}}{\Gamma(\varrho_1 + \varrho_2 + 1)} + \sum_{i=1}^m |\iota_i| \frac{(\Phi(v_i) - \Phi(\tilde{a}))^{\varrho_1+\varrho_2+\beta_i}}{\Gamma(\varrho_1 + \varrho_2 + \beta_i + 1)} \right] (\mathcal{K}_g \rho^\alpha + \mathcal{P}_g),
\end{aligned}$$

By leveraging the continuity property of the function Φ , the expression on the right-hand side of the preceding inequality approaches 0 as τ_2 approaches τ_1 . This indicates that $\mathcal{G}(\mathcal{Q})$ exhibits equicontinuity. Consequently, by applying the Arzelà-Ascoli theorem, we deduce that the operator $\mathcal{G}$ is compact. Furthermore, according to Proposition 2.13, $\mathcal{G}$ is Lipschitz with a constant 0. $\square$

We prove that our primary result, Theorem 3.3, is valid since all necessary conditions are fulfilled.

Proof. of Theorem 3.3 Consider the operators $\mathcal{G}$, $\mathcal{Z}$, and $\mathcal{C}$, which are, respectively, represented by (3.14), (3.15), and (3.16). These operators are bounded and continuous. Moreover, by employing Lemma 3.4, $\mathcal{Z}$ is κ -Lipschitz with constant Θ_2 , and by utilizing Lemma 3.6, $\mathcal{G}$ is κ -Lipschitz with constant 0. Consequently, $\mathcal{C}$ is a strict κ -contraction with constant Θ_2 . Lastly, since $\Theta_2 < 1$, $\mathcal{C}$ is κ -condensing.

After that, taking consideration for the subsequent set

$$\Lambda_\varepsilon = \{w \in X : w = \varepsilon \mathcal{C}w, \text{ for some } 0 \leq \varepsilon \leq 1\}. \quad (3.19)$$

The set Λ_ε must still be shown to be bounded in $\mathfrak{C}(\Upsilon, \mathbb{R})$. To do this, allow $w \in \Lambda_\varepsilon$. From there, we obtain $w = \varepsilon \mathcal{C}w = \varepsilon(\mathcal{G}w + \mathcal{Z}w)$. Then, Lemmas 3.5 and 3.4 are utilized to obtain

$$\begin{aligned}
\|w\| &= \varepsilon \|\mathcal{G}w + \mathcal{Z}w\| \\
&\leq \|\mathcal{G}w\| + \|\mathcal{Z}w\| \\
&\leq \Theta_1(\mathcal{K}_g \|w\|^\alpha + \mathcal{P}_g) + \Theta_2 \|w\| \\
&\leq \frac{\Theta_1(\mathcal{K}_g \|w\|^\alpha + \mathcal{P}_g)}{1 - \Theta_2},
\end{aligned}$$

Where Θ_1 and Θ_2 are determined by (3.10) and (3.11), respectively, then the set Λ_ε is bounded in $\mathfrak{C}(\Upsilon, \mathbb{R})$. In the case where Λ_ε is not bounded, then we assume

that $\chi := \|w\|$ tends to infinity. By utilizing the inequality above, we obtain

$$1 \leq \lim_{\chi \rightarrow +\infty} \frac{\Theta_1(\mathcal{K}_g \chi^\alpha + \mathcal{P}_g)}{\chi(1 - \Theta_2)} = 0 \quad (3.20)$$

This leads to a contradiction. Therefore, by employing Theorem 2.15, $\mathcal{C}$ has at least one fixed point, which serves as the solution to problem (1.1). Furthermore, the set of solutions to problem (1.1) is bounded in $\mathfrak{C}(\Upsilon, \mathbb{R})$. $\square$

Next, we employ Banach's Contraction Theorem to establish the uniqueness of solutions for problem (1.1).

Theorem 3.7. *Suppose $(\mathcal{H}_1)$ is valid. The problem (1.1) admits a unique solution on $\mathfrak{C}(\Upsilon, \mathbb{R})$ if $\mathcal{L}_g \Theta_1 + \Theta_2 < 1$.*

Proof. For every $w, v \in \mathfrak{C}(\Upsilon, \mathbb{R})$, and $\tau \in \Upsilon$ we obtain

$$\begin{aligned} & |\mathcal{C}w(\tau) - \mathcal{C}v(\tau)| \\ & \leq |\mathcal{G}w(\tau) - \mathcal{G}v(\tau) + \mathcal{Z}w(\tau) - \mathcal{Z}v(\tau)| \\ & \leq |\mathcal{G}w(\tau) - \mathcal{G}v(\tau)| + |\mathcal{Z}w(\tau) - \mathcal{Z}v(\tau)| \\ & \leq \mathfrak{J}^{\varrho_1 + \varrho_2; \Phi} |g(\tau, w(\tau)) - g(\tau, v(\tau))| + \frac{(\Phi(\tau) - \Phi(\tilde{a}))^{\delta_1 + \varrho_2 - 1}}{|\Delta| \Gamma(\delta_1 + \varrho_2)} \\ & \quad \times \left[\mathfrak{J}^{\varrho_1 + \varrho_2; \Phi} |g(b, w(\tilde{b})) - g(b, v(\tilde{b}))| \right. \\ & \quad \left. + \sum_{i=1}^m |\iota_i| \mathfrak{J}^{\varrho_1 + \varrho_2 + \beta_i; \Phi} |g(v_i, w(v_i)) - g(v_i, v(v_i))| \right] \\ & \quad + |\lambda| \mathfrak{J}^{\varrho_2; \Phi} |w(\tau) - v(\tau)| + \frac{(\Phi(\tau) - \Phi(\tilde{a}))^{\delta_1 + \varrho_2 - 1}}{|\Delta| \Gamma(\delta_1 + \varrho_2)} \left[|\lambda| \mathfrak{J}^{\varrho_2; \Phi} |w(\tilde{b}) - v(\tilde{b})| \right. \\ & \quad \left. + |\lambda| \sum_{i=1}^m |\iota_i| \mathfrak{J}^{\varrho_2 + \beta_i; \Phi} |w(v_i) - v(v_i)| \right], \\ & \leq \left\{ \frac{(\Phi(\tilde{b}) - \Phi(\tilde{a}))^{\varrho_1 + \varrho_2}}{\Gamma(\varrho_1 + \varrho_2 + 1)} + \frac{(\Phi(\tilde{b}) - \Phi(\tilde{a}))^{\delta_1 + \varrho_2 - 1}}{|\Delta| \Gamma(\delta_1 + \varrho_2)} \right. \\ & \quad \times \left[\frac{(\Phi(\tilde{b}) - \Phi(\tilde{a}))^{\varrho_1 + \varrho_2}}{\Gamma(\varrho_1 + \varrho_2 + 1)} + \sum_{i=1}^m |\iota_i| \frac{(\Phi(v_i) - \Phi(\tilde{a}))^{\varrho_1 + \varrho_2 + \beta_i}}{\Gamma(\varrho_1 + \varrho_2 + \beta_i + 1)} \right] \Big\} \mathcal{L}_g |w(\tau) - v(\tau)| \\ & \quad + |\lambda| \left\{ \frac{(\Phi(\tilde{b}) - \Phi(\tilde{a}))^{\varrho_2}}{\Gamma(\varrho_2 + 1)} + \frac{(\Phi(\tilde{b}) - \Phi(\tilde{a}))^{\delta_1 + \varrho_2 - 1}}{|\Delta| \Gamma(\delta_1 + \varrho_2)} \right. \\ & \quad \times \left[\frac{(\Phi(\tilde{b}) - \Phi(\tilde{a}))^{\varrho_2}}{\Gamma(\varrho_2 + 1)} + \sum_{i=1}^m |\iota_i| \frac{(\Phi(v_i) - \Phi(\tilde{a}))^{\varrho_2 + \beta_i}}{\Gamma(\varrho_2 + \beta_i + 1)} \right] \Big\} |w(\tau) - v(\tau)|, \\ & \leq (\mathcal{L}_g \Theta_1 + \Theta_2) \|w - v\|, \end{aligned}$$

where Θ_1 and Θ_2 are given by (3.10) and (3.11) respectively, then when we take the supremum over τ we obtain

$$\|\mathcal{C}w - \mathcal{C}v\| \leq (\mathcal{L}_g\Theta_1 + \Theta_2)\|w - v\|.$$

By leveraging the condition $\mathcal{L}_g\Theta_1 + \Theta_2 < 1$, it can be deduced that $\mathcal{C}$ is a contraction. Consequently, applying the Banach Fixed Point Theorem, $\mathcal{C}$ possesses a unique fixed point, which serves as the unique solution to problem (1.1). $\square$

4. EXAMPLE

Let us examine the following problem:

$$\begin{cases} {}^H\mathfrak{D}_{\frac{1}{5}, \frac{3}{5}; \frac{e^\tau}{6}} \left[{}^H\mathfrak{D}_{\frac{2}{3}, \frac{1}{3}; \frac{e^\tau}{6}} + \frac{1}{9} \right] w(\tau) = \frac{e^{-\tau}}{7 + e^\tau} \left[\frac{|w(\tau)|}{1 + |w(\tau)|} \right], & 0 \leq \tau \leq 1, \\ w(0) = 0, \quad w(1) = \frac{3}{8} \mathfrak{I}_{\frac{5}{2}; \frac{e^\tau}{6}} w\left(\frac{1}{3}\right) + \frac{5}{8} \mathfrak{I}_{\frac{7}{2}; \frac{e^\tau}{6}} w\left(\frac{1}{2}\right). \end{cases} \quad (4.1)$$

Where $\varrho_1 = \frac{1}{5}$, $\varrho_2 = \frac{2}{3}$, $\eta_1 = \frac{3}{5}$, $\eta_2 = \frac{1}{3}$, $\lambda = \frac{1}{9}$, $\tilde{a} = 0$, $\tilde{b} = 1$, $\Upsilon = [0, 1]$, $n = 2$, $\iota_1 = \frac{3}{8}$, $\iota_2 = \frac{5}{8}$, $\beta_1 = \frac{5}{2}$, $\beta_2 = \frac{7}{2}$, $v_1 = \frac{1}{3}$, $v_2 = \frac{1}{2}$ and $\Phi(\tau) = \frac{e^\tau}{6}$.

For $(\tau, w) \in \Upsilon \times \mathbb{R}_+$, let $g(\tau, w) = \frac{e^{-\tau}}{7 + e^\tau} \left(\frac{w}{1 + w} \right)$. This function g is continuous. Moreover, for each $\tau \in \Upsilon$ and every $w, v \in \mathbb{R}_+$, we obtain

$$\begin{aligned} |g(\tau, w) - g(\tau, v)| &\leq \left| \frac{e^{-\tau}}{7 + e^\tau} \right| \left| \frac{w - v}{(1 + w)(1 + v)} \right| \\ &\leq \frac{1}{8} |w - v|. \end{aligned}$$

Therefore, the assumption $(\mathcal{H}_1)$ holds with $\mathcal{L}_g = \frac{1}{8} > 0$. Moreover, for each $\tau \in \Upsilon$ and $w, v \in \mathbb{R}_+$, we get

$$\begin{aligned} |g(\tau, w)| &\leq \left| \frac{e^{-\tau}}{7 + e^\tau} \right| (|w| + 1) \\ &\leq \frac{1}{8} (|w| + 1). \end{aligned}$$

Thus, the conditions $(\mathcal{H}_2)$ hold with $\mathcal{K}_g = \mathcal{P}_g = \frac{1}{8} > 0$ and $\alpha = 1$. Furthermore, $\Theta_2 = 0.107986 < 1$. Consequently, all the requirements of Theorem 3.3 are met. Therefore, the problem (4.1) has at least one solution defined on $[0, 1]$.

For uniqueness, utilizing the provided data, we find that $\delta_1 = \varrho_1 + \eta_1 - \varrho_1\eta_1 = \frac{3}{5} = 0.6$, $|\Delta| \approx 0.801289$, $\Theta_1 = 0.369497$, $\Theta_2 = 0.107986 < 1$, and $\mathcal{L}_g = \frac{1}{8} = 0.125$.

Then $\mathcal{L}_g\Theta_1 + \Theta_2 = 0.125 \times 0.369497 + 0.107986 \simeq 0.154173 < 1$.

Accordingly, by Theorem 3.7 the problem (4.1) has a unique solution on $[0, 1]$.

5. CONCLUSION

In this study, we have explored the existence and uniqueness of solutions for a novel class of Φ -Hilfer fractional differential Langevin equations. Our investigation relies on fundamental definitions from fractional calculus and the topological degree method. The novelty of this research lies in its generality compared to previous works focusing on famous type of fractional derivatives such as Caputo, Riemann-Liouville, Φ -Riemann-Liouville, Hilfer, Hilfer-Hadamard, and Katugampola, considering various values of the function Φ and parameters η_j , $j = 1, 2$. By employing the topological degree method and Banach's fixed point theorem, we established both existence and uniqueness results for problem (1.1). Additionally, we provided a numerical example to illustrate the theoretical findings.

Acknowledgement. The authors would like to thank the referees for the valuable comments and suggestions that improve the quality of our paper.

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