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INTEGRAL OPERATOR ON WEIGHTED LORENTZ SPACE $\Gamma^p_X(\omega)$

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ABSTRACT. We study weighted Lorentz spaces, and by using a domination-like property for integrals, we prove the compactness of an integral operator defined on those spaces.

1. Introduction and Preliminaries

Lorentz spaces were introduced by George Gunter Lorentz (1910-2006) in [12, 13] as a generalization of the classical Lebesgue spaces L_p. Since then, Lorentz spaces have become a standard tool in mathematical analysis c.f. [1, 2, 3, 4, 6, 7, 8, 9, 11].

The space $\Gamma^p_\omega(X)$ was reintroduced in [5]. Some of its properties were studied, as well as the weighted composition operator defined on it. In the present paper, we would like to retake this concept to study the integral operator on $\Gamma^p_\omega(X)$ space.

In the sequel we gather some basic definitions and results which will be used throughout this paper. Let $(X, \mathcal{A}, \mu)$ be a ν-finite measure space and f be an $\mathcal{A}$-measurable function. The function that measures the level sets of f for $\lambda > 0$, is defined as

$$D_f(\lambda) = \mu\left(\{x \in X : |f(x)| > \lambda\}\right) \tag{1.1}$$

named distribution function of f. In the case we need to emphasize the underlying measure, we can write $D^\mu_f(\lambda)$.

It is worth noting that D_f depends solely on the absolute value $|f|$ of the function f, and it is possible for D_f to take the value $+\infty$. Additionally, it is essential to emphasize that D_f conveys information regarding the magnitude of f, but does not offer insight into the behavior of f at any specific point.

In fact, a function on $\mathbb{R}^n$ and all of its translates share the same distribution function. From (1.1), it follows that D_f is a (not necessarily strictly) decreasing function of λ and is right-continuous. For further details on distribution functions, refer to [2, 9, 11].

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The distribution function allows us to define the non-increasing rearrangement function, as follows

$$f^*(t) = \inf \{\lambda > 0 : D_f(\lambda) \leq t\}, \quad t > 0,$$

where we agree that $\inf \emptyset = +\infty$. Moreover, f^* is decreasing and right-continuous. Also, it should be clear that

$$f^*(0) \inf \{\alpha \geq 0 : D_f(\lambda) \leq 0\} = \|f\|_\infty,$$

since

$$\|f\|_\infty = \inf \{\alpha \geq 0 : \mu(\{x \in X : |f(x)| > \alpha\}) = 0\}.$$

If D_f is strictly decreasing, then

$$f^*(D_f(t)) = \inf\{\lambda > 0 : D_f(\lambda) \leq D_f(t)\} = t.$$

This fact shows us that indeed f^* is the inverse function of the distribution function D_f.

The coming result is interesting itself.

Proposition 1.1. *Let f be a strictly decreasing function $(f \geq 0)$ on $(0, \infty)$. Then $f^* = f(t)$.*

Proof. Let $\lambda > 0$ and consider the Lebesgue measure m, then

$$\begin{aligned} \mathrm{m}(\{x \in (0,\infty) : f(x) > \lambda\}) =& \mathrm{m}(\{x \in (0,\infty) : f^{-1}(f(x)) > f^{-1}(\lambda)\}) \\ =& \mathrm{m}(\{x \in (0,\infty) : 0 < x < f^{-1}(\lambda)\}) \\ =& f^{-1}(\lambda). \end{aligned}$$

Take $t = f^{-1}(\lambda)$, then $\lambda = f(t)$. Thus

$$f^*(t) = \inf\{\lambda > 0 : D_f(\lambda) \leq t\} = f(t),$$

that is

$$f^*(t) = f(t),$$

as we wanted to show. □

Let us denote by $\mathcal{F}(X, \mathcal{A})$ the set of all $\mathcal{A}$-measurable functions. The next definition is needed.

Definition 1.2. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two measure spaces. Two functions $f \in \mathcal{F}(X, \mathcal{A})$ and $g \in \mathcal{F}(Y, \mathcal{B})$ are said to be equimeasurable if they have the same distribution function, that is, if

$$D_f^\mu(\lambda) = D_g^\nu(\lambda)$$

for $\lambda \geq 0$.

The following proposition encapsulates some important properties of the distribution and non-increasing rearrangement functions.

Proposition 1.3. *Let f and g be two functions in $\mathcal{F}(X, \mathcal{A})$. For λ_1, $\lambda_2 > 0$ we have:*

(1) $|g| \leq |f|$ *μ-a.e implies* $D_g \leq D_f$;

(2) $D_{cf}(\lambda) = D_f\left(\frac{\lambda}{|c|}\right)$ *for all* $c \in \mathbb{C} \setminus \{0\}$;

(3) $D_{f+g}(\lambda_1+\lambda_2) \le D_f(\lambda_1) + D_g(\lambda_2)$;
(4) $D_{fg}(\lambda_1\lambda_2) \le D_f(\lambda_1) + D_g(\lambda_2)$;
(5) $f^*(t) > \lambda$ *if and only if* $D_f(\lambda) > t$.
(6) *If* $|f| \le \liminf_{n\to\infty} |f_n|$, *then* $f^* \le \liminf_{n\to\infty} f_n^*$;
(7) *if* $E \in \mathcal{A}$, *then* $(\chi_E)^*(t) = \chi_{[0,\mu(E)]}(t)$;
(8) *if* $E \in \mathcal{A}$, *then* $(f\chi_E)^*(t) \le f^*\chi_{[0,\mu(E)]}(t)$.
(9) f *and* f^* *are equimeasurable, that is*

$$D_f(\lambda) = D_{f^*}(\lambda).$$

For the proof of all nine properties see [10].

The non-increasing rearrangement is unique, as the following result shows.

Theorem 1.4. *There exits only one right-continuous decreasing function* f^* *equimeasurable with* f.

For a proof of the previous theorem see [10]. We shall end this section with the next result, which shows that the L_p-norms of f and f^* are equal.

Theorem 1.5. *Let* $(X, \mathcal{A}, \mu)$ *be a* σ*-finite measure space . Then for* $1 \le p < \infty$.

$$\int_X |f|^p \, d\mu = \int_0^\infty (f^*(t))^p \, dt. \tag{1.2}$$

Proof. (1.2) is obtained from the fact that f and f^* are equimeasurable. For details see [10]. $\square$

2. Weighted Lorentz space $\Gamma^p_X(\omega)$

Let f^{**} denote the maximal function of f^*, which is defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\, ds, \quad t > 0.$$

Some elementary properties of f^{**} are:

(1) $f^* \le f^{**}$;
(2) f^{**} is non decreasing;
(3) $(f+g)^{**} \le f^{**} + g^{**}$.

A weight ω is a non negative locally integrable function in $\mathbb{R}^n$ that takes values in $(0, \infty)$.

The weighted Lorentz space $\Gamma^p_X(\omega)$ are the space of all functions $f \in \mathcal{F}(X, \mathcal{A})$ for which

$$\|f\|_{\Gamma^p_X(\omega)} = \left(\int_0^\infty |f^{**}(t)|^p \omega(t)\, dt\right)^p$$

is finite. Here ω is a weight on $\mathbb{R}^+$ and $1 < p < \infty$. Specifically, in a set notation we define $\Gamma^p_X(\omega)$ as

$$\Gamma^p_X(\omega) = \left\{ f \in \mathcal{F}(X, \mathcal{A}) : \|f\|_{\Gamma^p_X(\omega)} < \infty \right\}.$$

3. A "Reciprocal" Domination Property for Integrals

The following is a well known result and its proof can be find in any Measure and Integration as well as in any Real Analysis books.

Theorem 3.1. *Let $(X, \mathcal{A}, \mu)$ be a measure space and f, g be two integrable functions on X. If $g \leq f$ a.e $[\mu]$ in X, then*

$$\int_X g\, d\mu \leq \int_X f\, d\mu.$$

We state and prove the theorem below, which we could not find among the literature. It is a kind of reciprocal of the domination property above. The aforementioned theorem will be the corner stone in the proof of our main result.

Theorem 3.2. *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. Suppose f and g measurable functions such that*

$$\int_X f\, d\mu \quad \text{and} \quad \int_X g\, d\mu$$

exist (through not necessarily finite). Also suppose

$$\int_E f\, d\mu \leq \int_E g\, d\mu$$

for all $E \in \mathcal{A}$. Then $f \leq g$ a.e $[\mu]$ in X.

Proof. First assume $\mu(X) < \infty$. Let

$$\begin{aligned} A =& \{x \in X : |g(x)| < \infty\} \\ B =& \{x \in X : g(x) = \infty\} \\ C =& \{x \in X : g(x) = -\infty\} \\ D =& \{x \in X : f(x) > g(x)\}. \end{aligned}$$

We have $X = A \cup B \cup C$, so

$$\begin{aligned} D =& D \cap X \\ =& (A \cap D) \cup (B \cap D) \cup (C \cap D). \end{aligned}$$

Clearly $B \cap D = \emptyset$, hence $D = (A \cap D) \cup (C \cap D)$. So

$$\mu(D) = \mu(A \cap D) + \mu(C \cap D). \tag{3.1}$$

We shall show $\mu(A \cap D) = 0 = \mu(C \cap D)$. Define for each $n \in \mathbb{N}$

$$E_n = \left\{x \in X : f(x) > g(x) + \frac{1}{n}\right\} \cap \{x \in X : |g(x)| \leq n\}$$

and

$$F_n = \{x \in X : f(x) > -n\} \cap C.$$

Then $E_n, F_n \in \mathcal{A}$ for all $n \in \mathbb{N}$. On E_n, $|g|$ is integrable since $\mu(X) < \infty$. Hence g is integrable on E_n. By hypothesis

$$\int\limits_{E_n} g\, d\mu \geq \int\limits_{E_n} f\, d\mu \geq \int\limits_{E_n} \left(g + \frac{1}{n}\right)\, d\mu,$$

that is

$$\int\limits_{E_n} g\, d\mu \geq \int\limits_{E_n} g\, d\mu + \frac{1}{n}\mu(E_n),$$

since

$$\int\limits_{E_n} g\, d\mu < \infty,$$

we conclude $\mu(E_n) \leq 0$.

Hence $\mu(E_n) = 0$ for all $n \in \mathbb{N}$ and so

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = 0.$$

But

$$\bigcup_{n=1}^{\infty} E_n = A \cap D,$$

thus

$$\mu(A \cap D) = 0. \tag{3.2}$$

Claim $\mu(F_n) = 0$ for all $n \in \mathbb{N}$.

Proof of Claim. Suppose that $\mu(F_n) \neq 0$ for some $n \in \mathbb{N}$, then there exists $m \in \mathbb{N}$ such that $\mu(F_m) > 0$, hence

$$\int\limits_{F_m} g\, d\mu = \int\limits_{F_m} (-\infty)\, d\mu = (-\infty)\mu(F_m) = -\infty.$$

But

$$\int\limits_{F_m} g\, d\mu \geq \int\limits_{F_m} f\, d\mu$$

(by hypothesis), that is $-\infty \geq -m\mu(F_m)$.

This entails $\mu(F_m) = \infty$ which is a contradiction since $\mu(F_m) < \infty$. Hence the claim is true. □

Thus $\mu(F_n) = 0$ for all $n \in \mathbb{N}$. So

$$\mu\left(\bigcup_{n=1}^{\infty} F_n\right) = 0.$$

But

$$\bigcup_{n=1}^{\infty} F_n = C \cap D,$$

then

$$\mu(C \cap D) = 0. \tag{3.3}$$

From (3.1), (3.2) and (3.3) we conclude $\mu(D) = 0$. Hence $f \leq g$ a.e $[\mu]$ in X. This proves Theorem 3.2 for the case $\mu(X) < \infty$.

Case $\mu(X) = \infty$. By σ-finiteness

$$X = \bigcup_{n=1}^{\infty} B_n$$

where $B_n \in \mathcal{A}$; $B_n \cap B_m = \emptyset$ if $n \neq m$ and $\mu(B_n) < \infty$ for all $n \in \mathbb{N}$. Then

$$D = \bigcup_{n=1}^{\infty} (D \cap B_n).$$

By finite case $\mu(D \cap B_n) = 0$ for all $n \in \mathbb{N}$. So $\mu(D) = 0$. Thus $f \leq g$ a.e $[\mu]$ in X. □

4. Compactness of an integral operator on $\Gamma_X^p(\omega)$ space

With all the previous results at hand, we are ready to prove the compactness of an integral operator on $\Gamma_X^p(\omega)$ space.

Theorem 4.1. *Let (X, d) be a metric space with $\mu(X) < \infty$. Let $K : X \times X \longrightarrow \mathbb{R}$ be a continuous function such that*

$$\int_X |K(x, y)|\, d\mu(x) \leq C \quad a.e\ y \in X$$

and

$$\int_X |K(x, y)|\, d\mu(y) \leq C \quad a.e\ x \in X.$$

For $1 \leq p < \infty$, the operator $T : \Gamma_X^p(\omega) \longrightarrow \Gamma_X^p(\omega)$ given by

$$Tf(x) = \int_X K(x, y) f(y)\, d\mu(y)$$

is compact.

Proof. Suppose $1 < p < \infty$. Taking q as the conjugate exponent of p, and using Hölder inequality in the product

$$|K(x, y) f(y)| = |K(x, y)|^{\frac{1}{q}} \left(|K(x, y)|^{\frac{1}{p}} |f(y)| \right)$$

we obtain

$$\begin{aligned} \int_X |K(x, y) f(y)|\, d\mu(y) \leq & \left[\int_X |K(x, y)|\, d\mu(y) \right]^{\frac{1}{q}} \left[\int_X |K(x, y)| |f(y)|^p\, d\mu(y) \right]^{\frac{1}{p}} \\ \leq & C^{\frac{1}{q}} \left[\int_X |K(x, y)| |f(y)|^p\, d\mu(y) \right]^{\frac{1}{p}} \end{aligned}$$

for almost $x \in X$. By Tonelli's Theorem we get

$$\begin{aligned}\int\limits_X \left[\int\limits_X |K(x,y)||f(y)|^p \, d\mu(y) \right]^p d\mu(x) \leq & C^{\frac{p}{q}} \int\limits_X \int\limits_X |K(x,y)||f(y)|^p \, d\mu(y) d\mu(x) \\ \leq & C^{\frac{p}{p+1}} \int\limits_X |f(y)|^p \, d\mu(y).\end{aligned}$$

Hence

$$\int\limits_X |Tf(x)|^p \, d\mu(x) \leq M \int\limits_X |f(y)|^p \, d\mu(y).$$

Since $\mu(X) < \infty$, then X is σ-finite. By Theorem 1.5, we obtain

$$\int_0^\infty ((Tf)^*(t))^p \, dt \leq M \int_0^\infty (f^*(t))^p \, dt.$$

Now, by Theorem 3.2, we get

$$((Tf)^*(t))^p \leq M(f^*(t))^p \quad \text{a.e [m] in } (0,\infty).$$

Thus

$$(Tf)^*(t) \leq M_0 f^*(t) \quad \text{a.e [m] in } (0,\infty),$$

where $M_0 = M^{1/p}$. Next, for $t > 0$ we arrive at

$$\frac{1}{t}\int_0^t (Tf)^*(s) \, ds \leq M_0 \frac{1}{t}\int_0^t f^*(s) \, ds$$

thus

$$(Tf)^{**}(t) \leq M_0 f^{**}(t),$$

which implies

$$\left(\int_0^\infty |(Tf)^{**}(t)|^p \omega(t) \, dt\right)^{\frac{1}{p}} \leq M_0 \left(\int_0^\infty |f^{**}(t)|^p \omega(t) \, dt\right)^{\frac{1}{p}}.$$

That is

$$\|Tf\|_{\Gamma^p_X(\omega)} \leq M_0 \|f\|_{\Gamma^p_X(\omega)}.$$

Which means that T is bounded. Now, let us fix a $y_0 \in X$ and let $\epsilon > 0$. By the continuity of K in $X \times X$, there exists $\delta > 0$ such that if $d(y, y_0) < \delta$ then

$$|K(x,y) - K(x,y_0)| < \epsilon$$

for each $x \in X$. Then if $d(y, y_0) < \delta$ and $f \in \Gamma_X^p(\omega)$ satisfies $\|f\|_{\Gamma_X^p(\omega)} \leq 1$; Hölder's inequality and Theorem 1.5 and the fact that $f^* \leq f^{**}$ imply

$$
\begin{aligned}
|Tf(x) - Tf(x_0)| =& \left| \int_X [K(x,y) - K(x_0,y)] f(y)\, d\mu(y) \right| \\
<& \epsilon \int_X |f(y)|\, d\mu \\
=& \epsilon \int_0^\infty f^*(t)\, dt \\
<& \epsilon \left(\int_0^\infty \omega(t)\, dt \right)^{\frac{1}{q}} \left(\int_0^\infty [f^*(t)]^p \omega(t)\, dt \right)^{\frac{1}{p}} \\
<& \epsilon \left(\int_0^\infty \omega(t)\, dt \right)^{\frac{1}{q}} \left(\int_0^\infty [f^{**}(t)]^p \omega(t)\, dt \right)^{\frac{1}{p}} \\
=& \epsilon \left(\int_0^\infty \omega(t)\, dt \right)^{\frac{1}{q}} \|f\|_{\Gamma_X^p(\omega)} \\
\leq& \epsilon \left(\int_0^\infty \omega(t)\, dt \right)^{\frac{1}{q}}.
\end{aligned}
$$

Therefore, we have proved that $\left\{ Tf : \|f\|_{\Gamma_X^p(\omega)} \leq 1 \right\}$ is a subset of $C(X)$, which is $\| \cdot \|_{\Gamma_X^p(\omega)}$-bounded and equicontinuous. The Arzelà-Ascoli theorem allows us to conclude that T is a compact operator. □

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