Gulf Journal of Mathematics
Vol 1, Issue 2 (2013) 148-170

BEST L_p-APPROXIMATION OF ANALYTIC FUNCTIONS AND GENERALIZED (α, α)-GROWTH

MOHAMMED HARFAOUI[1] AND DEVENDRA KUMAR[2*]

Abstract. The aim of this paper is the characterization of the generalized order and generalized type of analytic functions of several complex variables in the open set

$$\Omega_R = \left\{ z \in \mathbf{C}^n; \exp V_E(z) < R \right\},$$

in terms of the best polynomial approximation on a compact set E in L_p-norm and the coefficients of the development with respect to the extremal polynomial according to the set

$$\Omega_r = \left\{ z \in \mathbf{C}^n; \exp V_E(z) \leq r \right\}, \quad 1 < r < R,$$

where

$$V_E = \sup \left\{ \frac{1}{d} \ln |P_d|, P_d \quad \text{polynomial of degre} \leq d, \|P_d\|_E \leq 1 \right\}$$

is the Siciak extremal function of a L-regular compact E which defined the pluri-complex Green function with a pole at infinity.

1. Introduction.

Let E be a compact L-regular of $\mathbb{C}^n$. For an entire function f in $\mathbb{C}^n$ developed according an extremal polynomial basis $(A_k)_k$ (see [20]), M.Harfaoui (see [5] and [6]) have generalized growth in terms of coefficients with respect to the sequence $(A_k)_k$. The growth used by M. Harfaoui was defined according to the functions α and β (see [5], page 5, relation (2.14)), with respect to the set:

$$\Omega_r = \left\{ z \in \mathbb{C}^n, \ exp(V_E)(z) < r \right\},$$

where

$$V_E = \sup \left\{ \frac{1}{d} \ \log |P_d|, P_d \quad \text{polynomial of degree} \leq d, \|P_d\|_E \leq 1 \right\}$$

Date: Received: Jul 20, 2013; Accepted: Sep 12, 2013.

* Corresponding author.

2000 *Mathematics Subject Classification.* Primary 30E10; Secondary 41A21, 32E30.

Key words and phrases. Extremal function, L-regular, growth, best approximation of analytic function, L^p-norm.

is the Siciak's extremal function of E which is continuous in $\mathbb{C}^n$ (Because E is L-regular).

The generalized order and generalized type, called (α,β)-order and the (α,β)-type of an entire function f, used by M. Harfaoui ,are defined respectively by:

$$\rho(\alpha,\beta)=\limsup_{r\to+\infty}\frac{\alpha\Big(\log(\|f\|_{\overline{\Omega}_r})\Big)}{\beta(\log(r))}\quad\text{and}\quad\sigma(\alpha,\beta)=\limsup_{r\to+\infty}\frac{\alpha\Big(\|f\|_{\overline{\Omega}_r}\Big)}{\Big[\beta(r)\Big]^{\rho(\alpha,\beta)}}.$$

where

$$\big\|f\big\|_{\overline{\Omega}_r}=\sup_{\overline{\Omega}_r}\big|f(z)\big|.$$

These results have been used to establish the generalized growth in terms of the best approximation in L_p-norm for $p\geq 1$.

Let f be a function defined and bounded on E. For $k\in\mathbb{N}$ put

$$\pi_k^p(E,f)=\inf\Big\{\Big\|f-P\Big\|_{L^p(E,\mu)},\ P\in\mathcal{P}_k(\mathbb{C}^n)\Big\},$$

where $\mathcal{P}_k(\mathbb{C}^n)$ is the family of all polynomials of degree $\leq k$ and μ the well-selected measure (The equilibrium measure $\mu=(dd^cV_E)^n$ associated to a L-regular compact E) (see [1]) and $L^p(E,\mu)$, $p\geq 1$, is the class of all functions such that:

$$\Big\|f\Big\|_{L^p(E,\mu)}=\Big(\int_E\mid f\mid^p d\mu\Big)^{1/p}<\infty.$$

For an entire function f, M. Harfaoui have established a precise relationship between the general growth with respect to the set (see ([5])): $\Omega_r=\Big\{z\in\mathbb{C}^n:\exp(V_E)(z)<r\Big\}$, and the coefficients of the development of f with respect to the sequence $(A_k)_k$, called extremal polynomial (see [20]). After he used these results to give the relationship between the generalized α-growth of f and the sequence $(\pi_k^p(E,f))_k$.

Note that the result of M. Harfaoui does not study the case $0<p<1$ because the triangle inequality is not satisfied.

In this work we study the generalized growth (called α-growth), which will be defined later, for an analytic function in the open set Ω_R, where:

$$(R)\quad\Omega_R=\Big\{z\in\mathbb{C}^n,\exp(V_E(z))<R\Big\},\ \text{for some } R>1,$$

with respect to the set $\Omega_r=\Big\{z\in\mathbb{C}^n,\ \exp(V_E(z))<r\Big\}$, $1<r<R$.

So we obtain a generalization of the results of M.Harfaoui (see [5] and [6]), A.Janik (see [8]), and D. Kumar [9]), replacing $\mathbb{C}^n$ by Ω_R and the entire function in $\mathbb{C}^n$ by analytic function in Ω_R .

After studying the generalized α-growth of an analytic function in Ω_R, for some $R>1$,in terms of its coefficients, we use these results to characterize the generalized α-growth by means of best polynomial approximation on E in L_p-norm for $p>0$.

Recall that the generalized growth used by M.Harfaoui (see [5] and [6]) called (α,β)-growth was defined with respect to functions α and β as:

Let α and β are two positive functions defined on $]0,+\infty[$, strictly increasing and infinitely differentiable such that:

$$\begin{cases} \lim\limits_{x\to+\infty} \alpha(x) = \lim\limits_{x\to+\infty} \beta(x) = \infty, \\ \lim\limits_{x\to+\infty} \dfrac{\alpha(cx)}{\alpha(x)} = 1 \\ \lim\limits_{x\to+\infty} \dfrac{\beta\Big((1+\omega(x)).x\Big)}{\beta(x)} = 1,\ \lim\limits_{x\to0} \omega(x) = 0, \\ \alpha\Big(x/\beta^{-1}(c\alpha(x))\Big) = \big(1+o(x)\big)\alpha(x), \text{for } x\to+\infty. \\ \lim\limits_{x\to+\infty} \left|\dfrac{d\Big(\beta^{-1}\big(c\alpha(x)\big)\Big)}{\alpha(\log(x))}\right| \le b,\ b \text{ is a constant} > 0 \text{ and d(u) means the differential of u.} \end{cases} \tag{1.1}$$

The above result, obtained by M. Harfaoui, does not hold for $\alpha = \beta$.

To overcome this difficulty Kapoor and Nautiyal (see [7]) defined generalized order of slow growth with the help of general function as follows.

Let L denote the class of functions h satisfying the following conditions:

(i) h is defined on $]0,+\infty[$, strictly increasing to infinity differentiable such that $\lim\limits_{x\to+\infty} h(x) = \infty$.

(ii) $\lim\limits_{x\to+\infty} \dfrac{h\Big((1+\omega(x)).x\Big)}{h(x)} = 1$, for every function ω, such that $\lim\limits_{x\to0} \omega(x) = 0$

Let Λ denote the class of function h satisfying condition (i) and

$$\lim_{x\to+\infty} \frac{h(cx)}{h(x)} = 1,\ \text{for every } c > 0.$$

Let Ω be the class of functions h satisfying (i) and (iv) and $\overline{\Omega}$ be the class of functions satisfying (i) and (v) where

(iv) there exists a $\gamma \in \Omega$ and x_o, K_1 and K_2 such that, for all $x > xo$.

$$0 < K_1 \le \lim_{x\to+\infty} \frac{d(h(x))}{d\big(\gamma(logx)\big)} \le K_2 < +\infty$$

(v) $\lim\limits_{x\to+\infty} \dfrac{d(h(x))}{d(logx)} = K_3, 0 < K_3 < +\infty$.

For an entire function $f(z) = \sum\limits_{n=0}^{+\infty} a_n z^n$ and $\alpha = \beta$, R. Ganti and G. S. Srivastava (see [4]) obtained the characterization of generalized order and generalized type in terms of coefficients a_n (see Theorem 2.1 page 64) of the best approximation errors (see Theorem 2.2, page 68).

Recently, Kumar (see [9]) studied generalized order and generalized type in terms of the best approximation errors in L_p-norm, $0 < p \le +\infty$, in several complex variables for compact subset of $\mathbb{C}^n$ and $\alpha = \beta$. Thus, he improved and extended various results of M. Harfaoui [5].

The aim of this work to extended the results M. Harfaoui, R. Ganti and G. S. Srivastava, and D. Kumar, for an analytic function in the open set Ω_R, defined by the relation (R), with respect to the Lp-approximation on a compact subset of $\mathbb{C}^n$ and for $\alpha = \beta$.

2. Definitions and notations

Now we give some definitions and results which will be frequently used in this paper.

Definition 2.1. (see Siciak[17])

Let E be a compact set in $\mathbb{C}^n$ and $\|.\|_E$ the maximum norm on E. The function

$$V_E = \sup\left\{\frac{1}{d}\ \log|P_d|, P_d \quad \text{polynomial of degree} \leq d, \|P_d\|_E \leq 1, d \in \mathbb{N}\right\}$$

is called the Siciak's extremal function of the compact E.

Definition 2.2. A compact E in $\mathbb{C}^n$ is said to be L-regular if the extremal function, V_E, associated to E is continuous on $\mathbb{C}^n$.

Regularity is equivalent to the following Bernstein-Markov inequality (see [16]):
For any $\epsilon > 0$, there exists an open $U \supset E$ such that for any polynomial P

$$\|P\|_U \leq e^{\epsilon.deg(P)}\|P\|_E.$$

In this case we take $U = \left\{z \in \mathbb{C}^n; V_E(z) < \epsilon\right\}$.

Regularity also arises in polynomial approximation. For $f \in \mathcal{C}(E)$, we let

$$\epsilon_d(E, f) = \inf\left\{\|f - P\|_E,\ P \in \mathcal{P}_k(\mathbb{C}^n)\right\}$$

where $\mathcal{P}_k(\mathbb{C}^n)$ is the set of polynomials of degree at most d. Siciak (see [17]) showed:

If E is L-regular, then

$$\limsup_{d\to+\infty}\Big(\varepsilon_d(E, f)\Big)^{1/d} = \frac{1}{r} < 1$$

if and only if f has analytic continuation to

$$\left\{z \in \mathbf{C}^n; V_E(z) < \log\left(\frac{1}{r}\right)\right\}.$$

It is known that if E is an compact L-regular of $\mathbb{C}^n$, there exists a measure μ, called extremal measure, having interesting properties (see [16] and [17]), in particular, we have:

(P_1) Bernstein-Markov inequality:

$\forall\epsilon > 0$, there exists $C = C_\varepsilon$ is a constant such that

$$(BM) : \|P_d\|_E = C(1+\varepsilon)^{s_k}\|P_d\|_{L^2(E,\mu)}, \tag{2.1}$$

for every polynomial of n complex variables of degree at most d.

(P_2) Bernstein-Waish (B.W) inequality:

For every set L-regular E and every real $r > 1$ we have:

$$\big\|f\big\|_E \leq M.r^{deg(f)}\Big(\int_E \mid f \mid^p .d\mu\Big)^{1/p} \tag{2.2}$$

Note that the regularity is equivalent to the Bernstein-Markov inequality.
Let $s : \mathbb{N} \to \mathbb{N}^n, k \to s(k) = (s_1(k), ..., s_n(k))$ be a bijection such that

$$|s(k+1)| \geq |s(k)| \quad \text{where} \quad |s(k)| = s_1(k) + ... + s_n(k).$$

A. Zeriahi (see [20]) has constructed according to the Hilbert Shmidt method a sequence of monic orthogonal polynomials according to a extremal measure (see [16]), $(A_k)_k$, called extremal polynomial, defined by

$$A_k(z) = z^{\alpha(k)} + \sum_{j=1}^{k-1} a_j z^{\alpha(j)} \tag{2.3}$$

such that

$$\Big\|A_k\Big\|_{L^p(E,\mu)} = \left[\inf\left\{\Big\|z^{s(k)} + \sum_{j=1}^{k-1} a_j z^{s(j)}\Big\|_{L^2_{(E,\mu)}}, (a_1, a_2, ..., a_n) \in \mathbb{C}^n\right\}\right]^{1/s_k}.$$

We need the following notations which will be used in the sequel:

(N_1) $\quad \nu_k = \nu_k(E) = \Big\|A_k\Big\|_{L^2(K,\mu)}$.

(N_2) $\quad a_k = a_k(E) = \Big\|A_k\Big\|_E = \max_{z\in E}\big|A_k(z)\big|$ and $\tau_k = \Big(a_k\Big)^{1/s_k}$, where $s_k = deg(A_k)$.

With that notations and (B.W) inequality we have

$$\big\|A_k\big\|_{\overline{\Omega}_r} \leq a_k.r^{s_k} \quad \text{where } s_k = deg(A_k). \tag{2.4}$$

Definition 2.3. Let E be a compact L-regular, α in $\overline{\Omega}$, and f is an analytic function in

$$\Omega_R = \Big\{z \in \mathbb{C}^n : \ \exp(V_E(z)) < R\Big\}$$

for some $R > 1$.

We define the (α, α)-order and (α, α)-type of f (or generalized growth) by

$$\rho(\alpha, \alpha) = \limsup_{r\to R} \frac{\alpha\Big(\log(\|f\|_{\overline{\Omega}_r})\Big)}{\alpha(R/(R-r))},$$

$$\sigma(\alpha, \alpha) = \limsup_{r\to R} \frac{\alpha\Big(\log(\|f\|_{\overline{\Omega}_r})\Big)}{\Big[\alpha(R/(R-r))\Big]^{\rho(\alpha,\alpha)}},$$

where

$$\big\|f\big\|_{\overline{\Omega}_r} = \sup_{z\in\overline{\Omega}_r}\big|f(z)\big| = \sup\Big\{ \mid f(z) \mid : \exp V_E(z) \leq r, 1 < r < R\Big\}.$$

Note that in the classical case $\alpha(x) = \beta(x) = \log(x)$.
We need the following lemma (see [20]).

Lemma 2.1. If E is a compact L-regular subset of $\mathbb{C}^n$, then for every $\theta > 1$, there exists an integer $N_\theta \geq 1$ and a constant $C_\theta > 0$ such that:

$$\pi_k^p(E, f) \leq C_\theta \frac{(r+1)^{N_\theta}}{(r-1)^{2N-1}} \frac{\|f\|_{\overline{\Omega}_{r\theta}}}{r^k}. \tag{2.5}$$

for every $k \geq 1$, every $r > 1$ and every $f \in 0\Big(\overline{\Omega}_{r\theta}\Big)$.

If $f = \sum_{k=0}^{+\infty} f_k.A_k$ be an entire function, then for every $\theta > 1$, there exists $N_\theta \in \mathbb{N}^*$ and $C_\theta > 0$ such that

$$\Big|f_k\Big|\nu_k \leq C_\theta \frac{(r+1)^{N_\theta}}{(r-1)^{2N-1}} \frac{\|f\|_{\overline{\Omega}_{r\theta}}}{r^{s_k}}, \tag{2.6}$$

for every $k \geq 0$ and $r > 1$. C_θ and N_θ do not depend on r or k, or f.

Note that the second assertion of the Lemma is a consequence of the first assertion and it replaces Cauchy inequality for complex function defined on the complex plane $\mathbb{C}$.

3. GENERALIZED ORDER OF ANALYTIC FUNCTIONS AND COEFFICIENTS WITH RESPECT TO EXTREMAL POLYNOMIALS

The purpose of this section is to establish this relationship of the generalized growth of an analytic function in Ω_R with respect to the set

$$\Omega_r = \Big\{z \in \mathbb{C} : \ \exp(V_E(z)) < r\Big\}$$

and its coefficients in the development with respect to the sequence of extremal polynomials.

In all the sequel of this work the function α either belongs to Ω or to $\overline{\Omega}$ without recalling every time.

Let $(A_k)_k$ be a basis of extremal polynomial associated to the set E defined the relation (2.3). We recall that $(A_k)_k$ is a basis of $\mathcal{O}(\mathbb{C}^n)$ (the set of entire functions on $\mathbb{C}^n$). So if f is an entire function then

$$f = \sum_{k\geq 1} f_k.A_k$$

.

Let E be a compact L-regular , $f = \sum_{k\geq 1} f_k.A_k$ be an analytic function, and put:

$$\limsup_{k\to+\infty} \frac{\alpha(s_k)}{\alpha\left[s_k/\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)\right]} = \mu(\alpha, \alpha). \tag{3.1}$$

To prove the main result of this section we need the some Lemmas:

Lemma 3.1. (see [20])

Let E be a compact L-regular subset of $\mathbb{C}^n$. Then

$$\lim_{k\to+\infty}\left[\frac{|A_k(z)|}{\nu_k}\right]^{1/s_k} = \exp(V_E(z)), \tag{3.2}$$

for every $z \in \mathbb{C}^n \setminus \widehat{E}$ the connected component of $\mathbb{C}^n \setminus E$,

$$\lim_{k\to+\infty}\left[\frac{\|A_k\|_E}{\nu_k}\right]^{1/s_k} = 1. \tag{3.3}$$

Lemma 3.2. For every $r > 1$ and $\mu > 0$, the maximum of the function

$$x \to \omega(x,r) = x.\log\big(r/R\big) + \frac{x}{\alpha^{-1}\Big(\alpha(x)/\mu\Big)}$$

is reached for $x = x_r$ the unique solution of the equation

$$x = \alpha^{-1}\left\{\mu\alpha\left[\frac{1 - d\log\left(\Big(\alpha^{-1}(\alpha(x)/\mu)\Big)\right)/d(\log(x))}{\log\Big(R/r\Big)}\right]\right\}. \tag{3.4}$$

Proof.

Put $G(x,\mu) = \alpha^{-1}\Big(\alpha(x)/\mu\Big)$, then $\omega(x,r) = x.\log\big(r/R\big) + \dfrac{x}{G(x,\mu)}$.

The maximum of the function $x \to \omega(x,r)$ is reached for $x = x_r$ solution of the equation $\dfrac{d\omega(x,r)}{dx} = 0$.

We have

$$\frac{d\omega(x,r)}{dx} = 0 \Leftrightarrow \log\big(\frac{r}{R}\big) + \frac{G(x,\mu) - x.\dfrac{dG(x,\mu)}{dx}}{\Big(G(x,\mu)\Big)^2} = 0,$$

or

$$G(x,\mu) = \frac{1 - \dfrac{x}{G(x,\mu)}.\dfrac{dG(x,\mu)}{dx}}{\log\big(R/r\big)}.$$

Since $\dfrac{dG(x,\mu)}{dx} = \dfrac{dG(x,\mu)}{d\log(x)}.\dfrac{d\log(x)}{dx} = \dfrac{1}{x}.\dfrac{dG(x,\mu)}{d\log(x)}$, we get

$$G(x,\mu) = \frac{1 - \dfrac{1}{G(x,\mu)}.\dfrac{dG(x,\mu)}{d\log(x)}}{\log\big(R/r\big)} = \frac{1 - \dfrac{d\log G(x,\mu)}{d\log(x)}}{\log\big(R/r\big)}.$$

We deduce

$$x = x_r = \alpha^{-1}\left\{\mu\alpha\left[\frac{1 - d\log\left(\Big(\alpha^{-1}(\alpha(x)/\mu)\Big)\right)/d(\log(x))}{\log\Big(R/r\Big)}\right]\right\}.$$

Proposition 3.1. Let $f = \sum_{k\geq 0} f_k.A_k$ and E a L-regular compact. For every $r \in]1, R[$, we put:

$$\begin{cases} \overline{M}(f,r) = \sup_{k\in\mathbb{N}} \Big\{ \parallel f_k.A_k \parallel_E .r^{s_k},\ r>0 \Big\} \\ \overline{\rho}(\alpha,\alpha) = \limsup_{r\to R} \dfrac{\alpha\Big(\log(\overline{M}(f,r))\Big)}{\alpha(R/(R-r))} \end{cases}$$

then $\rho(\alpha,\alpha) \leq \overline{\rho}(\alpha,\alpha)$.

Proof. For $z \in \Omega_r$ and $k \in \mathbb{N}$ we have

$$\|f\|_{\overline{\Omega}_r} \leq \sum_{k\geq 0} \mid f_k \mid .\|A_k\|_{\Omega_r} \leq \sum_{k\geq 0} \mid f_k \mid .\|A_k\|_E.r^{s_k}.$$

Write $r = \sqrt{r.R}.\sqrt{r/R}$, then

$$\|f\|_{\overline{\Omega}_r} \leq \sum_{k\geq 0} \mid f_k \mid .\|A_k\|_E.(\sqrt{r.R})^{s_k}.(\sqrt{r/R})^{s_k}.$$

Because $\sqrt{r/R} < 1$ then

$$\|f\|_{\overline{\Omega}_r} \leq \sum_{k\geq 0} \sup_{k\in N} \Big(\mid f_k \mid .\|A_k\|_E.(\sqrt{r.R})^{s_k} \Big).(\sqrt{r/R})^{s_k}$$

thus

$$\|f\|_{\overline{\Omega}_r} \leq \overline{M}(f,r') \sum_{k\geq 0} \big(\sqrt{r/R}\big)^{s_k} \leq \overline{M}(f,r').\frac{1}{1-\sqrt{r/R}}.$$

where $r' = \sqrt{r.R}$
or

$$\log\Big(\|f\|_{\overline{\Omega}_r}\Big) \leq \log\Big(\overline{M}(f,r')\Big) - \log\Big(1-\sqrt{r/R}\Big).$$

Therefore

$$\frac{\alpha\Big(\log\big(\|f\|_{\overline{\Omega}_r}\big)\Big)}{\alpha(R/(R-r))} \leq \frac{\alpha\Big(\log\big(\overline{M}(f,\sqrt{r.R}) - \log\big(1-\sqrt{r/R}\big)\big)\Big)}{\alpha(R/(R-\sqrt{r.R}))}.\frac{\alpha(R/(R-\sqrt{r.R}))}{\alpha(R/(R-r))}$$

We have

$$\lim_{r\to R} \frac{\alpha(R/(R-\sqrt{r.R}))}{\alpha(R/(R-r))} = 1.$$

Hence, proceeding to the upper limit, we get

$$(*) \quad \rho(\alpha,\alpha) \leq \overline{\rho}(\alpha,\alpha)$$

Theorem 3.1. Let E be a compact L-regular and $f = \sum_{k\geq 1} f_k.A_k$ such that

$$\limsup_{k\to+\infty} \frac{\alpha(s_k)}{\alpha\left[s_k/\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)\right]} = \mu(\alpha,\alpha) < \infty. \tag{3.5}$$

.

Then f is analytic in Ω_R for some $R > 1$, and the its (α,α)-order is $\rho(\alpha,\beta) = \mu(\alpha,\alpha)$.

Proof.

By the definition of $\mu(\alpha,\alpha)$ (see relation(3.5)), we have, for every $\epsilon > 0$, there exists $k(\epsilon)$ such that for every $k > k(\epsilon)$,

$$\alpha\big(s_k\big) \leq (\mu+\epsilon)\alpha\Big(\frac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\Big).$$

or

$$\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right) \leq \frac{s_k}{\alpha^{-1}\left(\dfrac{1}{\mu+\epsilon}\big(\alpha(s_k)\big)\right)} \tag{3.6}$$

It is known that for every polynomial P(see [17])

$$|\ P(z)\ | \leq \|\ P\ \|_E \left(\exp\big(V_E(z)\big)\right)^{deg(P)}, \quad \text{for every} \quad z \in \mathbb{C}^n.$$

So for every $r \in]1,R[$, and for $P = f_k.A_k$ we get, for every $z \in \mathbb{C}^n$,

$$|\ f_k\ |\ .\ |\ A_k(z)\ | \leq |\ f_k\ |\ .\ \|\ A_k\ \|_E \left(\exp\big(V_E(z)\big)\right)^{s_k}. \tag{3.7}$$

Then for every $z \in \Omega_r$, we have

$$|\ f_k\ |\ .\ |\ A_k(z)\ | \leq |\ f_k\ |\ .\ \|\ A_k\ \|_E\ .r^{s_k}.$$

So, for every $r \in]1,R[$ the series $\sum f_k.A_k$ is convergent in Ω_r, whence $\sum_{k\geq 1} f_k.A_k$ is analytic in Ω_R.

(1) We show that $\rho(\alpha,\alpha) \leq \mu(\alpha,\alpha)$. By the definition of $\mu(\alpha,\alpha)$ (see relation(3.5)), we have, for every $\epsilon > 0$, there exists $k(\epsilon)$ such that for every $k > k(\epsilon)$,

$$\alpha\big(s_k\big) \leq (\mu(\alpha,\alpha)+\epsilon)\alpha\Big(\frac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\Big),$$

since

$$\log\left(|f_k|.\tau_k^{s_k}.r^{s_k}\right) = s_k \log\left(\frac{r}{R}\right) + \log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right),$$

putting $\overline{\mu} = \mu(\alpha,\alpha)+\epsilon$, then, by 3.6

$$\log\left(|f_k|.\tau_k^{s_k}.r^{s_k}\right) \leq s_k \left[\log\left(\frac{r}{R}\right) + \frac{1}{\alpha^{-1}\Big(\dfrac{1}{\overline{\mu}}.\alpha\big(\alpha(s_k)\big)\Big)}\right]. \tag{3.8}$$

Choose $s_k = x_r = \alpha^{-1}\left\{\overline{\mu}\alpha\left[\dfrac{1-d\Big(\alpha^{-1}(\alpha(x)/\mu)\Big)/d(\log(x))}{\log\Big(R/r\Big)}\right]\right\}$.

Using the proprieties of the function α $\left(\dfrac{d\log(G(x,\overline{\mu}))}{d\log(x)} = O(1)\right)$, and the function $t \to \log(t)$, $(\log(1+x) = (1+o(1)).x, \quad x \to 0)$, we have

$$s_k = x_r = \Big(1+o(1)\Big)\alpha^{-1}\left[\overline{\mu}\alpha\Big(R/(R-r)\Big)\right].$$

and replacing in the relation (3.8), we have

$$\log\Big(\mid f_k \mid . \parallel A_k \parallel_E .r^{s_k}\Big) \leq \big(1+o(1)\big).\alpha^{-1}\Big(\overline{\mu}.\alpha\big(R/(R-r)\big)\Big)\left(\log\big(r/R\big) + \frac{1}{R/(R-r)}\right), \quad k \in \mathbb{N}.$$

Since $\dfrac{R}{R-r} > 1$, then

$$\log\Big(\mid f_k \mid . \parallel A_k \parallel_E .r^{s_k}\Big) \leq C_0.\alpha^{-1}\Big(\overline{\mu}.\alpha\big(R/(R-r)\big)\Big), \quad k \in \mathbb{N}.$$

Proceeding to the maximum for the variable $k \in \mathbb{N}$ we obtain, for r sufficiently close to R

$$\log\Big(\overline{M}(f,r)\Big) \leq C_0.\alpha^{-1}\Big(\overline{\mu}.\alpha\big(R/(R-r)\big)\Big), \quad k \in \mathbb{N}.$$

Then, by the proprieties of α, we obtain

$$\frac{\alpha\Big(\log(\overline{M}(f,r))\Big)}{\alpha(R/(R-r))} \leq \overline{\mu}.$$

Proceeding to upper limit for $r \to R$ we have

$$(**) \quad \overline{\rho}(\alpha,\alpha) \leq \mu(\alpha,\alpha).$$

By the relations (*) of the proposition 3.1 and (**) we obtain $\rho(\alpha,\alpha) \leq \mu$.

(2) Now we show that $\rho(\alpha,\alpha) \geq \mu$. By the definition of $\rho(\alpha,\alpha)$ (see definition 2.3), for every $\epsilon > 0$, there exists $r_\epsilon \in]1,R[$ such that for every $r \geq r_\epsilon$

$$\log\Big(\|f\|_{\overline{\Omega}_r}\Big) \leq \alpha^{-1}\Big[(\rho(\alpha,\alpha)+\epsilon).\alpha(R/(R-r))\Big] \tag{3.9}$$

For every $r \in]1,R[$ we have

$$\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big) = -s_k\log\Big(r/R\Big) + \log\Big(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\Big).$$

Applying the relation (2.6) of the Lemma 2.1 and relation (3.3) of the Lemma 3.1 we get, for every $k \in \mathbb{N}$ and r sufficiently close to R:

$$\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big) \leq -s_k\log\Big(r/R\Big) + \log\left(C_0.\frac{(r-1)^{N_\theta}}{(r+1)^{2N-1}}\right) + \log\Big(\|f\|_{\overline{\Omega}_r}\Big).$$

By the relation (3.9), we have

$$\frac{\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big)}{s_k} \leq \varphi(r,k)$$

where

$$\varphi(r,k) \;=\; \log\big(R/r\big) + \frac{1}{s_k}\log\left(C_0\frac{(r-1)^{N_\theta}}{(r+1)^{2N-1}}\right) + \frac{1}{s_k}\alpha^{-1}\Big[(\rho(\alpha,\alpha)+\epsilon).\alpha(R/(R-r))\Big]$$

For r sufficiently close to R and k sufficiently large, $\varphi(r,k)$ is equivalent to $\log\big(R/r\big)$ for $k \to +\infty$, and $\log\big(R/r\big)$ is equivalent to $\dfrac{R-r}{r} = \dfrac{R}{r} - 1$ for $r \to R$. Then

$$\varphi(r,k) = \big(1+o(1)\big)\log\big(R/r\big)\;\; (k \to +\infty) \;\text{ and }\; \log\big(R/r\big) = \big(1+o(1)\big)\frac{R-r}{r}\;\; (r \to R)$$

Therefore for r sufficiently close to R and k sufficiently large

$$\frac{\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big)}{s_k} \leq \big(1+o(1)\big).\big(\frac{R}{r}-1\big). \tag{3.10}$$

Put $\rho = \rho(\alpha,\alpha)$ and

$$r_k = \frac{R}{1+\alpha^{-1}\Big(\dfrac{1}{\rho+\epsilon}.\alpha(s_k)\Big)}$$

Using the same reasoning as that used in M.Harfaoui (see [5], page 9), and replacing r by r_k in the relation (3.10), we obtain

$$\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big) \leq C_1.\frac{s_k}{\alpha^{-1}(\alpha(s_k)/(\rho+\epsilon))}$$

where C_1 is a constant. Therefore

$$\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big) \leq C_1.\frac{s_k}{\alpha^{-1}(\alpha(s_k)/(\rho+\epsilon))},$$

thus

$$\frac{C_1.s_k}{\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big)} \geq \alpha^{-1}(\alpha(s_k)/(\rho+\epsilon)),$$

or

$$\frac{\alpha(s_k)}{\alpha\left(\dfrac{C_1.s_k}{\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big)}\right)} \leq \rho+\epsilon.$$

Using the proprieties of the function α and proceeding to the upper limit, we obtain

$$\mu(\alpha,\alpha) \leq \rho(\alpha,\alpha).$$

4. GENERALIZED TYPE OF ANALYTIC FUNCTIONS AND COEFFICIENTS WITH RESPECT TO EXTREMAL POLYNOMIALS

The purpose of this section is to establish the relationship of the generalized type of an analytic function in Ω_R with respect to the set

$$\Omega_r = \Big\{ z \in \mathbb{C} : \ \exp(V_E(z)) < r \Big\}$$

and its coefficients in the development according to the sequence of extremal polynomials.

Let E be a compact L-regular and $f = \sum_{k\geq 1} f_k.A_k$ be an analytic function of (α,α)-order $\rho(\alpha,\alpha)$, such that:

$$\tau_E(\alpha, f) = \limsup_{k\to+\infty} \frac{\alpha(s_k/\rho)}{\left\{ \alpha\left(\frac{\rho}{\rho-1}.\frac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)} \right) \right\}^{\rho-1}}. \tag{4.1}$$

We need the following proposition

Proposition 4.1. Let $f = \sum_{k\geq 0} f_k.A_k$ and E a L-regular compact For every $r > 1$.

If we put

$$\begin{cases} \overline{M}(f,r) = \sup_{k\in\mathbb{N}} \Big\{ \ \| \ f_k.A_k \ \|_E \ .r^k, \ r > 0 \Big\} \\ \overline{\sigma_1}(\alpha,\alpha) = \limsup_{r\to R} \dfrac{\alpha\Big(\log(\overline{M}(f,r)) \Big)}{\Big(\alpha(R/(R-r)) \Big)^{\rho}} \end{cases}$$

then $\sigma(\alpha,\alpha) \leq \overline{\sigma_1}(\alpha,\alpha)$.

Proof. For $z \in \Omega_r$ and $k \in \mathbb{N}$, using the similar arguments and inequalities as that used in page 9 in this work, we get

$$\frac{\alpha\Big(\log\left(\|f\|_{\overline{\Omega}_r} \right) \Big)}{\left[\alpha(R/(R-r)) \right]^{\rho(\alpha,\alpha)}} \leq \frac{\alpha\Big(\log\left(\overline{M}(f,\sqrt{r.R}) - \log\left(1-\sqrt{r/R}\right) \right) \Big)}{\left[\alpha(R/(R-\sqrt{r.R})) \right]^{\rho(\alpha,\alpha)}} . \frac{\left[\alpha(R/(R-\sqrt{r.R})) \right]^{\rho(\alpha,\alpha)}}{\left[\alpha(R/(R-r)) \right]^{\rho(\alpha,\alpha)}}$$

Passing to the upper limit we get

$$(*) \quad \sigma(\alpha,\alpha) \leq \overline{\sigma_1}(\alpha,\alpha).$$

Theorem 4.1. Let E be a compact L-regular and $f = \sum_{k\geq 1} f_k.A_k$. If f is of finite generalized (α,α)-order $\rho(\alpha,\alpha)$, and

$$\tau_E(\alpha,f)=\limsup_{k\to+\infty}\frac{\alpha(s_k/\rho)}{\left\{\alpha\left(\frac{\rho(\alpha,\alpha)}{\rho(\alpha,\alpha)-1}.\frac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right)\right\}^{\rho(\alpha,\alpha)-1}}<+\infty \quad (4.2)$$

Then f is analytic in Ω_R, for some $R>1$, and its (α,α)-type $\sigma(\alpha,\alpha)=\tau_E(\alpha,f)$

Proof.

Put $\tau=\tau_E(\alpha,f)$, $\rho=\rho(\alpha,\alpha)$, and $\sigma=\sigma(\alpha,\alpha)$

The function is analytic by the definition $\tau_E(\alpha,f)$ and the arguments used in Theorem 3.1.

(1) We shall show that $\sigma(\alpha,\alpha)\leq\tau_E(\alpha,f)$.

If $\tau<\infty$, by the definition of τ, for every $\epsilon>0$, there exists $k_0\in\mathbb{N}$ such that for every $k\geq k_0$

$$\alpha(s_k/\rho)\leq(\tau+\epsilon).\left\{\alpha\left(\frac{\rho}{\rho-1}.\frac{s_k}{\log\left(\mid f_k\mid\tau_k^{s_k}.R^{s_k}\right)}\right)\right\}^{\rho-1}$$

A simple calculus gives for, $\overline{\tau}=\tau+\epsilon$.

$$\log\left(\mid f_k\mid\tau_k^{s_k}.R^{s_k}\right)\leq\frac{\rho}{\rho-1}\frac{s_k}{\alpha^{-1}\left(\left(\frac{1}{\overline{\tau}}\alpha(s_k/\rho)\right)^{1/(\rho-1)}\right)}. \quad (4.3)$$

Since

$$\log\left(\mid f_k\mid.\tau_k^{s_k}.r^{s_k}\right)\leq s_k\log(r/R)+\log\left(\mid f_k\mid\tau_k^{s_k}.R^{s_k}\right).$$

By the relation (4.3), we get

$$\log\left(\mid f_k\mid\tau_k^{s_k}.r^{s_k}\right)\leq s_k\log(r/R)+\frac{\rho}{\rho-1}\frac{s_k}{\alpha^{-1}\left(\left(\frac{1}{\overline{\tau}}\alpha(s_k/\rho)\right)^{1/(\rho-1)}\right)}. \quad (4.4)$$

For every $r\in]1,R[$, and r and r sufficiently close to R, we put

$$\phi(x,r)=x\log\left(r/R\right)+\frac{\rho}{\rho-1}.\alpha^{-1}\left(\left(\frac{1}{\overline{\tau}}\alpha(x/\rho)\right)^{1/(\rho-1)}\right).$$

then

$$\frac{\partial\phi}{\partial x}(x,r)=\log\left(r/R\right)+\frac{\rho}{\rho-1}\frac{d}{dx}\left\{\frac{x}{\alpha^{-1}\left(\left(\frac{1}{\overline{\tau}}\alpha(x/\rho)\right)^{1/(\rho-1)}\right)}\right\}.$$

Then the maximum of the function $x\to\phi(x,r)$ is reached for $x=x_r$, where x_r is the unique solution of the equation

$$\frac{\partial\phi}{\partial x}(x,r)=0.$$

If we put $F = F\big(x, \overline{\tau}, \frac{1}{\rho-1}\big) = \alpha^{-1}\left(\left(\frac{1}{\overline{\tau}}\alpha(x/\rho)\right)^{1/\rho-1}\right)$ then

$$\phi(x,r) = x\log\big(r/R\big) + \frac{x}{F}.$$

We have

$$\frac{\partial\phi}{\partial x}(x,r) = 0 \Leftrightarrow \log\big(r/R\big) + \frac{\rho}{\rho-1}\left(\frac{F - x\dfrac{dF}{dx}}{F}\right) = 0.$$

We obtain

$$\log\big(R/r\big) = \frac{\rho}{\rho-1}\left(\frac{1 - \dfrac{d\log(F)}{d\log(x)}}{F}\right),$$

because $\dfrac{dF}{dx} = \dfrac{dF}{d\log(x)}.\dfrac{d\log(x)}{dx} = \dfrac{1}{x}.\dfrac{dF}{d\log(x)}$. Since

$$\log\left(\frac{R}{r}\right) = -\log\left(\frac{r}{R}\right) = \log\left(\frac{r-R}{R}+1\right) \sim \frac{r-R}{R} \quad \left(\text{because } \frac{r-R}{R} \to 0\right),$$

and $\left|\dfrac{d\left[\log\left(\alpha^{-1}\left(\left(\alpha(x/\rho)\right)^{1/(\rho-1)}\right)\right)\right]}{d\log(x)}\right| \leq b$, where b is a constant positive.

Then by the proprieties of α, we have

$$F = \big(1+o(1)\big)\frac{\rho}{\rho-1}.\frac{R}{R-r},$$

thus

$$\alpha^{-1}\left(\left(\frac{1}{\overline{\tau}}\alpha(x/\rho)\right)^{1/\rho-1}\right) = \big(1+o(1)\big)\frac{\rho}{\rho-1}.\frac{R}{R-r},$$

therefore

$$x_r = \big(1+o(1)\big)\rho.\alpha^{-1}\left(\overline{\tau}\big(\alpha(R/(R-r))\big)^{\rho-1}\right).$$

Replacing in the relation (4.4), we have

$$\log\left(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\right) \leq \sup_{r\in\mathbb{N}} \phi(x,r) = \phi(x_r, r)$$

Replacing s_k by x_r in this last relation we obtain

$$\log\left(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\right) \leq \frac{\big(1+o(1)\big)\dfrac{\rho-1}{\rho}\alpha^{-1}\left(\overline{\tau}\big(\alpha(R/(R-r))\big)^{\rho-1}\right)}{R/(R-r)}$$

Since $\dfrac{R}{R-r} > 1$ and $\dfrac{\rho-1}{\rho} < 1$, then

$$\log\Big(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\Big) \leq C.\alpha^{-1}\left(\overline{\tau}.\big(\alpha(R/(R-r))\big)^{\rho-1}\right).$$

Then

$$\sup_{k\in\mathbb{N}} \log\Big(\mid f_k \mid \tau_k^{s_k}.r^{s_k} \leq C.\alpha^{-1}\left(\overline{\tau}.\big(\alpha(R/(R-r))\big)^{\rho-1}\right).$$

or

$$\log(\overline{M}(f,r)) \leq C.\alpha^{-1}\left(\overline{\tau}.\big(\alpha(R/(R-r))\big)^{\rho-1}\right).$$

Therefore

$$\frac{\alpha\big(\log(\overline{M}(f,r))\big)}{\big(\alpha(R/(R-r))\big)^{\rho}} \leq \overline{\tau}.$$

Proceeding to the upper limit for $r \to R$, get

$$\overline{\sigma}_1(\alpha,\alpha) = \lim_{r\to R} \frac{\alpha\big(\log(\overline{M}(f,r))\big)}{\big(\alpha(R/(R-r))\big)^{\rho}} \leq \tau.$$

By the relations (*) of the proposition 4.1 we obtain

$$\sigma(\alpha,\alpha) = \lim_{r\to R} \frac{\alpha\big(\log(\overline{M}(f,r))\big)}{\big(\alpha(R/(R-r))\big)^{\rho}} \leq \tau.$$

Thus

$$\sigma(\alpha,\alpha) \leq \tau_E(\alpha,f).$$

The result is obviously holds for $\tau = +\infty$.

(2) Now we show that $\sigma(\alpha,\alpha) \geq \tau_E(\alpha,f)$. Put $\overline{\sigma} = \sigma(\alpha,\alpha) + \epsilon$, $\rho = \rho(\alpha,\alpha)$

Suppose that $\sigma < \infty$. By definition of $\sigma(\alpha,\alpha)$, we have for every $\epsilon > 0$, there exist $r_0 \in]1,R[$, such that for every $r > r_0$ ($R > r > r_0 > 1$)

$$\log\Big(\|f\|_{\overline{\Omega}_r}\Big) \leq \alpha^{-1}\Big[(\overline{\sigma}).\Big(\alpha(R/(R-r))\Big)^{\rho}\Big]$$

Applying the relation (2.6) of the Lemma2.1 and relation (3.3) of the Lemma3.1 we get, for every $k \in \mathbb{N}$ and r sufficiently close to R:

$$\log\Big(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\Big) \leq -s_k\log(r) + \log\big(C_0.\frac{(r+1)^{N_\theta}}{(r-1)^{(2N+1)}} + \log\Big(\|f\|_{\overline{\Omega}_r}\Big).$$

As for every $r \in]1,R[$

$$\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big) = -s_k\log\Big(r/R\Big) + \log\Big(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\Big)$$

then

$$\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big) \leq -s_k\log\big(r/R\big) + \log\Big(C_0.\frac{(r+1)^{N_\theta}}{(r-1)^{(2N+1)}}\Big) + \log\Big(\|f\|_{\overline{\Omega}_r}\Big).$$

or

$$\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big) \leq -s_k \log\Big(r/R\Big)+\log\Big(C_0.\frac{(r+1)^{N_\theta}}{(r-1)^{(2N+1)}}\Big)+\alpha^{-1}\Big[\overline{\sigma}.\Big(\alpha(R/(R-r))\Big)^{\rho}\Big]$$

Therefore

$$\frac{\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big)}{s_k} \leq \omega(r,k)$$

where

$$\omega(r,k) = -\log\big(r/R\big)+\frac{1}{s_k}\log\big(C_0.\frac{(r+1)^{N_\theta}}{(r-1)^{(2N+1)}}\big)+\frac{1}{s_k}\alpha^{-1}\Big[\overline{\sigma}.\Big(\alpha(R/(R-r))\Big)^{\overline{\rho}}\Big]$$

For r sufficiently close to R we have

$$\lim_{k\to+\infty}\omega(r,k) = -\log\big(r/R\big) = \log\big(R/r\big).$$

Then for k sufficiently large and r sufficiently close to R, we have

$$\omega(r,k) = (1+o(1))\log\big(R/r\big), \quad k\to+\infty,$$

then

$$\frac{1}{s_k}\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big) \leq (1+o(1))\log\big(R/r\big). \tag{4.5}$$

Choose

$$r_k = \frac{R.(\rho-1)\alpha^{-1}\Big(\frac{1}{\overline{\sigma}}\alpha(s_k/\rho)\Big)^{1/(\rho-1)}}{\rho+(\rho-1)\alpha^{-1}\Big(\frac{1}{\overline{\sigma}}\alpha(s_k/\rho)\Big)^{1/(\rho-1)}}$$

Using the relation (4.5) and the proprieties of the function $t\to\log(t)$, we obtain, for r sufficiently close to R

$$\frac{\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big)}{s_k} \leq \big(1+o(1)\big)\Big(\frac{R}{r}-1\Big).$$

because

$$\log\Big(\frac{R}{r}\Big) = \log\big(\frac{R-r+r}{r}\big) = \log\big(1+\frac{R-r}{r}\big) \sim \frac{R-r}{r} \quad (r\to R).$$

Replacing r by the chosen r_k in this last relation we have

$$\begin{aligned}\frac{R-r_k}{r_k} &= \frac{R}{r_k}-1\\ &= \frac{\rho+(\rho-1)\alpha^{-1}\Big(\frac{1}{\overline{\sigma}}\alpha(s_k/\rho)\Big)^{1/(\rho-1)}}{(\rho-1)\alpha^{-1}\Big(\frac{1}{\overline{\sigma}}\alpha(s_k/\rho)\Big)^{1/(\rho-1)}}-1\\ &= \frac{\rho}{(\rho-1)\alpha^{-1}\Big(\frac{1}{\overline{\sigma}}\alpha(s_k/\rho)\Big)^{1/(\rho-1)}}.\end{aligned}$$

Then, for r sufficiently close to R and k r sufficiently large we get

$$\frac{\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big)}{s_k} \leq \frac{\rho}{(\rho-1)\alpha^{-1}\Big(\frac{1}{\overline{\sigma}}\alpha(s_k/\rho)\Big)^{1/(\rho-1)}},$$

thus

$$\frac{\rho-1}{(\rho)\alpha^{-1}\Big(\frac{1}{\overline{\sigma}}\alpha(s_k/\rho)\Big)^{1/(\rho-1)}} \leq \frac{s_k}{\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big)}$$

or

$$\Big(\frac{1}{\overline{\sigma}}\alpha(s_k/\rho)\Big)^{1/(\rho-1)} \leq \alpha\left(\frac{\rho}{(\rho-1)}.\frac{s_k}{\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big)}\right)$$

Therefore

$$\frac{1}{\overline{\sigma}}\alpha(s_k/\rho) \leq \left\{\alpha\left(\frac{\rho}{(\rho-1)}.\frac{s_k}{\log\Big(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\Big)}\right)\right\}^{\rho-1}$$

or

$$\frac{\alpha(s_k/\rho)}{\left\{\alpha\left(\frac{\rho}{(\rho-1)}.\frac{s_k}{\log\Big(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\Big)}\right)\right\}^{\rho-1}} \leq \overline{\sigma} = \sigma + \epsilon.$$

Proceeding to the upper limit we obtain $\sigma(\alpha,\alpha) \geq \tau_E(\alpha, f)$.
The result is obviously holds for $\sigma(\alpha,\alpha) = +\infty$.

5. Generalized growth and best polynomial approximation of analytic functions in L^p-norm.

Let E a L-regular compact of $\mathbb{C}^n$. The purpose of this section is to give the relationship between the generalized order and generalized type of an analytic function and rate of convergence to 0 in the best polynomial in L_p-norm on E. We need the following Lemma.

Lemma 5.1. .
Let $f = \sum_{k\geq 0} f_k.A_k$ an element of $L^p(E,\mu)$, for $0 < p \leq +\infty$, and

$$\pi_k^p(E,f) = \inf\left\{\Big\|f - P\Big\|_{L^p(E,\mu)},\ P \in \mathcal{P}_k(\mathbb{C}^n)\right\}$$

then

$$\overline{\lim}_{k\to+\infty}\frac{\alpha(s_k)}{\alpha\left[\dfrac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right]}=\overline{\lim}_{k\to+\infty}\frac{\alpha(k)}{\alpha\left[\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]}. \tag{5.1}$$

and

$$\overline{\lim}_{k\to+\infty}\frac{\alpha(s_k/\rho)}{\left\{\alpha\Big(\dfrac{\rho}{\rho-1}.\dfrac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\Big)\right\}^{\rho-1}}=\overline{\lim}_{k\to+\infty}\frac{\alpha(k/\rho)}{\left\{\alpha\Big(\dfrac{\rho}{\rho-1}.\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\Big)\right\}^{\rho-1}}. \tag{5.2}$$

Proof.

(1) Assume that $p\geq 2$.

If $f\in L^p(E,\mu)$ where $p\geq 2$, then $f=\sum_{k=0}^{+\infty}f_k.A_k$ with convergence in $L^2(E,\mu)$, hence for $k\geq 0$

$$f_k=\frac{1}{{\nu_k}^2}\int_E f.\overline{A}_k d\mu$$

and therefore

$$f_k=\frac{1}{{\nu_k}^2}\int_E (f-P_{k-1}).\overline{A}_k d\mu$$

(because $deg(A_k)=s_k$).

Since the relation,

$$\big|f_k\big|\leq\frac{1}{{\nu_k}^2}\int_E\big|f-P_{k-1}\big|.\big|\overline{A}_k\big|\mu$$

is satisfied, is easily verified by using inequalities Bernstein-Walsh and Holder that we have for all $\varepsilon>0$

$$\big|f_k\big|.\nu_k\leq C_\varepsilon.(1+\varepsilon)^{s_k}.\pi^p_{s_k-1}(E,f). \tag{5.3}$$

for all $k\geq 0$.

(2) Assume that $1\leq p<2$.

Let p' such that $\dfrac{1}{p}+\dfrac{1}{p'}=1$, we have $p'\geq 2$. According to the inequality of Hölder we have:

$$\big|f_k\big|.\nu_k^2\leq\Big\|f-P_{k-1}\Big\|_{L^p(E,\mu)}.\Big\||A_k\Big\|_{L^{p'}(E,\mu)}$$

But

$$\Big\|A_k\Big\|_{L^{p'}(E,\mu)}\leq C.\big\|A_k\big\|_E=C.a_k(E).$$

This shows, according to inequality (BM), that:

$$\big|f_k\big|.\nu_k^2\leq C.C_\varepsilon.(1+\varepsilon)^{s_k}.\Big\|f-P_{s_k-1}\Big\|_{L^p(E,\mu)}.$$

Hence the result

$$\big|f_k\big|.\nu_k^2 \leq C'_\varepsilon.(1+\varepsilon)^{s_k}.\pi_k^{s_k-1}(E,f).$$

In both cases we have therefore

$$\big|f_k\big|.\nu_k^2 \leq A_\varepsilon.(1+\varepsilon)^{s_k}.\pi^p_{s_k-1}(E,f) \tag{5.4}$$

where A_ε is a constant which depends only on ε.

Therefore, for every $R>1$, we have

$$\big|f_k\big|.\nu_k^2.R^{s_k} \leq A_\varepsilon.(1+\varepsilon)^{s_k}.R^{s_k}.\pi^p_{s_k-1}(E,f) \tag{5.5}$$

After proceeding to the upper limit in the relation (5.5) and applying the relation (3.3) of the Lemma 3.1 we get

$$\limsup_{k\to+\infty} \frac{\alpha(s_k)}{\alpha\left[\dfrac{s_k}{\log\Big(|f_k|.\tau_k^{s_k}.R^{s_k}\Big)}\right]} \leq \limsup_{k\to+\infty} \frac{\alpha(k)}{\alpha\left[\dfrac{k}{\log\Big(\pi_k^p(E,f).R^k\Big)}\right]}$$

and

$$\limsup_{k\to+\infty} \frac{\alpha(s_k/\rho)}{\left\{\alpha\Big(\dfrac{\rho}{\rho-1}.\dfrac{s_k}{\log\big(|f_k|.\tau_k^{s_k}.R^{s_k}\big)}\Big)\right\}^{\rho-1}} \leq \limsup_{k\to+\infty} \frac{\alpha(k/\rho)}{\left\{\alpha\Big(\dfrac{\rho}{\rho-1}.\dfrac{k}{\log\Big(\pi_k^p(E,f).R^k\Big)}\Big)\right\}^{\rho-1}}.$$

To prove the other inequality we consider the polynomial of degree s_k,

$$P_k(z)=\sum_{s_j=0}^{k} f_j.A_j$$

then

$$\pi^p_{s_k-1}(E,f) \leq \sum_{s_j=s_k}^{+\infty} \big|f_j\big|.\Big\|A_j\Big\|_{L^p(E,\mu)} \leq C_0 \sum_{s_j=s_k}^{+\infty} \big|f_j\big|.\big\|A_j\big\|_E.$$

By Bernstein-Walsh inequality we have

$$\pi_k^p(E,f) \leq C_\epsilon \sum_{s_j=s_k}^{+\infty} (1+\epsilon)^{s_j}\big|f_j\big|.\nu_j$$

for $k\geq 0$ and $p\geq 1$. If we take as a common factor $(1+\varepsilon)^{s_k}.\big|f_k\big|.\nu_k$ the other factor is convergent thus we have

$$\pi_k^p(K,f) \leq C(1+\epsilon)^{s_k}.\big|f_k\big|.\nu_k$$

and by (3.3) of Lemma 3.1 we have, then

$$\pi_k^p(E,f) \leq C(1+\epsilon)^{2s_k}.\big|f_k\big|.\tau_k^{s_k}.$$

or

$$R^k.\pi_k^p(E,f) \leq C(1+\epsilon)^{2s_k}.R^{s_k}.\left|f_k\right|.\tau_k^{s_k}. \tag{5.6}$$

We deduce

$$\limsup_{k\to+\infty} \frac{\alpha(s_k)}{\alpha\left[\dfrac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right]} \geq \limsup_{k\to+\infty} \frac{\alpha(k)}{\alpha\left[\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]}$$

and

$$\limsup_{k\to+\infty} \frac{\alpha(s_k/\rho)}{\left\{\alpha\Big(\dfrac{\rho}{\rho-1}.\dfrac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\Big)\right\}^{\rho-1}} \geq \limsup_{k\to+\infty} \frac{\alpha(k/\rho)}{\left\{\alpha\Big(\dfrac{\rho}{\rho-1}.\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\Big)\right\}^{\rho-1}}.$$

(3) Let $0<p<1$, of course, for $0<p<1$ the L_p-norm does not satisfy the triangle inequality. But our relations (5.3) and (5.4) are also satisfied for $0<p<1$ (see [9]), because using Holder's inequality we have, for some $M>0$ and all $r>p$ (p fixed)

$$\|\| f \|_{L^p(E,\mu)} \leq M.\ \| f \|_{L^r(E,\mu)}\ .$$

Using the inequality

$$\int_E |\ f\ |^p\ d\mu \leq \| f \|_E^{p-r}\ .\int_E |\ f\ |^r\ d\mu$$

we get

$$\| f \|_{L^p(E,\mu)} \leq \| f \|_E^{1-(r/p)}\ .\ \| f \|_{L^r(E,\mu)}^{r/p}\ .$$

We deduce that (E,μ) satisfies the Bernstein.Markov inequality. For $\epsilon>0$ there is a constant $C=C(\epsilon,p)>0$ such that, for all (analytic) polynomials P we have

$$\|\| P \|_E \leq C(1+\epsilon)_{deg(P)}.\ \| P \|_{L^p(E,\mu)}\ .$$

Thus if (E,μ) satisfies the Bernstein-Markov inequality for one $p>0$, then (5.3) and (5.4) are satisfied for all $p>0$.

Applying this Lemma we get the following main results:

Theorem 5.1. Let $f\in L^p(E,\mu)$, for $0<p\leq+\infty$, then

(1) f is μ-almost-surely the restriction to E of an analytic function in Ω_R of finite generalized order $\rho(\alpha,\alpha)$ if and only if

$$\rho(\alpha,\alpha) = \limsup_{k\to+\infty} \frac{\alpha(k)}{\alpha\left[\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]} + \infty. \tag{5.7}$$

(2) f is μ-almost-surely the restriction to E of an an analytic function in Ω_R of finite generalized order $\rho(\alpha,\alpha)$ and finite generalized type $\sigma(\alpha,\alpha)$ if and only if

$$\sigma(\alpha,\alpha)=\limsup_{k\to+\infty}\frac{\alpha(k/\rho)}{\left\{\alpha\Big(\frac{\rho}{\rho-1}.\frac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\Big)\right\}^{\rho-1}}+\infty. \tag{5.8}$$

Proof. We prove only the first assertion of the theorem, the second assertion follows in similar way of the first with help of the second assertion of Lemma5.1 and the arguments used above.

Suppose that f is μ-almost-surely the restriction to E of an entire function g of general order ρ $(0<\rho<+\infty)$ and show that $\rho=\rho(\alpha,\alpha)$.

We have $g\in L^p(E,\mu)$, $p\geq 2$ and $g=\sum_{k\geq 0} g_k.A_k$ in $L^2(E,\mu)$ Since g is an element of $L^2(E,\mu)$ then $g=\sum_{k=0}^{+\infty} g_k.A_k$ and according to the theorem 3.1

$$\rho(g,\alpha,\alpha)=\limsup_{k\to+\infty}\frac{\alpha(s_k)}{\alpha\left[\dfrac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right]}.$$

and with the Lemma 5.1 we have

$$\limsup_{k\to+\infty}\frac{\alpha(s_k)}{\alpha\left[\dfrac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right]}=\limsup_{k\to+\infty}\frac{\alpha(k)}{\alpha\left[\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]}$$

But $g=f$ on E hence

$$\rho=\limsup_{k\to+\infty}\frac{\alpha(k)}{\alpha\left[\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]}<+\infty.$$

Now suppose that $f\in L^p(E,\mu)$, for $p>0$ such that the relation (5.7) hold. The proof is done in three steps $p\geq 2$, $1\leq p<2$ and $0<p<1$.

Step.1. Let $p\geq 2$, then $f=\sum_{k=0}^{+\infty} f_k.A_k$, because f is an element of $L^2(E,\mu)$ $\Big((L^p(E,\mu))_{p\geq 1}$ is decreasing sequence$\Big)$.

Consider in $\mathbb{C}^n$ the series $\sum f_k.A_k$, $k\geq 0$. By the relation (5.7) and the inequality (BW) we have the inequality on coefficients $\mid A_k \mid$ (2.4) it can be seen that this series converges normally on all compact of $\mathbb{C}^n$, to an analytic function denoted f_1. We have $f_1=f$, obviously, μ-almost surly on E.

and by Theorem 3.1 we have

$$\limsup_{k\to+\infty}\frac{\alpha(s_k)}{\alpha\left[\dfrac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right]}=\limsup_{k\to+\infty}\frac{\alpha(k)}{\alpha\left[\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]}<+\infty.$$

According to the Lemma 5.1 we get

$$\rho(f_1)=\limsup_{k\to+\infty}\frac{\alpha(k)}{\alpha\left[\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]}<+\infty.$$

Let $f_1=\sum_{k\geq 0} f_k.A_k$, then $f_1(z)=f(z)$ μ-almost surely for every z in E. Therefore the (α,α)-order of f_1 is:

$$\rho(f_1,\alpha,\alpha)=\limsup_{k\to+\infty}\frac{\alpha(k)}{\alpha\left[\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]}<+\infty.$$

(see Theorem 3.1). By Lemma 5.1 we check $\rho(f_1)=\rho$ so the proof is completed.

Step.2. Now let $p\in[1,2[$ and $f\in L^p(E,\mu)$. By (BM) inequality and Hölder inequality we have again the inequality the relation (5.4) and by the previous arguments we obtain the result.

Step.3. Let $0<p<1$, relations (5.3) and (5.4) are also satisfied for $0<p<1$ (see Lemma5.1).

We deduce that (E,μ) satisfies the Bernstein.Markov inequality. For $\epsilon>0$ there is a constant $C=C(\epsilon,p)>0$ such that, for all (analytic) polynomials P we have

$$\|\| P \|_E\leq C(1+\epsilon)_{deg(P)}.\ \| P \|_{L^p(E,\mu)}\ .$$

Thus if (E,μ) satisfies the Bernstein-Markov inequality for one $p>0$

Therefore the inequalities (5.3) and (5.4) are satisfied for all $p>0$.

Hence proceeding to the upper limit in the inequalities (5.5) and (5.6) we obtain the desired result.

The rest of proof is easily deduced using the same reasoning as in step.1 and step.2, with help of the Lemma 5.1 and the Theorem 3.1.

Acknowledgement. The authors are very much thankful to learned referee for giving fruitful comments to improve the paper.

References

[1] E.Bedford and B.A. Taylor,*A new capacity for plurisubharmonic function*,Acta Math 149, (1982); 1-40.

[2] S.N. Bernstein, Lessons on the properties and extremal best approximation of analytic functions of one real variable,Gautier-Villars, Paris,1926 .

[3] R.P. Boas, Entire functions,Academic Press, New York,1954 .

[4] R. Ganti and G. S.Srivastava, *Approximation of entire functions of slow growth*, Gen. Math. 14 (2006), no. 2, 59–76.

[5] M. Harfaoui, *Generalized order and best approximation of entire function in L_p- norm*, Internat. J. Math. Math. Sci. 2010 (2010), 15 pages, Article ID 360180. doi:10.1155/2010/360180.

[6] M. Harfaoui, *Generalized growth of entire function by means best approximation in L_p-norm*, J.P J. Math. Sci. 1(2) (2011), 111-126.

[7] G.P. Kapoor, A. Nautiyal : *Polynomial approximation of an entire function of slow growth.* J. Approx. Theory 32, 6475 (1981)

[8] A. Janik, *A characterisation of the growth of analytic functions by means of polynomial approximation*, Univ.Iagel. Acta Math. 24(1984), 295-319.

[9] D. Kumar *Generalized growth and best approximation of entire functions in L^p-norm in several complex variables.* Ann. Univ. Ferrara Sez. VII Sci. Mat. 57 (2011), no. 2, 353–372

[10] B. K.Lahiri and Dibyendu Banerjee, *Generalized relative order of entire functions*, Proc. Nat. Acad. Sci. India, 72(A), IV(2002) , 351-371.

[11] B.Mitiagin, *Approximate dimension and bases in nuclear spaces*, no. 4, 59-128, Uspehi.Mat. Nauk (1961), 63-132.

[12] T.V. Nguyen , *Bases de schauder dans certains espaces de fonctions holomorphes*, Ann. Polon. Math. 24,(1982), 325-333 .

[13] T.V. Nguyen , *growth and best polynomial approximation of entire functions of several variables*, Ann. Polon. Math. 24,(1972), 169-253 .

[14] R. Reddy A. R, *Approximation of entire function*, J. Approx. Theory 3 (1970), 128-137.

[15] R. Reddy A. R, *Best polynomial approximation of entire functions*, J. Approx. Theory 5 (1972), 97-112.

[16] J. Siciak, *On some extremal functions and their applications in the theory of analytic function of serval complex wariables*, Trans. Amer. Math. Soc. 105(1962), 332-357.

[17] J. Siciak, *Extremal plurisubharmonic functions in $\mathbb{C}^n$*, Priceedings of the first Fininish-Polish Summer School in Complex Analysis at Prodlesice (Lodz 1977), University of Lodz, 115-152.

[18] J.L. Walsh, *Interpolation and Approximation by Rationnal Functions in the complex domain*, Colloq. Publications 20; Amer. Math. Soc. ProvidenceR.I. 1969.

[19] T. Winiarski, *Approximation and interpolation of functions*, Ann. Polon. Math. 23,(1970), 259-273.

[20] A. Zeriahi,*Best increasing polynomial approximation of entire functions on affine algebraic varieties* Ann. Inst. Fourier (Grenoble) 37 (1987), no. 2, 79-104

[21] A. Zeriahi,*Families of almost everywhere bounded polynomials* Bull. Sci. Math. (2) 107 (1983), no. 1, 81-91

[1] Mohammed Harfaoui

University Hassan II-Mohammedia,, Laboratory of Mathematics, cryptography and mechanics, F.S.T, B.O.Box 146, Mohammedia,, Morocco.

E-mail address: mharfaoui04@yahoo.fr

[2] Department of Mathematics Faculty of Science, [Kingdom of Saudi Arabia, Ministry of Higher Education], Al-Baha University,P.O.Box.1988, Alaqiq. AL-Bah-65431, Saudi Arabia, K.S.A.

E-mail address: d_kumar001@rediffmail.com