SUBMANIFOLDS OF SOME INDEFINITE CONTACT AND PARACONTACT MANIFOLDS

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Abstract. The purpose of the present paper is to study some types of submanifolds of indefinite contact and paracontact manifolds and a few properties of contact CR-submanifolds of an indefinite Sasakian manifold, indefinite Kenmotsu manifold, indefinite trans-Sasakian manifold and $\epsilon$-paracontact Sasakian manifold.

1. Introduction and preliminaries

Submanifolds of cosymplectic manifolds were discussed by G.D. Ludden in 1970 [9]. After him, B.Y. Chen studied geometry of submanifolds in [5]. B.Y. Chen and K. Ogiue in 1974 have given the notion of totally real submanifolds and differential geometry of Kaehler submanifolds (see [6], [11]). Blair and Shower introduced an almost contact manifold ([3], [4]). M. Kon and K. Yano investigated invariant and anti-invariant submanifolds in [8] and [13]. Later on, A. Bejancu in 1978 and 1979 introduced the notion of CR-submanifolds of a Kaehlerian manifold (see [1], [2]). M. Kobayashi discussed CR-submanifolds of a Sasakian manifold in 1981 [7]. Topology of 3-cosymplectic manifolds was discussed by B.C. Montano in [10]. Recently, in 2010 M. Tarafdar et.al. studied contact CR-submanifolds of an indefinite Sasakian manifold [12].

In this paper we study some types of submanifolds of some indefinite contact and paracontact manifolds and further make an analysis of the properties of contact CR-submanifolds of an indefinite Sasakian manifold, indefinite Kenmotsu manifold, indefinite trans-Sasakian manifold and $\epsilon$-paracontact Sasakian manifold.

Definition 1.1. An $(2n+1)$-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be an indefinite almost contact manifold if it admits an indefinite almost contact structure $(\phi, \xi, \eta)$, where $\phi$ is a tensor field of type $(1, 1)$, $\xi$ is a vector field and $\eta$ is a 1-form, satisfying

\[ \phi^2 X = -X + \eta(X)\xi, \quad \eta \circ \phi = 0, \quad \phi \xi = 0, \quad \eta(\xi) = 1, \]

(1.1)

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\end{itemize}
\[ \tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) - \epsilon \eta(X)\eta(Y), \quad (1.2) \]

\[ \tilde{g}(X, \xi) = \epsilon \eta(X), \quad (1.3) \]

for all vector fields \( X, Y \) on \( \tilde{M} \) and where \( \epsilon = \tilde{g}(\xi, \xi) = \pm 1 \) and \( \tilde{\nabla} \) is the Levi-Civita (L-C) connection for a semi-Riemannian metric \( \tilde{g} \).

**Definition 1.2.** An almost contact structure \((\phi, \xi, \eta)\) is said to be normal if the almost complex structure \( J \) on the product manifold \( \tilde{M} \times \mathbb{R} \) is given by

\[ J(X, f \frac{d}{dt}) = (\phi X - f \xi, \eta(X)\frac{d}{dt}), \quad (1.4) \]

where \( f \) is a \( C^\infty \)-function on \( \tilde{M} \times \mathbb{R} \) having no torsion, i.e., \( J \) is integrable, the condition for normality in terms of \( \phi, \xi \) and \( \eta \) is \([\phi, \phi] + 2d\eta \otimes \xi = 0 \) on \( \tilde{M} \), where \( [\phi, \phi] \) is the Nijenhuis tensor of \( \phi \).

Lastly, the fundamental 2-form \( \Phi \) is defined by

\[ \Phi(X, Y) = \tilde{g}(X, \phi Y). \quad (1.5) \]

**Definition 1.3.** An indefinite almost contact metric structure \((\phi, \xi, \eta, \tilde{g})\) is said to be indefinite cosymplectic structure, if it is normal and both \( \Phi \) and \( \eta \) are closed which is characterized by

\[ \tilde{\nabla}_X \phi = 0, \quad \tilde{\nabla}_X \eta = 0. \quad (1.6) \]

Now we define contact CR-submanifold of an indefinite cosymplectic manifold.

**Definition 1.4.** An \( m \)-dimensional Riemannian submanifold \( M \) of an indefinite cosymplectic manifold \( \tilde{M} \) is called a contact CR-submanifold if

1. \( \xi \) is tangent to \( M \),
2. there exists on \( M \) a differentiable distribution \( D : x \rightarrow D_x \subset T_x(M) \), such that \( D_x \) is invariant under \( \phi \); i.e., \( \phi D_x \subset D_x \), for each \( x \in M \) and the orthogonal complementary distribution \( D^\perp : x \rightarrow D^\perp_x \subset T_x(M) \) of the distribution \( D \) on \( M \) is totally real; i.e., \( \phi D^\perp_x \subset T^\perp_x(M) \), where \( T_x(M) \) and \( T^\perp_x(M) \) are the tangent space and the normal space of \( M \) at \( x \).

We call \( D \) (resp. \( D^\perp \)) the horizontal (resp. vertical) distribution. Also the contact CR-submanifold of an indefinite cosymplectic manifold is called \( \xi \)-horizontal (resp. \( \xi \)-vertical) if \( \xi_x \in D_x \) (resp. \( \xi_x \in D^\perp_x \)) for each \( x \in M \) by [7]. The distribution \( D \) (resp. \( D^\perp \)) can be defined by a projector \( P \) (resp. \( Q \)), satisfying

\[ P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0, \quad g \circ (P \times Q) = 0. \quad (1.7) \]

For a vector field \( X \) tangent to \( M \), we write

\[ \phi X = PX + QX, \quad (1.8) \]

where \( PX \) (resp. \( QX \)) is a tangential (resp. normal) component of \( \phi X \).
Again we write
\[ \phi N = BN + CN, \] (1.9)
for a vector field \( N \) in the normal bundle, where \( BN \) (resp. \( CN \)) is a tangential (resp. normal) component of \( \phi N \). The Gauss and Weingarten formulae are given by
\[ \nabla_X Y = \nabla_X Y + h(X, Y), \] (1.10)
\[ \nabla_X N = -A_N X + \nabla^\perp_X N, \] (1.11)
where \( \nabla \) is the Riemannian connection on \( M \) and \( \nabla^\perp \) is the connection on the normal bundle induced by \( \nabla \) and \( h \) is the second fundamental form of the immersion, satisfying the condition
\[ g(A_N X, Y) = g(h(X, Y), N). \] (1.12)

**Definition 1.5.** A submanifold \( M \) of a Riemannian manifold \( \tilde{M} \) is said to be totally umbilical if
\[ h(X, Y) = g(X, Y) H, \] (1.13)
where \( H \) is the mean curvature vector.

**Definition 1.6.** An indefinite almost contact metric structure \((\phi, \xi, \eta, \tilde{g})\) is called an indefinite Sasakian structure if
\[ (\tilde{\nabla}_Z \phi) W = \epsilon \eta(W) Z - g(Z, W) \xi, \] (1.14)

**Definition 1.7.** An indefinite almost contact metric structure \((\phi, \xi, \eta, \tilde{g})\) is called an Kenmotsu structure if
\[ (\tilde{\nabla}_Z \phi) W = g(\phi Z, W) \xi - \epsilon \eta(W) \phi Z, \] (1.15)

**Definition 1.8.** An indefinite almost contact metric structure \((\phi, \xi, \eta, \tilde{g})\) is called an indefinite trans-Sasakian structure if
\[ (\tilde{\nabla}_Z \phi) W = \alpha [g(Z, W) \xi - \epsilon \eta(W) Z] + \beta [g(\phi Z, W) \xi - \epsilon \eta(W) \phi Z] \] (1.16)
for functions \( \alpha \) and \( \beta \) on \( \tilde{M} \) of type \((\alpha, \beta)\),

**Definition 1.9.** An indefinite almost contact metric structure \((\phi, \xi, \eta, \tilde{g})\) is called an \( \epsilon \)-paracontact Sasakian structure if
\[ (\tilde{\nabla}_Z \phi) W = -g(\phi Z, \phi W) \xi - \epsilon \eta(W) \phi^2 Z, \] (1.17)
for all vector fields \( Z, W \) in respective \( \tilde{M} \).
2. Contact CR-submanifold of an indefinite Sasakian manifold

Here we consider a contact CR-submanifold $M$ of an indefinite Sasakian manifold $\tilde{M}$. Then we prove the following theorem and corollary:

**Theorem 2.1.** Let $M$ be a totally umbilical contact CR-submanifold of an indefinite Sasakian manifold $\tilde{M}$. Then the anti invariant distribution $D^\perp$ is one dimensional i.e. $\dim D^\perp = 1$.

**Proof.** For an indefinite Sasakian structure we have

$$ (\tilde{\nabla}_Z\phi)W = \epsilon\eta(W)Z - g(Z,W)\xi. \quad (2.1) $$

Also we know

$$ \tilde{\nabla}_Z\phi W = (\tilde{\nabla}_Z\phi)W + \phi\tilde{\nabla}_Z W. \quad (2.2) $$

From (2.1) and (2.2) we get

$$ \tilde{\nabla}_Z\phi W = \phi\tilde{\nabla}_Z W + \epsilon\eta(W)Z - g(Z,W)\xi. \quad (2.3) $$

Since $M$ is totally umbilical, by Weingarten and Gauss formula we have

$$ \nabla^\perp_Z\phi W - g(H,\phi W)Z = \phi[\nabla_Z W + g(Z,W)H] + \epsilon\eta(W)Z - g(Z,W)\xi \quad (2.4) $$

for any $Z, W \in \Gamma(D^\perp)$.

Taking inner product with $Z \in \Gamma(D^\perp)$ in (2.4) we obtain

$$ -g(H,\phi W)||Z||^2 = g(Z,W)g(\phi H, Z) + \epsilon\eta(W)||Z||^2 - g(Z,W)\epsilon\eta(Z). \quad (2.5) $$

Interchanging $Z$ and $W$ we have

$$ -g(H,\phi Z)||W||^2 = g(W,Z)g(\phi H, W) + \epsilon\eta(Z)||W||^2 - g(W,Z)\epsilon\eta(W). \quad (2.6) $$

Substituting (2.5) in (2.6) and simplifying we have

$$ g(H,\phi Z)[1 - \frac{g(Z,W)^2}{||Z||^2||W||^2}] + \epsilon\eta(Z)[1 - \frac{g(Z,W)^2}{||Z||^2||W||^2}] = 0. \quad (2.7) $$

The equation (2.7) has a solution if $Z \parallel W$ i.e. $\dim D^\perp = 1$. Hence the theorem is proved. $\square$

**Corollary 2.2.** A contact CR-submanifold of an indefinite Sasakian manifold reduces to a cosymplectic manifold provided the vector field $Z$ becomes the structure vector field $\xi$.

**Proof.** In a contact CR-submanifold of an indefinite Sasakian manifold we have

$$ (\tilde{\nabla}_Z\phi)W = \epsilon\eta(W)Z - g(Z,W)\xi. \quad (2.8) $$

Taking inner product with $Z$ and using (1.3) in (2.8) we obtain

$$ (\tilde{\nabla}_Z\phi)W = \epsilon\eta(W)g(Z, Z) - \epsilon\eta(Z)g(Z, W). \quad (2.9) $$

The equation vanishes provided $Z = \xi$ and therefore the contact CR-submanifold reduces to a cosymplectic manifold. $\square$
3. Contact CR-submanifold of an indefinite Kenmotsu manifold

In this section we take a contact CR-submanifold $M$ of an indefinite Kenmotsu manifold $\tilde{M}$. Then we obtain the following theorem and corollary:

**Theorem 3.1.** Let $M$ be a totally umbilical contact CR-submanifold of an indefinite Kenmotsu manifold $\tilde{M}$. Then the anti invariant distribution $D^\perp$ is one dimensional i.e. $\dim D^\perp=1$.

**Proof.** From (1.15) and (2.2) we get

$$\tilde{\nabla}_Z\phi W = \phi\tilde{\nabla}_Z W + g(\phi Z, W)\xi - \epsilon\eta(W)\phi Z.$$  \hspace{1cm} (3.1)

Since $M$ is totally umbilical, by Weingarten and Gauss formula we have

$$\nabla^*_Z\phi W - g(H, \phi W)Z = \phi[\nabla Z W + g(Z, W)H] + g(\phi Z, W)\xi - \epsilon\eta(W)\phi Z$$  \hspace{1cm} (3.2)

for any $Z, W \in \Gamma(D^\perp)$.

Taking inner product with $Z \in \Gamma(D^\perp)$ in (3.2) we have

$$g(H, \phi W)\|Z\|^2 = g(Z, W)g(H, \phi Z) + \epsilon\eta(W)g(\phi Z, Z) - g(\phi Z, W)\epsilon\eta(Z).$$  \hspace{1cm} (3.3)

Interchanging $Z$ and $W$ we have

$$g(H, \phi Z)\|W\|^2 = g(W, Z)g(H, \phi W) + \epsilon\eta(Z)g(\phi W, W) - g(\phi W, Z)\epsilon\eta(W).$$  \hspace{1cm} (3.4)

Substituting (3.3) in (3.4) and after a brief calculation we have

$$g(H, \phi W)[1 - \frac{g(Z, W)^2}{\|Z\|^2\|W\|^2}] + \frac{\epsilon}{\|Z\|^2\|W\|^2}[g(\phi Z, W)\eta(Z) - g(\phi Z, Z)\eta(W)] +$$

$$\frac{\epsilon g(Z, W)}{\|Z\|^2\|W\|^2}[\eta(W)g(\phi W, Z) - \eta(Z)g(\phi W, W)] = 0.$$ \hspace{1cm} (3.5)

The Equation (3.5) has a solution if $Z \parallel W$ i.e. $\dim D^\perp=1$. Hence the theorem.

**Corollary 3.2.** A contact CR-submanifold of an indefinite Kenmotsu manifold reduces to a cosymplectic manifold if vector field $Z$ becomes the structure vector field $\xi$.

**Proof.** Taking inner product with $Z$ in the equation (1.15) and using (1.3) we obtain

$$(\tilde{\nabla}_Z\phi)W = g(\phi Z, W)\epsilon\eta(Z) - \epsilon\eta(W)g(\phi Z, Z).$$  \hspace{1cm} (3.6)

The equation vanishes provided $Z = \xi$ and hence the contact CR-submanifold reduces to a cosymplectic manifold.

\hspace{1cm} $\square$

4. Contact CR-submanifold of an indefinite trans-Sasakian manifold

We consider a contact CR-submanifold $M$ of an indefinite trans-Sasakian manifold $\tilde{M}$. Then we obtain the following theorem and corollary.
**Theorem 4.1.** Let $M$ be a totally umbilical contact CR-submanifold of an indefinite trans-Sasakian manifold $\tilde{M}$. Then the anti invariant distribution $D^\perp$ is one dimensional i.e. $\dim D^\perp = 1$.

**Proof.** Using equations (1.16) and (2.2) we get

$$\tilde{\nabla}_Z \phi W = \alpha [g(Z,W)\xi - \epsilon_\eta(W)Z] + \beta [g(\phi Z,W)\xi - \epsilon_\eta(W)\phi Z] + \phi \tilde{\nabla}_Z W. \quad (4.1)$$

Since $M$ is totally umbilical, by Weingarten and Gauss formula we have

$$\nabla^\perp_Z \phi W - g(H,\phi W)Z = \alpha g(Z,W)\xi - \alpha \epsilon_\eta(W)Z + \beta g(\phi Z,W)\xi - \beta \epsilon_\eta(W)\phi Z + \phi \nabla^\perp_Z W + g(Z,W)\phi H \quad (4.2)$$

for any $Z, W \in \Gamma(D^\perp)$. Taking inner product with $Z \in \Gamma(D^\perp)$ in Equation (4.2) we have

$$-g(H,\phi W)g(Z,Z) = \alpha g(Z,W)\epsilon_\eta(Z) - \alpha \epsilon_\eta(Z)g(Z,Z) + \beta g(\phi Z,W)\epsilon_\eta(Z) - \beta \epsilon_\eta(Z)g(\phi Z,Z) + g(Z,W)g(\phi H,Z). \quad (4.3)$$

Interchanging $Z$ and $W$ we have

$$-g(H,\phi Z)||W||^2 = \alpha g(W,Z)\epsilon_\eta(W) - \alpha \epsilon_\eta(Z)||W||^2 + \beta g(\phi W,Z)\epsilon_\eta(W) - \beta \epsilon_\eta(Z)g(\phi W,W) + g(W,Z)g(\phi H,W). \quad (4.4)$$

$$g(H,\phi Z) = -\frac{\alpha g(W,Z)\epsilon_\eta(W)}{||W||^2} + \alpha \epsilon_\eta(Z) - \frac{\beta g(\phi W,Z)\epsilon_\eta(W)}{||W||^2} + \frac{\beta \epsilon_\eta(Z)g(\phi W,W)}{||W||^2} + \frac{g(W,Z)g(H,\phi W)}{||W||^2}. \quad (4.5)$$

Substituting Equation (4.3) in Equation (4.5) we have after some steps of calculations

$$g(H,\phi Z) = -\frac{\alpha g(W,Z)\epsilon_\eta(W)}{||W||^2} + \alpha \epsilon_\eta(Z) - \frac{\beta g(\phi W,Z)\epsilon_\eta(W)}{||W||^2} + \frac{\beta \epsilon_\eta(Z)g(\phi W,W)}{||W||^2} + \frac{g(W,Z)g(H,\phi W)}{||W||^2} - \frac{\alpha g(Z,W)\epsilon_\eta(Z)}{||Z||^2} + \alpha \epsilon_\eta(W) - \frac{\beta g(\phi Z,W)\epsilon_\eta(Z)}{||Z||^2} + \frac{\beta \epsilon_\eta(W)g(\phi Z,Z)}{||Z||^2} - \frac{g(Z,W)g(\phi H,Z)}{||Z||^2}. \quad (4.6)$$
\[ g(H, \phi Z)[1 - \frac{g(Z, W)^2}{||Z||^2 ||W||^2}] - \alpha \epsilon \eta(Z)[1 - \frac{g(Z, W)^2}{||Z||^2 ||W||^2}] - \frac{\beta g(\phi Z, W) \epsilon}{||W||^2} [\eta(W) - \frac{\eta(Z) g(Z, W)}{||Z||^2} - \frac{\beta \epsilon}{||W||^2} [\eta(Z) g(\phi W, W) - \frac{\eta(W) g(\phi Z, Z) g(W, Z)}{||Z||^2}] = 0. \] 

(4.7)

The equation Equation (4.7) has a solution if \( Z \parallel W \) i.e. \( \text{dim} D^\perp = 1 \). Hence the proof is complete.

**Corollary 4.2.** A contact CR-submanifold of an indefinite trans-Sasakian manifold reduces to a cosymplectic manifold when the vector field \( Z \) becomes the structure vector field \( \xi \).

**Proof.** Taking inner product with \( Z \) in the equation (1.16) and using (1.3) we obtain

\[(\nabla_Z \phi) W = \epsilon \eta(Z) [\alpha g(Z, W) + \beta g(\phi Z, W)] - \epsilon \eta(W) [\alpha ||Z||^2 + \beta g(\phi Z, Z)]. \] 

(4.8)

The equation vanishes provided \( Z = \xi \) and thus this contact CR-submanifold reduces to a cosymplectic manifold.

This is an example of a trans-Sasakian manifold, which proves the above corollory.

**Example 4.3.** Let \( R^3 \) be a 3-dimensional Euclidean space with rectangular coordinates \((x, y, z)\). In \( R^3 \) we define

\[ \eta = dz - ydx, \xi = \frac{\partial}{\partial z}, \]

(4.9)

\[ \phi(\frac{\partial}{\partial x}) = \frac{\partial}{\partial y}, \phi(\frac{\partial}{\partial y}) = -\frac{\partial}{\partial x} - y \frac{\partial}{\partial z}, \phi(\frac{\partial}{\partial z}) = 0. \] 

(4.10)

The Riemannian metric \( g \) is defined by the matrix:

\[
\begin{bmatrix}
\epsilon y^2 & 0 & -\epsilon y \\
0 & 0 & 0 \\
-\epsilon y & 0 & \epsilon
\end{bmatrix}
\]

Then it can be easily seen that \((\phi, \xi, \eta, g)\) forms an indefinite trans-Sasakian structure in \( R^3 \) for some functions \( \alpha \) and \( \beta \) and it reduces to a cosymplectic manifold when \( Z = \xi \). The corollary (3.1) can be verified by assuming \( \alpha = 0 \) and \( \beta = 1 \).

5. **Contact CR-submanifold of an epsilon-paracontact Sasakian manifold**

In this section we consider a contact CR-submanifold \( M \) of an \( \epsilon \)-paracontact Sasakian manifold \( \tilde{M} \). Then we get the following theorem and corollary.
Theorem 5.1. Let $M$ be a totally umbilical contact CR-submanifold of an $\epsilon$-paracontact Sasakian manifold $\tilde{M}$. Then the anti invariant distribution $D^\perp$ is one dimensional i.e. $\dim D^\perp=1$.

Proof. Substituting (1.17) in (2.2) we have
\begin{equation}
\tilde{\nabla}_Z\phi W = \phi \tilde{\nabla}_Z W - g(\phi Z, \phi W)\xi - \epsilon\eta(W)\phi^2 Z.
\end{equation}
Since $M$ is totally umbilical by Weingarten and Gauss formula we have
\begin{equation}
\nabla^\perp_Z\phi W - g(H, \phi W)Z = \phi[\nabla_Z W + g(Z, W)H] - g(\phi Z, \phi W)\xi - \epsilon\eta(W)\phi^2 Z
\end{equation}
for any $Z, W \in \Gamma(D^\perp)$. Taking inner product with $Z \in \Gamma(D^\perp)$ in (5.2) we have
\begin{equation}
g(H, \phi W)||Z||^2 = g(W, Z)g(H, \phi W) - \epsilon\eta(Z)||W||^2 + g(Z, W)\epsilon\eta(W).
\end{equation}
Interchanging $Z$ and $W$ we have
\begin{equation}
g(H, \phi Z)||W||^2 = g(W, Z)g(H, \phi W) - \epsilon\eta(Z)||W||^2 + g(Z, W)\epsilon\eta(W).
\end{equation}
Substituting (5.3) in (5.4) and simplifying we have
\begin{equation}
g(H, \phi Z)[1 - \frac{g(Z, W)^2}{||Z||^2||W||^2}] + \epsilon\eta(Z)[1 - \frac{g(Z, W)^2}{||Z||^2||W||^2}] = 0.
\end{equation}
The equation (5.5) has a solution if $Z \parallel W$ i.e. $\dim D^\perp=1$. Hence the proof follows.

Corollary 5.2. A contact CR-submanifold of an $\epsilon$-paracontact Sasakian manifold reduces to a cosymplectic manifold if the vector field $Z$ becomes the structure vector field $\xi$.

Proof. Taking inner product with $Z$ in equation (1.17) and using (1.3) we get
\begin{equation}
(\nabla_Z\phi)W = -g(\phi Z, \phi W)\epsilon\eta(Z) - \epsilon\eta(W)g(\phi^2 Z, Z).
\end{equation}
The equation vanishes provided $Z = \xi$ and hence this contact CR-submanifold reduces to a cosymplectic manifold.

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