Gulf Journal of Mathematics
Vol 1, Issue 2 (2013) 171-179

FUZZY $\mathcal{H}$-WEAK CONTRACTIONS AND FIXED POINT THEOREMS IN FUZZY METRIC SPACES

SATISH SHUKLA

Abstract. The purpose of this paper is to introduce the notion of a fuzzy $\mathcal{H}$-weak contraction and prove a fixed point theorem for fuzzy $\mathcal{H}$-weak contractions without exploiting the notion of continuity in M-complete fuzzy metric spaces. Examples are given to show that our result is a proper extension of some known results in the literature.

1. Introduction and preliminaries

Zadeh [11] introduced the notion of fuzzy sets and considered the nature of uncertainty in the behavior of a given system possesses fuzzy rather than stochastic nature. Kramosil and Michálek [7] introduced the concept of fuzzy metric spaces. They defined the Cauchy (G-Cauchy) sequence in a fuzzy metric space and completeness (G-completeness) of a fuzzy metric space. George and Veeramani [1] modified the definition of fuzzy metric spaces due to Kramosil and Michálek, also they defined M-Cauchy sequence and M-completeness of a fuzzy metric space. The fixed point theory in fuzzy metric spaces was started by Grabiec [3] which has become of interest for several authors. Gregori and Sapena [4] introduced the concept of fuzzy contractive mappings and proved some fixed point results for fuzzy contractive mappings.

Fuzzy contractive mappings play a very essential role in proving of the fixed point theorems in fuzzy metric spaces. Very recently, Wardowski [10] generalized the concept of fuzzy contractive mappings by introducing the concept of a fuzzy $\mathcal{H}$-contractive mapping. He proved a fixed point result for fuzzy $\mathcal{H}$-contractive mappings in an M-complete fuzzy metric space.

In this paper, we introduce the notion of fuzzy $\mathcal{H}$-weak contractions and generalize the $\mathcal{H}$-contractive mappings. A fixed point result for fuzzy $\mathcal{H}$-weak contractions in M-complete fuzzy metric spaces is proved which generalizes the results of Wardowski [10] and Gregori and Sapena [4]. Some examples are given which illustrate the results.

Date: Received: Aug 20, 2013; Accepted: Sep 4, 2013.
2010 *Mathematics Subject Classification.* Primary 54H25; Secondary 47H10.
Key words and phrases. Fuzzy metric space; fuzzy $\mathcal{H}$-weak contraction; fixed point.

2. Preliminaries

Firstly, we recall some known definitions and the properties about fuzzy metric spaces.

Definition 2.1 (Schweizer and Sklar [8]). A binary operation $T\colon [0,1]\times[0,1]\to[0,1]$ is called a t-norm if the following conditions are satisfied:

(T1) $T(a,b)=T(b,a)$;
(T2) $T(a,b)\le T(c,d)$ for $a\le c, b\le d$;
(T3) $T(T(a,b),c)=T(a,T(b,c))$;
(T4) $T(a,0)=0, T(a,1)=1$;

for all $a,b,c,d\in[0,1]$.

For $a,b\in[0,1]$, instead of $T(a,b)$ we will use the infix notation $a*b$. For $a_1,a_2,\dots,a_n\in[0,1]$ and $n\in\mathbb{N}$, the product $a_1*a_2*\cdots*a_n$ will be denoted by $\prod_{i=1}^{n} a_i$. For the details concerning t-norms the reader is referred to [5, 6].

In the present paper we will use the following definition of a fuzzy metric space:

Definition 2.2 (George and Veeramani [1]). A triple $(X,M,*)$ is called a fuzzy metric space if X is a nonempty set, $*$ is a continuous t-norm and $M\colon X^2\times(0,\infty)\to[0,1]$ is a fuzzy set satisfying following conditions:

(GV1) $M(x,y,t)>0$;
(GV2) $M(x,y,t)=1$ if and only if $x=y$;
(GV3) $M(x,y,t)=M(y,x,t)$;
(GV4) $M(x,z,t+s)\ge M(x,y,t)*M(y,z,s)$;
(GV5) $M(x,y,\cdot)\colon(0,\infty)\to[0,1]$ is a continuous mapping;

for all $x,y,z\in X$ and $s,t>0$.

For topological properties of a fuzzy metric space in the sense of George and Veeramani the reader is referred to [1].

Remark 2.3 (George and P. Veeramani [2]). Let $(X,M,*)$ be a fuzzy metric space, then the function $M(x,y,\cdot)$ is a nondecreasing function.

Definition 2.4 (George and Veeramani [1]). Let $(X,M,*)$ be a fuzzy metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is called an M-Cauchy sequence if for all $\varepsilon\in(0,1), t>0$ there exists $n_0\in\mathbb{N}$ such that

$$M(x_n,x_m,t)>1-\varepsilon \quad \text{for all} \quad n,m>n_0.$$

Definition 2.5 (Grabiec [3]). Let $(X,M,*)$ be a fuzzy metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is called a G-Cauchy sequence if; for all $t>0, p>0$ we have

$$\lim_{n\to\infty} M(x_{n+p},x_n,t)=1.$$

The above definitions of Cauchy sequences are different (see [9]).

Theorem 2.6 (George and Veeramani [1]). *Let $(X, M, *)$ be a fuzzy metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ converges to x if and only if*

$$\lim_{n\to\infty} M(x_n, x, t) = 1 \quad \text{for all} \quad t > 0.$$

Definition 2.7 (See [10] and the references therein). An M-complete (G-complete) fuzzy metric space is a fuzzy metric space in which every M-Cauchy (G-Cauchy) sequence is convergent.

Definition 2.8 (Gregori and Sapena [4]). Let $(X, M, *)$ be a fuzzy metric space. A mapping $T\colon X \to X$ is called t-uniformly continuous if for all $r \in (0, 1)$ there exists $s \in (0, 1)$ such that

$$M(x, y, t) \geq 1 - s \text{ implies } M(Tx, Ty, t) \geq 1 - r \text{ for all } x, y \in X, t > 0.$$

Remark 2.9. If T is t-uniformly continuous then it is uniformly continuous for the uniformity generated by M, thus it is continuous for the topology deduced from M. For the details concerning a uniform structure in a fuzzy metric space, see [4].

Definition 2.10 (Gregori and Sapena [4]). Let $(X, M, *)$ be a fuzzy metric space. A mapping $T\colon X \to X$ is called a fuzzy contractive mapping if

$$\frac{1}{M(Tx, Ty, t)} - 1 \leq k\left[\frac{1}{M(x, y, t)} - 1\right] \tag{2.1}$$

for all $x, y \in X, t > 0$, where $k \in (0, 1)$.

Now we state some definitions and results about $\mathcal{H}$-contractions.

Definition 2.11 (Wardowski [10]). Let $\mathcal{H}$ be the family of mappings $\eta\colon (0, 1] \to [0, \infty)$ satisfying the following two conditions:

(H1) η transforms $(0, 1]$ onto $[0, \infty)$;
(H2) for all $s, t \in (0, 1]$, $s < t$ implies $\eta(s) > \eta(t)$ (that is η is strictly decreasing).

Note that (H1) and (H2) imply $\eta(1) = 0$.

Proposition 2.12 (Wardowski [10]). Let $(X, M, *)$ be a fuzzy metric space and let $\eta \in \mathcal{H}$. A sequence $\{x_n\}$ in X is M-Cauchy if and only if, for all $\varepsilon > 0, t > 0$ there exists $n_0 \in \mathbb{N}$ such that $\eta(M(x_m, x_n, t)) < \varepsilon$ for all $n, m > n_0$.

Proposition 2.13 (Wardowski [10]). Let $(X, M, *)$ be a fuzzy metric space and let $\eta \in \mathcal{H}$. A sequence $\{x_n\}$ in X converges to $x \in X$ if and only if $\lim_{n\to\infty} \eta(M(x_n, x, t)) = 0$ for all $t > 0$.

Proposition 2.14 (Wardowski [10]). Let $(X, M, *)$ be a fuzzy metric space, $T\colon X \to X$ be a mapping and let $\eta \in \mathcal{H}$. The mapping T is t-uniformly continuous if and only if, for all $r > 0$ there exists $s > 0$ such that $\eta(M(x, y, t)) \leq s$ implies $\eta(M(Tx, Ty, t)) \leq r$ for all $x, y \in X, t > 0$.

Definition 2.15 (Wardowski [10])**.** Let $(X, M, *)$ be a fuzzy metric space. A mapping $T\colon X \to X$ is said to be fuzzy $\mathcal{H}$-contractive with respect to $\eta \in \mathcal{H}$ if the following holds:

$$\eta(M(Tx, Ty, t)) \leq k\eta(M(x, y, t)) \tag{2.2}$$

for all $x, y \in X$, $t > 0$, where $k \in (0, 1)$.

By Proposition 2.14, it is clear that every fuzzy $\mathcal{H}$-contractive mapping is t-uniformly continuous, therefore continuous. Now we give an example of function η.

Example 2.16. Let $f\colon [0, \infty) \to [1, \infty)$ be any onto and strictly increasing function. Define $g\colon [0, \infty) \to (0, 1]$ by $g(t) = \dfrac{1}{f(t)}$ for all $t \in [0, \infty)$. Then g is a strictly decreasing and onto function, therefore g is also one-to-one and so its inverse exists. Then $\eta \in \mathcal{H}$, where $\eta(t) = g^{-1}(t)$ for all $t \in (0, 1]$.

Remark 2.17. For $f(t) = 1 + t$ for all $t \in [0, \infty)$ in the above example, the fuzzy $\mathcal{H}$-contractive mapping reduces into the fuzzy contractive mapping. An example (see Example 3.3 of [10]) shows that the fuzzy $\mathcal{H}$-contractive mappings need not be a fuzzy contractive mapping. Therefore the fuzzy $\mathcal{H}$-contractive mappings are a proper generalization of the fuzzy contractive mappings.

Remark 2.18. Let $\eta \in \mathcal{H}$, then by (H1) and (H2) it is easy to see that the function η is necessarily continuous on $(0, 1)$.

Now we define the fuzzy $\mathcal{H}$-weak contractions.

Definition 2.19. Let $(X, M, *)$ be a fuzzy metric space and $T\colon X \to X$ be a mapping. Suppose the following condition is satisfied:

$$\eta(M(Tx, Ty, t)) \leq k \max\{\eta(M(x, y, t)), \eta(M(x, Tx, t)), \eta(M(y, Ty, t))\} \tag{2.3}$$

for all $x, y \in X, t > 0$, where $k \in (0, 1)$. Then T is called a fuzzy $\mathcal{H}$-weak contraction.

Note that a fuzzy $\mathcal{H}$-weak contraction is a generalization of $\mathcal{H}$-contractive mappings. Also, a fuzzy $\mathcal{H}$-contractive mappings is necessarily t-uniformly continuous, but the fuzzy $\mathcal{H}$-weak contractions need not be even continuous.

Example 2.20. Let $X = [0, 1]$, $*$ be the product t-norm, that is, $a * b = ab$ for all $a, b \in [0, 1]$ and let $M(x, y, t) = \dfrac{t}{t + |x - y|}$ for all $x, y \in X, t > 0$. Then $(X, M, *)$ is a fuzzy metric space. Define $T\colon X \to X$ by $Tx = \dfrac{x}{2}$ if $x \in [0, 1)$ and $T1 = 0$. Then T is not continuous therefore not a fuzzy $\mathcal{H}$-contractive mapping, but for $\eta(t) = \dfrac{1}{t} - 1$ for all $t \in (0, 1]$, T is a fuzzy $\mathcal{H}$-weak contractions with respect to $\eta \in \mathcal{H}$ and for $k \in [\dfrac{1}{2}, 1)$.

Now we can state the fixed point theorem for the fuzzy $\mathcal{H}$-weak contractions.

3. Fixed Point Theorem

Theorem 3.1. *Let $(X, M, *)$ be an M-complete fuzzy metric space and let $T\colon X \to X$ be a fuzzy $\mathcal{H}$-weak contraction with respect to $\eta \in \mathcal{H}$ such that:*

[(a)]

(1) $\prod_{i=1}^{k} M(x, Tx, t_i) \neq 0$ *for all* $x \in X, k \in \mathbb{N}$ *and any sequence* $\{t_i\}$ *in* $(0, \infty)$ *with* $t_i \searrow 0$;
(2) $r * s > 0$ *implies* $\eta(r * s) \leq \eta(r) + \eta(s)$ *for all* $r, s \in \{M(x, Tx, t)\colon x \in X, t > 0\}$;
(3) $\{\eta(M(x, Tx, t_i))\colon i \in \mathbb{N}\}$ *is bounded for all* $x \in X$ *and any sequence* $\{t_i\}$ *in* $(0, \infty)$ *with* $t_i \searrow 0$.

Then T has a unique fixed point $x^* \in X$ and for each $x_0 \in X$ the sequence $\{T^n x_0\}$ converges to x^*.

Proof. Let $x_0 \in X$ be arbitrary and define a sequence $\{x_n\}$ by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Now for all $t > 0$ it follows from (2.3) that

$$\begin{aligned} \eta(M(x_n, x_{n+1})) &= \eta\left(M(Tx_{n-1}, Tx_n)\right) \\ &\leq k \max\left\{\eta(M(x_{n-1}, x_n, t)), \eta(M(x_{n-1}, Tx_{n-1}, t)), \eta(M(x_n, Tx_n, t))\right\} \\ &= k \max\left\{\eta(M(x_{n-1}, x_n, t)), \eta(M(x_{n-1}, x_n, t)), \eta(M(x_n, x_{n+1}, t))\right\}. \end{aligned}$$

Setting $M_n(t) = M(x_n, x_{n+1}, t)$ for all $t > 0$, we have

$$\eta(M_n(t)) \leq k \max\left\{\eta(M_{n-1}(t)), \eta(M_n(t))\right\}. \tag{3.1}$$

If $\max\{\eta(M_{n-1}(t)), \eta(M_n(t))\} = \eta(M_n(t))$ then it follows from (3.1) that $\eta(M_n(t)) \leq k\eta(M_n(t)) < \eta(M_n(t))$, which is a contradiction. Therefore we must have

$$\max\left\{\eta(M_{n-1}(t)), \eta(M_n(t))\right\} = \eta(M_{n-1}(t))$$

and by (3.1) we have

$$\eta(M_n(t)) \leq k\eta(M_{n-1}(t)).$$

Repetition of this process gives

$$\eta(M_n(t)) \leq k^n \eta(M_0(t)). \tag{3.2}$$

From the above it is easy to conclude that

$$M_n(t) \geq M_0(t) \text{ for all } n \in \mathbb{N}, t > 0. \tag{3.3}$$

We shall show that $\{x_n\}$ is a Cauchy sequence. Let $m, n \in \mathbb{N}$ and $m < n$, then we can choose a strictly decreasing sequence $\{a_n\}$ of positive numbers such that

$\sum_{i=1}^{\infty} a_i = 1$. Now it follows from (GV2), (GV4), (3.3) and (a) that

$$\begin{aligned} M(x_m, x_n, t) &\geq M\left(x_m, x_m, t - \sum_{i=m}^{n-1} a_i t\right) * M\left(x_m, x_n, \sum_{i=m}^{n-1} a_i t\right) \\ &= 1 * M\left(x_m, x_n, \sum_{i=m}^{n-1} a_i t\right) = M\left(x_m, x_n, \sum_{i=m}^{n-1} a_i t\right) \\ &\geq \prod_{i=m}^{n-1} M\left(x_i, x_{i+1}, a_i t\right) = \prod_{i=m}^{n-1} M_i(a_i t) \\ &\geq \prod_{i=m}^{n-1} M_0(a_i t) > 0. \end{aligned}$$

Therefore, from the assumption (b) we have

$$\eta(M(x_n, x_m, t)) \leq \eta\left(\prod_{i=m}^{n-1} M_i(a_i t)\right) \leq \sum_{i=m}^{n-1} \eta\left(M_i(a_i t)\right)$$

which together with (3.2) yields

$$\eta(M(x_n, x_m, t)) \leq \sum_{i=m}^{n-1} k^i \eta(M_0(a_i t)).$$

Since $\{a_n\}$ is strictly decreasing sequence of positive numbers therefore by Remark 2.3, the sequence $\{\eta(M_0(a_i t))\}$ is nondecreasing and also bounded (by (c)) and so the series $\sum_{i=1}^{\infty} k^i \eta(M_0(a_i t))$ is convergent. Therefore, for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{i=m}^{n-1} k^i \eta(M_0(a_i t)) < \varepsilon \quad \text{for all} \quad n, m > n_0, m < n.$$

Thus by Proposition 2.12, sequence $\{x_n\}$ is an M-Cauchy sequence. By M-completeness of the space X and Theorem 2.6, there exists $x^* \in X$ such that $\lim_{n\to\infty} M(x_n, x^*, t) = 1$ for all $t > 0$. By Proposition 2.13 we have $\lim_{n\to\infty} \eta(M(x_n, x^*, t)) = 0$ for all $t > 0$. We shall show that x^* is the fixed point of T. For this we must show that $M(x^*, Tx^*, t) = 1$ for all $t > 0$. Suppose there exists $s > 0$ such that $M(x^*, Tx^*, s) < 1$ then $\eta(M(x^*, Tx^*, s)) > 0$. Now it follows from (2.3) that

$$\begin{aligned} \eta(M(Tx^*, x_{n+1}, s)) &= \eta(M(Tx^*, Tx_n, s)) \\ &\leq k \max\left\{\eta(M(x^*, x_n, s)), \eta(M(x^*, Tx^*, s)), \eta(M(x_n, Tx_n, s))\right\} \\ &= k \max\left\{\eta(M(x^*, x_n, s)), \eta(M(x^*, Tx^*, s)), \eta(M(x_n, x_{n+1}, s))\right\}. \end{aligned}$$

As $\lim_{n\to\infty} \eta(M(x_n, x^*, t)) = 0$ for all $t > 0$ and $\{x_n\}$ is an M-Cauchy sequence, therefore there exists $n_1 \in \mathbb{N}$ such that $\eta(M(x_n, x^*, s)) < \eta(M(x^*, Tx^*, s))$ and

$\eta(M(x_n, x_{n+1}, s)) < \eta(M(x^*, Tx^*, s))$ for all $n > n_1$. So, it follows from the above inequality that

$$\eta(M(Tx^*, x_{n+1}, s)) \leq k\eta(M(x^*, Tx^*, s)) \quad \text{for all} \quad n > n_1. \tag{3.4}$$

As $k < 1$ and $\eta \in \mathcal{H}$, we have $M(Tx^*, x_{n+1}, s) > M(x^*, Tx^*, s)$ for all $n > n_1$. Also for all $m \in \mathbb{N}, n > n_1$ we have

$$\begin{aligned} M(x^*, Tx^*, s + \frac{s}{m}) &\geq M\left(x^*, x_{n+1}, \frac{s}{m}\right) * M(Tx^*, x_{n+1}, s) \\ &> M\left(x^*, x_{n+1}, \frac{s}{m}\right) * M(x^*, Tx^*, s). \end{aligned}$$

Since $\lim_{n\to\infty} M(x_n, x^*, t) = 1$ for all $t > 0$ and the above inequalities are true for all $m \in \mathbb{N}$ therefore we must have

$$\lim_{n\to\infty} M(Tx^*, x_{n+1}, s) = M(x^*, Tx^*, s). \tag{3.5}$$

By Remark 2.18, (3.4) and (3.5) we have

$$\eta(M(x^*, Tx^*, s)) \leq k\eta(M(x^*, Tx^*, s)) < \eta(M(x^*, Tx^*, s)).$$

This contradiction shows that $M(x^*, Tx^*, t) = 1$ for all $t > 0$. Therefore $Tx^* = x^*$, that is, x^* is a fixed point of T. If y^* is another fixed point of T which is distinct from x^* that is, $Ty^* = y^*$ and $x^* \neq y^*$. Then from (2.3) we have

$$\begin{aligned} \eta(M(x^*, y^*, t)) &= \eta(M(Tx^*, Ty^*, t)) \\ &\leq k \max\{\eta(M(x^*, y^*, t)), \eta(M(x^*, Tx^*, t)), \eta(M(y^*, Ty^*, t))\} \\ &= k\eta(M(x^*, y^*, t)) < \eta(M(x^*, y^*, t)). \end{aligned}$$

This contradiction shows that $x^* = y^*$, and uniqueness follows. □

We illustrate the above result by the following example, which additionally shows that Theorem 3.1 essentially generalizes Theorem 3.2 of Wardowski [10].

Example 3.2. Let $N \geq 4$ be a fixed positive integer, $X = \left\{\frac{1}{2^n} : n \in \mathbb{N}, n \leq N\right\}$ and let $*$ be the product norm, that is, $a * b = ab$ for all $a, b \in [0, 1]$. Define the fuzzy set $M : X^2 \times (0, \infty) \to [0, 1]$ by

$$M(x, y, t) = \begin{cases} 1, & \text{if} \quad x = y\ ; \\ xy, & \text{otherwise.} \end{cases} \quad \text{for all } t > 0$$

Let $T : X \to X$ be a mapping defined by

$$T\left(\frac{1}{2^n}\right) = \begin{cases} \frac{1}{16}, & \text{if} \quad n \in \mathbb{N}_e, n \leq N\ ; \\ \frac{1}{2}, & \text{if} \quad n \in \mathbb{N}_o, n \leq N, \end{cases}$$

where $\mathbb{N}_e$ and $\mathbb{N}_o$ are the set of all even and odd natural numbers respectively. Then T is a fuzzy $\mathcal{H}$-weak contraction with respect to the function $\eta(t) = \ln\left(\frac{1}{t}\right)$

for all $t \in (0,1]$ and the constant $k = \frac{5}{6}$. To see this, we consider the following cases:

(i) If $x = \frac{1}{2^n}, y = \frac{1}{2^m}$, where $n, m \in \mathbb{N}_e$ or $n, m \in \mathbb{N}_o$ with $n, m \leq N$ then $\eta(M(Tx, Ty, t)) = \ln(1) = 0$ therefore (2.3) is satisfied trivially.

(ii) If $x = \frac{1}{2^n}, y = \frac{1}{2^m}$, where $n \in \mathbb{N}_e$, $m \in \mathbb{N}_o$ with $n, m \leq N$ then

$$\eta(M(Tx, Ty, t)) = \ln(32) = 5\ln(2)$$

and

$$\eta(M(x, Tx, t)) = \ln(2^n \cdot 16) = \ln\left(2^{(4+n)}\right) = (4+n)\ln(2).$$

Therefore (2.3) is satisfied for $k = \frac{5}{6}$.

(iii) If $x = \frac{1}{2^n}, y = \frac{1}{2^m}$, where $n \in \mathbb{N}_o$, $m \in \mathbb{N}_e$ with $n, m \leq N$ then by symmetry of $M(\cdot, \cdot, t)$ for all $t > 0$ and the case (ii) the condition (2.3) satisfied for $k = \frac{5}{6}$.

Note that, all the assumptions of Theorem 3.1 are satisfied which ensure the existence of unique fixed point of T, and $x^* = \frac{1}{2}$ is the unique fixed point of T. Now, if we consider the same fuzzy metric space $(X, M, *)$ and the same mapping T then we obtain that the mapping T is not a fuzzy $\mathcal{H}$-contractive mapping. To observe this, it is sufficient to take $x = \frac{1}{2^2}, y = \frac{1}{2}$ and for these values we have $M(Tx, Ty, t) = M\left(\frac{1}{16}, \frac{1}{2}, t\right) = \frac{1}{32}$ and $M(x, y, t) = M\left(\frac{1}{2^2}, \frac{1}{2}, t\right) = \frac{1}{8}$. Therefore there exist no $k \in (0,1)$ and $\eta \in \mathcal{H}$ such that

$$\eta(M(Tx, Ty, t)) \leq k\eta(M(x, y, t)) \quad \text{for all} \quad x, y \in X, t > 0.$$

Thus T is not a fuzzy $\mathcal{H}$-contractive mapping and result of Wardowski [10] is not applicable.

Corollary 3.3. *Let $(X, M, *)$ be an M-complete fuzzy metric space and let $T\colon X \to X$ be a mapping and $\eta \in \mathcal{H}$ are such that:*

(a) *there exist nonnegative constants k_1, k_2, k_3 such that $k_1 + k_2 + k_3 < 1$ and*

$$\eta(M(Tx, Ty, t)) \leq k_1\eta(M(x, y, t)) + k_2\eta(M(x, Tx, t)) + k_3\eta(M(y, Ty, t))$$

for all $x, y \in X, t > 0$;

(b) $\prod_{i=1}^{k} M(x, Tx, t_i) \neq 0$ *for all $x \in X, k \in \mathbb{N}$ and any sequence $\{t_i\}$ in $(0, \infty)$ with $t_i \searrow 0$;*

(c) *$r * s > 0$ implies $\eta(r * s) \leq \eta(r) + \eta(s)$ for all $r, s \in \{M(x, Tx, t)\colon x \in X, t > 0\}$;*

(d) *$\{\eta(M(x, Tx, t_i))\colon i \in \mathbb{N}\}$ is bounded for all $x \in X$ and any sequence $\{t_i\}$ in $(0, \infty)$ with $t_i \searrow 0$.*

Then T has a unique fixed point in $x^* \in X$ and for each $x_0 \in X$ the sequence $\{T^n x_0\}$ converges to x^*.

Acknowledgement. Author is thankful to Professor Dariusz Wardowski for providing his paper for this work.

References

1. A. George, P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems **64** (1994), 395-399.
2. A. George and P. Veeramani, *On some results of analysis for fuzzy metric spaces*, Fuzzy Sets Systems, **90** (1997), 365-368.
3. M. Grabiec, *Fixed points in fuzzy metric spaces*, Fuzzy Sets and Systems **27** (1988), 385-389.
4. V. Gregori, A. Sapena, *On fixed-point theorems in fuzzy metric spaces*, Fuzzy Sets and Systems **125** (2002), 245-252.
5. O. Hadžić, E. Pap, Fixed Point Theory in Probabilistic Metric Spaces, Mathematics and its Applications, vol. 536, Kluwer Academic Publishers, Dordrecht, Boston, London, 2001.
6. E.P. Klement, R. Mesiar, E. Pap, Triangular Norms, Trends in Logics, vol. 8, Kluwer Academic Publishers, Dordrecht, Boston, London, 2000.
7. I. Kramosil, J. Michlek, *Fuzzy metrics and statistical metric spaces*, Kybernetika **11** (1975), 336-344.
8. B. Schweizer, A. Sklar, *Statistical metric spaces*, Pacific J. Math. **10** (1960), 313-334.
9. R. Vasuki, P. Veeramani, *Fixed point theorems and Cauchy sequences in fuzzy metric spaces*, Fuzzy Sets and Systems **135** (2003), 415-417.
10. D. Wardowski, *Fuzzy contractive mappings and fixed points in fuzzy metric spaces*, Fuzzy Sets and Systems **222** (2013), 108-114.
11. L. A. Zadeh, *Fuzzy sets*, Information and Control, **89** (1965), 338-353.

Department of Applied Mathematics, Shri Vaishnav Institute of Technology & Science, Gram Baroli, Sanwer Road, Indore (M.P.) 453331, India.
E-mail address: satishmathematics@yahoo.co.in